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THE NEW
MATRICULATION ALGEBRA

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THE NEW MATRICULATION ALGEBRA

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PREFACE

This book is a revised edition of my Elementary Treatise on Algebra, Part I, Fourteenth Edition In the work of revision, I have throughout kept in view the Syllabus prescribed by the University though I have not hesitated to follow largely the suggestions of the Mathematical Association Thus the fundamental laws "have been inferred from examples in arithmetic" before their formal proofs are given, "the use of the minus sign to denote a negative quantity has been introduced early" and illustrated graphically, Long Multiplication and Long Division have been "postponed till after problems on x, y equations," and graphs have been "introduced as early as possible" and "used extensively" for purposes of illustration.

It has, therefore, been thought advisable to provide an easy course (i) of Multiplication and Division, (ii) of Formulæ and Factors, (iii) of Equations with numerical coefficients only and (iv) of Problems, their fuller and more copious treatment having been reserved for later Chapters [*vide* Chapters xiii, xiv, xxi, and xxii]

In the Chapter on Long Multiplication and Long Division, the Method of Detached Coefficients, Functional Notation and the Remainder Theorem have been introduced, while the next Chapter which deals with harder cases of Factorization, contains methods of Multiplication and Division by rearrangement of terms, Division by resolution into factors, and Identities

In Chapter xxiv many useful theorems with their applications have been given, and here and in the Appendix will be found many examples of what is commonly called Algebraical Artifices

Thus while an attempt has been made to conform to modern views as to order and treatment, the main features of the original work, which commended it to the notice of the public for over twenty years have been left intact

To meet the requirements of students taking up Additional Mathematics, Chapters on Quadratic Equations, Problems leading to them, Progressions and allied series, and graphs of Quadratic Equations, all of which have been treated very fully, are included in the present work

In the treatment of the whole subject, I have followed what to me seemed to be the most approved method of writing text-books, *viz*, to illustrate every article by examples, to give others for exercise, and where possible, to direct the student to work out the same example by different methods The solution of equations has been based on, "first principles (the four axioms)" and verification of the solution of both an equation and a problem has been insisted upon

The examples are numerous, many of which are original, while the rest have been taken from Cambridge and other Examination Papers The book thus contains many new examples, besides some old ones which have been inserted on account of their special interest

It is thus hoped that the present work will be of material help to students preparing for the Matriculation Examinations of Indian Universities

Any suggestions towards improvement from gentlemen engaged in the cause of education, if kindly communicated, will be most thankfully received

CALCUTTA,
December 18, 1911

PREFACE TO THE FOURTH EDITION

In this edition, the few misprints that had crept in have been corrected, all the Answers to the Examples carefully verified and some of the diagrams replaced by new ones It is thus hoped that the book in its present form will prove more useful to students

CALCUTTA,
November 25, 1915 } S C BASU

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PRINCIPAL FORMULÆ

1.	$(a+b)^2=a^2+2ab+b^2$	Art.	64.
2	$(a-b)^2=a^2-2ab+b^2$	Art.	64
3	$(a+b)(a-b)=a^2-b^2$	Art.	65.
4	$(a+b)^3=a^3+3a^2b+3ab^2+b^3$ $=a^3+b^3+3ab(a+b)$	Art	66.
5	$(a-b)^3=a^3-3a^2b+3ab^2-b^3$ $=a^3-b^3-3ab(a-b)$	Art	66
6	$(a+b)(a^2-ab+b^2)=a^3+b^3$	Art	67
7	$(a-b)(a^2+ab+b^2)=a^3-b^3$	Art.	67
8	$(x+a)(x+b)=x^2+(a+b)x+ab$	Art.	68
9	$\begin{cases}(a+b)^2=(a-b)^2+4ab\\(a-b)^2=(a+b)^2-4ab\end{cases}$	Art.	143
10	$a^2+b^2=(a+b)^2-2ab$ $=(a-b)^2+2ab$ $=\frac{1}{2}(a+b)^2+\frac{1}{2}(a-b)^2$	Art.	143.
11	$\begin{cases}4ab=(a+b)^2-(a-b)^2\\ab=\left(\dfrac{a+b}{2}\right)^2-\left(\dfrac{a-b}{2}\right)^2\end{cases}$	Art.	
12	$\begin{cases}a^3+b^3=(a+b)^3-3ab(a+b)\\(a+b)^3-a^3-b^3=3ab(a+b)\end{cases}$	Art	144.
13	$\begin{cases}a^3-b^3=(a-b)^3+3ab(a-b)\\(a-b)^3-(a^3-b^3)=-3ab(a-b)\end{cases}$	Art.	144.
14	$(ax+b)(cx+d)=acx^2+(bc+ad)x+bd.$	Art.	145.
15	$(x+a)(x+b)(x+c)=x^3+(a+b+c)x^2+(bc+ca+ab)x+abc$	Art.	147.
16	$\begin{cases}(a+b+c)^2=a^2+b^2+c^2+2bc+2ca+2ab\\(bc+ca+ab)^2=b^2c^2+c^2a^2+a^2b^2+2abc(a+b+c).\end{cases}$	Art	148.
17	$\begin{cases}(b-c)+(c-a)+(a-b)=0\\a(b-c)+b(c-a)+c(a-b)=0\\(b^2-c^2)+(c^2-a^2)+(a^2-b^2)=0\end{cases}$	Art	149.
18	$(a^2+ab+b^2)(a^2-ab+b^2)=a^4+a^2b^2+b^4$	Art.	150

19. $\left.\begin{cases} a^2(b-c)+b^2(c-a)+c^2(a-b) \\ bc(b-c)+ca(c-a)+ab(a-b) \\ a^2b-ab^2+b^2c-bc^2+c^2a-ca^2 \end{cases}\right\} = -(b-c)(c-a)(a-b).$

$a(b^2-c^2)+b(c^2-a^2)+c(a^2-b^2)=(b-c)(c-a)(a-b)$ Art. 154.

20 $a^3+b^3+c^3-3abc=(a+b+c)(a^2+b^2+c^2-bc-ca-ab)$

$=\frac{1}{2}(a+b+c)\{(b-c)^2+(c-a)^2+(a-b)^2\}$

Art 155

21. $S+2abc=(b+c)(c+a)(a+b)$ Art. 160, Ex. 1.

22. $S+3abc=(a+b+c)(bc+ca+ab)$ Art 160, Ex. 2

23. $(a+b+c)(bc+ca+ab)-abc=(b+c)(c+a)(a+b)$

Art. 160, Ex. 2.

24. $(a+b+c)^3=a^3+b^3+c^3+3(b+c)(c+a)(a+b)$ Art. 160, Ex. 3.

25. $(a+b+c)^3-a^3-b^3-c^3=3(b+c)(c+a)(a+b)$. Art 160, Ex. 3.

26. $S+a^3+b^3+c^3=(a+b+c)(a^2+b^2+c^2)$ Art 160, Ex 4.

27. $S-a^3-b^3-c^3-2abc=(b+c-a)(c+a-b)(a+b-c)$

Art 160, Ex 5.

28 $2b^2c^2+2c^2a^2+2a^2b^2-a^4-b^4-c^4$

$=(a+b+c)(b+c-a)(c+a-b)(a+b-c)$. Art. 160, Ex. 6.

29 If $ax+by+cz=0$ and $a'x+b'y+c'z=0$,

then $\frac{x}{bc'-b'c}=\frac{y}{ca'-c'a}=\frac{z}{ab'-a'b}$. Art 245.

30. If $\frac{a}{b}=\frac{c}{d}$, then $\frac{a+b}{a-b}=\frac{c+d}{c-d}$. Art 272.

31. If $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}=\ldots$, then

each ratio $=\left\{\frac{pa^n+qc^n+re^n+\ldots}{pb^n+qd^n+rf^n+\ldots}\right\}^{\frac{1}{n}}$. Art 276.

32 $x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$ Art 303.

33 Distance of a pt from origin $=\sqrt{x^2+y^2}$. Art. 326.

34 Distance bet two pts $=\sqrt{(x'-x'')^2+(y'-y'')^2}$ Art 327.

35. In an A P, n^{th} term $=a+(n-1)d$ Art 339.

36. In an A. P, $S=\frac{n}{2}(a+l)=\frac{n}{2}\{2a+(n-1)d\}$. Art 342

37 In a G P, n^{th} term $= ar^{n-1}$. Art 352.

38 In a G P, $S = \dfrac{a(r^n - 1)}{r-1} = \dfrac{rl - a}{r-1}$. Art. 355.

39. Sum of a G P to infinity $= \dfrac{a}{1-r}$ Art. 356.

40 Sum of n natural numbers $= \dfrac{n(n+1)}{2}$. Art. 363

41. Sum of *squares* of n natural nos $= \dfrac{n(n+1)(2n+1)}{6}$ Art. 364.

42 Sum of *cubes* of n natural nos $= \left\{\dfrac{n(n+1)}{2}\right\}^2$. Art. 365

THE NEW MATRICULATION ALGEBRA

INTRODUCTION

1 Origin of the Science It is now generally admitted that the Hindus were the inventors of the Science of Algebra It was introduced into Europe about the beginning of the thirteenth century, from the works of the Arabic writers "who certainly derived their knowledge from the Hindus" Hence the name Algebra "is the European corruption of the first words of an Arabic phrase, which may be thus written, *al jebr e al mokabaleh*, meaning *restoration and reduction*"

2 Unit, Measure The concrete quantities with which we are concerned in Algebra are *values*, *lengths*, *areas*, *volumes*, *weights*, &c They are all measured by the *number of times* each contains a *smaller quantity of its own kind*, which is called its unit Thus the quantity "five rupees" is *five* times its unit *one rupee*, "eight yards" is *eight* times its unit *one yard*, and so on The unit is represented by the figure 1, which is read *one* or *unity*

The number, which indicates *how often* the unit is contained in a quantity is called the **measure** of that quantity Thus 5 is the measure of the sum of money "5 rupees", 8 is the measure of the length "8 yards", &c

A quantity is represented by the number which is its measure and therefore **symbols of quantities, whether figures or letters, always represent numbers**

Numbers are either whole or fractional A whole number is called an **integer**; the corresponding adjective is *integral*

CHAPTER I

DEFINITIONS—SYMBOLICAL EXPRESSIONS

3 Definitions Algebra is the science that treats of numbers

In Arithmetic (which is also the science of numbers), quantities are represented by **figures** each of which has a *fixed* value as 1, 12, 54, 125, &c But in Algebra, quantities are represented

by **symbols** which are letters, to which we may assign *any valu*
we please Thus by 5 marbles a fixed number of marbles is mean
but by n marbles we may mean 4 marbles, or 15 marbles, or an
other number of marbles

The symbols used in Algebra to represent numbers are generall
small letters, such as $a, b, c, \ldots m, n, \ldots x, y, z$ If necessary, capit
letters and letters of the Greek alphabet are used for this purpose
as $A, B, C, \ldots$ α (alpha), β (beta), γ (gamma), ...

Quantities which have determinate values are called **know**
quantities or **constants** and are generally represented by th
first letters, as $a, b, c, d, \ldots$ Quantities whose values hav
to be determined are called **unknown quantities** or **variable**
and are usually represented by the *last* letters, as $x, y, z, v, \ldots$

4 **Signs of Operation** The five signs $+, -, \times, \div$ an
$=$ are called signs of operation and have the same meanings as i
Arithmetic, except the second sign which has a peculiar meanin
in Algebra, in addition to its ordinary meaning

Besides these, there is another sign $\sim$ (sign of **difference**
which is placed between two quantities to express their *differen*
when we do not know which of them is greater Thus $x \sim y$ i
either $x-y$ or $y-x$, according as x or y is the greater, if $x=$
and $y=5$, $x \sim y=3$

5 **Product, Factor, Coefficient** The sign of multiplicatio
is sometimes replaced by a **dot**, thus $a \cdot b$ is the same as $a \times$
In Algebra the sign of multiplication is *often* omitted betwee
a figure and a letter or between two letters, and these ar
then placed in close succession Thus $5a = 5 \times a$, $ab = a \times b$, $3xy$
$= 3 \times x \times y \times z$

It should be noted here that the sign of multiplication mu
not be omitted between two *figures*, thus the product of 2 and
should be written either as 2×3 or as $2 \cdot 3$, but never as 23, whic
means, not 2×3, but $20+3$

To distinguish a decimal point from a dot as a sign of multiplicatior
the former is placed *higher up* Thus $2{\cdot}3$ means $2\frac{3}{10}$ and 2.3 mear
2×3 or 6

In the product ab, the *first* number a is the *multiplicand* an
the *second* number b is the *multiplier* When, however, the mu
tiplier is a figure, it is always placed first, thus $a \times 5 = 5a$

When two or more quantities are multiplied together, the resul
is called their **product** Thus $2abc$ is the product of 2, a, b and c

Each of the quantities multiplied together to form a product
is called a **factor** of the product Thus 2, a, b and c are each a
factor of $2abc$

When a product is considered as divided into two sets of factors, each set is called the **coefficient** (*i e*, *co-factor*) of the other. Thus in $2abc$, 2 is the coefficient of abc, $2a$ of bc, and $2ab$ of c, in xyz, x is the coefficient of yz, y of xz, and z of xy

When the coefficient is expressed in *number*, it is called a **numerical co-efficient**, and is generally placed *first*, thus 2 in $2abc$ The numerical coefficient may be either *integral* or *fractional*, thus 3, $\frac{1}{2}$, $\frac{5}{7}$ in $3a$, $\frac{1}{2}x$ and $\frac{5}{7}mn$ respectively

When the coefficient is expressed in *letter*, it is called a **literal coefficient**, thus m in mn

Note When the coefficient is *unity*, it is generally omitted, thus it is usual to write a instead of $1a$

6 Division otherwise expressed. Division is often expressed by placing the dividend *over* the divisor in the form of a fraction, or by writing the dividend *before* the divisor with a slant line between them

Thus $84 \div 7$ is expressed either as $\frac{84}{7}$ or as $84/7$.

7. Other signs The following signs are used as abbreviations for the words or phrases, written against them —

(i) $\pm$ for **plus or minus.**

(ii) $>$ . . **is greater than**

(iii) $<$ . . **is less than**

(iv) **since or because**

(v) **hence or therefore.**

8 Symbolical Expressions As $10+3$ denotes the sum of 10 and 3, so $a+b$ denotes the sum of the numbers represented by the symbols a and b

If $a=5$, $b=8$, then $a+b=5+8=13$,
if $a=10$, $b=7$, then $a+b=10+7=17$;
if $a=2\frac{1}{3}$, $b=1\frac{2}{5}$, then $a+b=2\frac{1}{3}+1\frac{2}{5}=3\frac{11}{15}$,

and so on

Thus while $10+3$ has a *definite* value, viz, 13, $a+b$ may have *any number of values* Hence the symbolical expression $a+b$ is the *general form* for expressing the sum of *any two* numbers.

Similarly it will be seen that the symbolical expression $a-b$ is the general form to express the difference between any two numbers, $a \times b$ their product, and $a \div b$ the quotient of one by the other

Thus it appears that symbols in Algebra are perfectly *general* and may therefore stand for *any arithmetical number*. Thus *an algebraical result is the symbolical expression for all corresponding arithmetical problems*. Hence in its simple applications, Algebra may be regarded as *General* or *Universal* Arithmetic*.

The following examples will further illustrate the above remark.

Ex 1 If x stand for the tens' digit and y for the units' digit how would a number of two digits be expressed symbolically?

Just as 43 denotes $4\times10+3$, so $x\times10+y$ represents any number, whose tens' digit is x and units' digit is y. Thus $x\times10+y$ or $10x+y$ expresses symbolically *any number* consisting of two digits.

Thus if $x=2, y=5$, then $10x+y$ represents $10\times2+5$ or 25.

Ex 2 Express a rupees b annas c pies (i) in pies, (ii) in rupees and (iii) in annas.

$2R\ 3a\ 5p$ is a short way of writing $2R+3a+5p$.

(i) Now as $2R=(2\times192)$ pies, so $aR=(a\times192)p=192a$ pies; also as $3a=(3\times12)$ pies, so $ba=(b\times12)p=12b$ pies.

$$aR\ ba\ cp=(a\times192)p+(b\times12)p+cp$$
$$=(192a+12b+c)\text{pies}$$

(ii) Just as $3a=\frac{3}{16}R$, so $ba=\frac{b}{16}R$;

and as $5p=\frac{5}{192}R$, so $cp=\frac{c}{192}R$

$$aR\ ba\ cp=aR+\frac{b}{16}R+\frac{c}{192}R$$
$$=\left(a+\frac{b}{16}+\frac{c}{192}\right)R$$

(iii) As $2R=(2\times16)a$, so $aR=(a\times16)a=16a$ annas;

and as $5p=\frac{5}{12}a$, so $cp=\frac{c}{12}$ annas

$$aR\ ba\ cp=\left(16a+b+\frac{c}{12}\right)\text{annas}$$

Ex 3 I have a rupees in excess of x rupees; how many annas have I?

* Sir ISAAC NEWTON calls it Universal Arithmetic

Take a similar arithmetical example "I have 5 rupees in excess of 14 rupees, how many rupees have I?" Evidently I have $(14+5)$ rupees Similarly using the symbols a and x for 5 and 14, it is easily seen that I have $(x+a)$ rupees

the number of annas I have $=16(x+a)$

Ex 4 The age of a man is x years How old was he 5 years ago and how old will he be 5 years hence?

5 years ago his age was 5 years *less* than x, hence his age then was $(x-5)$ years

5 years hence his age will be 5 years *more* than x, hence his age then will be $(x+5)$ years

Ex 5 A man receives a rupees from each of x persons How much does he receive? If he spends y rupees out of the sum, how much has he left?

If the man receives 3 rupees from each of 10 persons, then he receives altogether (3×10) rupees, so here he receives ax rupees

If he spends y rupees, his total money will be deminished by y rupees; hence he will then have $(ax-y)$ rupees

Examples I

1 Express £m (i) in shillings, (ii) in pence, (iii) in crowns and (iv) in half-crowns

2 Express x shillings (i) in £, (ii) in pence, and (iii) in florins

3 Express y pence (i) in £, (ii) in shillings, and (iii) in half-pence

4 Express a pies (i) in rupees, (ii) in annas, and (iii) in half-rupees.

5 Express x yards (i) in feet, (ii) in inches, and (iii) in miles

6 Express x ft (i) in inches, and (ii) in yards

7 Express £m ns. pd. (i) in shillings, and (ii) in £

8 Express Rx ya zp (i) in rupees, and (ii) in annas

9 Express x metres (i) in decimetres, (ii) in centimetres, (iii) in millimetres and (iv) in kilometres

10 Express y millimetres (i) in centimetres, (ii) in metres and (iii) in kilometres

11. If a represent the units' digit and b the tens' digit, express the number symbolically.

12 If x represent the hundreds' digit, y the tens' digit and z the units' digit, express the number symbolically

13 Find the number which is greater than x by y

14 Find the number which is less than x by y

15 *By how much does a exceed 15 ?*

16 By how much is x less than 25 ?

17 If 100 is divided into two parts one of which is x, what is the other part ?

18 A man has x rupees and he spends 5 rupees out of it, how much has he left ?

19 A has x rupees and B has n rupees How many rupees have they together ? If A spends 8 rupees and B 10 rupees, how many rupees will each have and how many will they have altogether ?

20 An article costs a rupees and another costs 3 times as much, what is its price ?

21 If x represent a number, how would you represent (i) twice that number, (ii) half of it, (iii) 5 times it, and (iv) n times ?

22 I pay x shillings to each of y men, how much is that altogether (i) in £ and (ii) in pence ?

23 A sum of x rupees is divided equally among 8 boys, how many rupees does each get ? How many pies does each get ?

24 Find the cost, at a pence each, of (i) 5 articles and (ii) x articles Give the answer in shillings as well as in £

25 Twelve is greater than x by 5 Express this statement symbolically

26 Fifteen is less than m by 2 Express this statement symbolically

27 How far does a man walk, if he walks (i) 4 hours at 3 miles an hour, (ii) 5 hours at x miles, (iii) a hours at m miles, (iv) half an hour at $2x$ miles ?

28 How many men there are altogether, (i) if 8 benches seat x men each, (ii) if a benches seat 6 men each, (iii) if x benches seat y men each ?

29 A man is now c years old, how old was he m years ago and how old will he be y years hence ?

30 Four years hence a boy will be x years old, what was his age 6 years ago ?

31 A man bought a horse for x rupees and sold it for y rupees, how much did he gain ?

32 A man sold a horse for £a and thereby gained £b, what did the horse cost him ?

33 I engaged a man for x weeks and paid him $2y$s every week, how many £ did I pay him ? How many pence did I pay him ?

34 If a person owe m rupees to A, n rupees to B and $2p$ rupees to C, what is the amount of his debt?

35 I have a rupees, out of which I give x and y rupees to two persons, what have I left?

36 A is x years older than B, if A's age be a years, what is B's age?

37 A house and a garden together cost R1500, if the value of the house be x rupees, what is the value of the garden?

38 A man bought a house for x rupees and sold it for y rupees, gaining thereby a rupees Express this statement symbolically

9 **Expression, Term** Any collection of symbols connected by the signs of operation is called an *Algebraical Expression* or simply an **Expression** Thus $a+b-m+fd$ is an expression An expression is sometimes called a *Quantity*

The parts of an expression connected by the signs + and − are called its **terms** Thus the expression $3a+b+1-4cx$ contains 4 terms $3a$, b, 1 and $4cx$, the expression $a-b\times c+2d-b-1-c\times 5f$ contains 4 terms a, $b\times c$, $2d-b$ and $1-c\times 5f$, and so on

Observe that it is usual to consider the symbols connected by the signs × and ÷ as *one* term Thus in the second expression $b\times c$, $2d-b$ and $1-c\times 5f$ are each a term

If no sign is prefixed to a term, the sign + is always understood

An expression consisting of *one* term is called a *simple expression*, or a **Monomial**, as a, $-5b$, $3px$, &c An expression consisting of *two* terms is called a **Binomial**, as $a+b$, $ax-by$, &c An expression consisting of *three* terms is called a **Trinomial**, as $a+b+c$, $ax+2b-3z$, &c An expression consisting of *several* terms is called a *Multinomial Expression* or briefly a **Polynomial**, as $a+b+c-d+e+f$, $2x-by-2cz+5px-3q$, &c

A polynomial is sometimes termed a *Compound Quantity* or *Compound Expression*

10 **Substitution** We have seen [Art 8] that an algebraical expression will have a **numerical value** when the value of each of its symbols is given in *numbers* To find this value, we *substitute* the given numbers for the symbols and proceed as in Arithmetic

Ex 1 If $x=8$, $y=5$ and $z=3$, find the numerical value of $x+y+z$

$$x+y+z=8+5+3=16$$

12 Order of operation The expression $a+m+z$ means that m is to be *first* added to a and then to the sum z is to be added For example, $5+8+1=13+1=14$, *i e*, we first take 8 which is to the *right* of 5, and then take 1 which is to the *right* of 8 Thus in simplifying an expression we proceed from *left to right* Similar remarks hold in the case of the expression $a-m-z$, $a\times b\times c$, $a-b-c$, $a-b-c$, &c

Thus we see that *the order of the operations is from* **left** *to* **right** Hence to simplify an expression, we find (1) the value of *each term* by proceeding from left to right, and then (2) the value of the *whole expression* by proceeding also from left to right Thus to find the value of $15\times 8\div 12+36\div 4\times 14\div 6-72\times 5\div 8\div 3$

The expression has 3 terms, viz,

$$15\times 8\div 12,\ 36\div 4\times 14\div 6, \text{ and } 72\times 5\div 8\div 3$$

First term $=15\times 8\div 12=120\div 12=10$,

second term $=36\div 4\times 14\div 6=9\times 14\div 6=126\div 6=21$,

third term $=72\times 5\div 8\div 3=360\div 8\div 3=45\div 3=15$

Value reqd $=10+21-15=31-15=16$

REMARK The case in which the sign of multiplication is omitted between two or more quantities deserves *special notice* Here the omission of the sign and the consequent closeness of the quantities cause the product to be regarded as a *single* quantity Thus $a\div b\times c$ means that a is to be *first* divided by b and then the result multiplied by c, but $a\div bc$ means that a is to be divided by the product of b and c at once Hence if $a=36$ $b=3$ and $c=4$, $a\div b\times c=36\div 3\times 4=12\times 4=48$, but $a\div bc=36\div 12=3$ Similarly $a\div m\times n\times p$ and $a\div mnp$ give different results, and so on

Examples IV

Find the value of

1 $20-4\times 3$ 2 $35-7-5$ 3 $25+6\times 4$ 4 $18-8-4$

5 $5\times 3-16-4+6-2\times 5-3$

6 $2\times 6-4+8-2\times 5-4-15-3\times 2-5$

If $a=12$, $b=2$, $c=3$, $d=4$, find the numerical value of

7 $a-b-c$ 8 $a\times b-c$ 9 $ab\div c$ 10 $a-c\times d$

11 $a-cd$ 12 $a+d-b$ 13 $a-b\times c$ 14 $a-b-c\times d$

15 $4a-b-cd$ 16 $a-b+d-b-2c$ 17 $a\times b-c+a-b\times c$

18 $a-3b+a-3\times b+1$ 19 $2b\times c-3d\times 4c-a$

20 $3a\div 2b-c\times 5d-5\times d$ 21 $a\times b-c+a-b\times c-a-bc$

22 $6ab\times 4b-c-ad+5ad-bc-3ac-2b-36\times 5d$

23 $8a-3b\times 5c-6d-2-12a-8b-32c\times 16a-9d+a-2bcd$

If $a=4$, $b=8$, $p=10\frac{1}{4}$, $q=\frac{5}{6}$, $x=4\frac{1}{3}$, find the value of

24 $10+2q-b$ 25 $3p-a-8q$. 26 $\frac{b}{a}-\frac{q}{p}+1$

27 $a+\frac{b}{p}-\frac{1}{q}$ 28 $15a+10b-2p+4q$ 29 $\frac{a}{b}-\frac{b}{x}+\frac{82}{p}-q$.

30 $\frac{2a}{3b}+\frac{2p}{b}-\frac{13\frac{2}{3}\times aq}{5p}-\frac{x}{12a}$

If $r=\cdot 02$, $s=3\cdot 5$, $x=\cdot 005$, find the value of

31 $5r+3s-10x$ 32 $rs-15x+2r$ 33 $4x-3r+s-rx$

13 Power, Index, Exponent If a quantity be multiplied by *itself* any number of times, the **product** is called a **Power** of that quantity Thus aa is called the *second power* or the **square** of a, aaa is called the *third power* or the **cube** of a, $aaaa$ is called the *fourth power* of a, and so on The quantity *itself* is called its *first power*, thus the *first power* of a is a

For the sake of convenience, aa is written a^2, aaa is written a^3, $aaaa$ is written a^4, and generally $aaaa$. to n factors is written a^n The small figure or letter placed above a quantity and to its right, *to indicate how often that quantity is to be taken as a factor in a power*, is called the **Index** or **Exponent** of that power Thus 2, 3, 4, n are the *indices* or *exponents* of a^2, a^3, a^4 and a^n respectively

The first power of a being a or a^1, the *index of the first power of a quantity is unity.*

In the above examples, a^2 is read "a raised to the *second power*," or "a *squared*", a^3 is read "a raised to the *third power*," or "a *cubed*", a^4 is read, "a raised to the *fourth power*," or briefly "a to the *fourth*", and a^n is read "a to the nth," or "a nth"

Note **Any power of 0 is 0, and any power of 1 is 1**

For if $a=0$, $a^2=0$, $a^3=0$, &c

if $a=1$, $a^2=1\times 1=1$, $a^3=1\times 1\times 1=1$, &c

14 Bracket, Vinculum Brackets ( ), { }, [ ], are used to enclose the terms of an expression which are meant to be taken *collectively* Thus $a+(b+c)$ means that the sum of b and c is to be added to a, similarly $(a+b)\div x^2$ means that the sum of a and b is to be divided by x^2, $(ab)^2$ means that the product of a and b is to be squared, and so on

Sometimes a *line* is placed over the terms which are meant to be taken as a whole, thus $a+\overline{b+c}$ is the same as $a+(b+c)$ The line is then termed a VINCULUM

The three kinds of Brackets are sometimes conveniently called *Parentheses*, *Braces* and *Crochets* respectively

If $a=3$, $x=2$, find the value of

Ex 1 $4^5=4\times4\times4\times4\times4=1024$

Ex 2 $(15)^a=(15)^3=15\times15\times15=3375$

Ex 3 $7^{xa}=7^{2\times3}=7^6=7\times7\times7\times7\times7\times7=117649$

Ex 4 $4a^4=4\times3^4=4\times3\times3\times3\times3=324$

Ex. 5 $(5ax)^x=(5\times3\times2)^2=(30)^2=30\times30=900$

Examples V

If $a=3$, $x=2$, $y=4$, find the value of

1 5^3 2 8^x 3 $(2^y)^2$ 4 $(12)^a$ 5 $(50)^y$ 6 3^{ax}

If $a=2$, $b=0$, $c=8$, $x=3$, $y=4$, find the value of

7 a^5 8 $3c^3$ 9 $14y^2$ 10 $12ax^5$ 11. $24bx^3$ 12 $10c^2x^4$

13 $\frac{3}{8}a^2xy^2$ 14 $\frac{11}{16}x^2y^3$ 15 $\frac{29}{34}a^3b^2x^4$ 16 $\frac{5}{12}a^4cy^5$

17 $\frac{1}{9}x^3c^2y^5$ 18 $\frac{5}{12}a^7b^3x^5$

If $a=5$, $b=1$, $c=4$, $m=2$, $n=3$, $r=7$, $x=6$, find the value of

19 a^m 20 m^x 21 $3x^n$ 22 $5b^r$ 23 $4c^a$ 24 $(2ab)^x$

25 $(4a^2x)^c$ 26 $(3ar)^{mb}$ 27 $3m^2x^c$ 28 $\frac{3}{4}xa^m$ 29 $\frac{5}{18}nx^n$

30 $\frac{3}{7}r^3m^r$ 31 $\frac{4}{9}(a^2x^3)^c$ 32 $\frac{14}{325}a^2c^mn^x$ 33 $b^rc^2r^bx^n$

If $a=4$, $x=r=2$, $y=3$, find the value of

34 $\left(\frac{y}{x}\right)^5$ 35 $\frac{5}{6}\left(\frac{y}{a}\right)^r$ 36 $\left(\frac{5a}{30}\right)^y$ 37 $\left(\frac{3r}{5x}\right)^4$

38 $\frac{3}{4}\left(\frac{3x^4}{4y^3}\right)^a$ 39 $\frac{9}{16}\left(\frac{2a^2}{3r^3}\right)^{xy}$

15 Square Root, Cube Root The **Square Root** of a given quantity is that quantity whose *square* or *second power* gives the proposed quantity Thus 2 is a square root of 4, for $2^2=4$, a is a square root of a^2, &c The square root of a quantity, say a, is written $\sqrt[2]{a}$, or more commonly $\sqrt{a}$, hence $\sqrt{4}=2$, $\sqrt{9}=3$, &c $\sqrt{a}$ is read "*square root of* a" or more commonly "*root* a"

The **Cube Root** of a given quantity is that quantity whose *cube* or *third power*, gives the proposed quantity Thus 3 is a cube root of 27, for $3^3=27$, a is a cube root of a^3, &c The cube root of a quantity, say a, is represented by $\sqrt[3]{a}$, hence $\sqrt[3]{8}=2$, $\sqrt[3]{125}=5$, &c $\sqrt[3]{a}$ is read "*cube root of a*"

The sign $\sqrt{\ }$ by means of which the root of a given quantity is expressed, is called the RADICAL SIGN, and is a corruption of the initial letter *r* of the word *radix*

Note From Art 14, we at once see that $\sqrt{\overline{a+b}}$ is equivalent to $\sqrt{(a+b)}$, the line over $a+b$ serving as a *vinculum* Hence there is a distinction between $\sqrt{\overline{a+b}}$ and $\sqrt{a}+b$, for $\sqrt{\overline{a+b}}$ means the square root of the sum of a and b, whereas $\sqrt{a}+b$ means that b is to be added to the square root of a, so also $\sqrt{\overline{3x}}$ means the square root of the product $3x$, but $\sqrt{3}x$ means that the square root of 3 is to be multiplied by x Thus **where there is no bracket or vinculum, the radical sign refers only to the quantity before which it is placed** Hence to express the root of a *number*, bracket or vinculum is unnecessary, thus $\sqrt{35}$ is sufficient to express $\sqrt{(35)}$ or $\sqrt{\overline{35}}$

If $a=12$ and $x=9$, find the value of

Ex. 1 $\sqrt{(3ax)}=\sqrt{(3\cdot12\cdot9)}=\sqrt{(3\cdot3\cdot4\cdot3\cdot3)}=3\cdot2\cdot3=18$

Ex 2 $\sqrt[3]{(24x^4)}=\sqrt[3]{(3\cdot8\cdot9\cdot9\cdot9\cdot9)}=\sqrt[3]{(27\cdot8\cdot9\cdot9\cdot9)}=3\cdot2\cdot9=54$

Ex. 3 If $x=4$ and $c=5$, find the value of $\sqrt{x^c}$

$\sqrt{x^c}=\sqrt{4^5}=\sqrt{(4\cdot4\cdot4\cdot4\cdot4)}=2\cdot2\cdot2\cdot2\cdot2=32$

Ex 4 If $x=8$, $a=2$, $b=32$, find the value of $\sqrt[3]{(2bx^a)}$

$\sqrt[3]{(2bx^a)}=\sqrt[3]{(2\cdot32\cdot8^2)}=\sqrt[3]{(64\cdot8\cdot8)}=\sqrt[3]{\overline{(4\cdot4\cdot4)(4\cdot4\cdot4)}}=4\cdot4=16$

Ex 5 Find the value of $\sqrt{\overline{4x+y}}$, $\sqrt{\overline{4x}}+y$ and $\sqrt{4}x+y$, when $x=16$ and $y=36$

$\sqrt{\overline{4x+y}}=\sqrt{(4\times16+36)}=\sqrt{100}=10$

$\sqrt{\overline{4x}}+y=\sqrt{(4\times16)}+36=\sqrt{64}+36=8+36=44$

$\sqrt{4}x+y=2\times16+36=32+36=68$

Examples VI

If $a=12$, $b=15$, $x=1$, $y=9$, find the value of

1 $\sqrt{(16xy)}$ 2 $\sqrt{(12a^3y)}$ 3 $\frac{2}{5}\sqrt{(5ab^3)}$ 4 $3x\sqrt{(a^2xy)}$

5 $3\sqrt{(20aby^3)}$ 6 $\sqrt[3]{(24y)}$ 7 $2\sqrt[3]{(80ab^3)}$

8 $2b\sqrt[3]{(3y^4)}$ 9. $\frac{3}{4}\sqrt[3]{(4a^2y^2)}$ 10 $\frac{8}{15}\sqrt[3]{(45b^2xy)}$

If $a=2$, $b=4$, $c=1$, $d=9$, $x=8$, $y=3$, $z=0$, find the value of

11. $\sqrt{\left(\frac{4b^3}{ax}\right)}$ 12 $\sqrt{\left(\frac{3ay}{8cd}\right)}$ 13 $\sqrt{\left(\frac{3dx^2}{b^2y}\right)}$.

14. $\sqrt{\left(\frac{1}{2dx}\right)}$. 15 $\frac{1}{\sqrt{(3d^2y)}}$ 16. $\frac{1}{\sqrt{(6a^3x^2y)}}$

17 $\dfrac{8a^2z}{\sqrt{(8c^3d^3)}}$ 18 $\sqrt[3]{\left(\dfrac{64x}{9y}\right)}$ 19 $\sqrt[3]{\left(\dfrac{2bx}{27dy}\right)}$

20 $\sqrt[3]{\left(\dfrac{16by^3}{27c^4}\right)}$ 21 $\sqrt[3]{\left(\dfrac{1}{125ab}\right)}$ 22 $\sqrt[3]{\left(\dfrac{1}{8cx^2}\right)}$

If $a=9$, $b=1$, $c=25$, $d=8$, $x=16$, $y=36$, find the value of

23 $c\sqrt{x+a}$ 24 $c\sqrt{x}+a$ 25 $\sqrt{cx}+\sqrt{a}$

26 $\sqrt{(ax+bc)}$ 27 $\sqrt{ax}+\sqrt{(bc)}$ 28 $\sqrt{(ax)}+\sqrt{bc}$

29 $\sqrt{ax}+\sqrt{bc}$ 30 $\sqrt[3]{3(b+d)}$ 31 $\sqrt[3]{3a}(b+d)$

32 $2\sqrt[3]{d(x-a)}$ 33 $x\sqrt[3]{3a+b+y}$ 34 $2\sqrt[3]{4x+y+c}(2a-x)^3$

16 We here give a few examples of expressions involving powers and roots

Ex 1 If $a=5, b=2, c=\frac{2}{3}$, find the value of $a(4b-3c)^2-c^2(a-2b)$

Given expn $=5(4\cdot2-3\times\frac{2}{3})^2-(\frac{2}{3})^2(5-2\cdot2)$

$=5(8-2)^2-\frac{4}{9}(5-4)$

$=5\cdot6^2-\frac{4}{9}\times1=180-\frac{4}{9}=179\frac{5}{9}$

Ex 2 If $a=4$, $b=6$, $c=2$, find the value of $5a^2\dfrac{b-c}{b+c}-c^3\dfrac{2b}{(b-2c)^2}$

Given expn $=5\cdot4^2\dfrac{6-2}{6+2}-\dfrac{2\cdot6\cdot2^3}{(6-2\cdot2)^2}$

$=5\cdot4^2\cdot\frac{4}{8}-\dfrac{12\cdot2^3}{2^2}=5\cdot8-12\cdot2=40-24=16$

Ex 3 If $y=x^2-3x+15$, find the values of y, when x has the values 0, 1, 2, 3, 5, 8

Arrange the values of x and y as in the following table

$x=$	0	1	2	3	5	8
$x^2=$	0	1	4	9	25	64
$-3x=$	0	-3	-6	-9	-15	-24
$15=$	15	15	15	15	15	15
$y=$	15	13	13	15	25	55

Thus the required values of y are 15, 13, 13, 15, 25 and 55

This is called **Tabulating** the values of an expression

Examples VII

If $a=2$, $b=3$, $c=5$, $d=0$, find the value of

1 a^2+b^2+2ab 2 $a^3+b^3+c^3+3abc$ 3 $a^3b+b^3c+c^3d+d^3a$

4 $a^4+b^4+c^4+d^4+4abcd$ 5 $3a^3+4bc^2-4abc$

6 $10a^4c^3-5b^2cd+2a^2bc^2-3ad^2+27c^3$

7 $\frac{1}{2}ac^2+\frac{1}{2}b^2c-\frac{3}{5}abc-a^3+b^3$

8 $\frac{3}{10}a^3c+2a^6d-80d^4+\frac{1}{5}c^4-\frac{5}{6}b^3a^3$

9 $\frac{2}{3}abc+\frac{3}{5}bc^2d-15a^2c+\frac{4}{5}bc^2-\frac{5}{7}d^2a^2$

If $a=5$, $b=2$, $c=3$, $d=1$, find the value of

10 $a^2+b^2-c^2+d^2$.   11 $a^3+b^3+c^3+d^3$ 12. $a^4+b^4+c^4+d^4$

13 $(a+b)^2+(c+d)^2$ 14 $(a-d)^2-(c-b)^2$ 15. $(a^2-c^2)^2-(b^2-d^2)^2$

16 $(a+b+c)^2-a^2-b^2-c^2$.   17. $(a+b+c)^3-a^3-b^3-c^3$

18 $(ad+bc)^2+(ac-bd)^2$ 19 $(2a-b^2)^3+(3c-bc)^2$.

20 $\frac{3}{8}(2c^2-ab)^3+\frac{5}{32}(a^2d^4-c^3)^2$

If $a=2$, $b=\frac{1}{3}$, $c=\frac{1}{4}$ and $d=\frac{1}{3}$, find the value of

21. $a^4-4a^3b+9b^2c^2-48cd^4$.   22 $3(ab-6cd)^2+4(ad-bc)^2$

23. $\dfrac{3c^2}{a^2}+\dfrac{b^2}{d^2}-\dfrac{3-b^2}{2(1+2c^2)}$.   24 $\dfrac{a^2-6b^2}{4d^2+3b^2}-\dfrac{18c^3+\frac{1}{12}a^2}{12-a^3}$

If $a=4$, $b=3$, $c=5$, $d=0$, find the value of

25 $6\sqrt{(a^2+b^2)}-(c-bd)^2$ 26 $\sqrt{(5ac^3)}+\dfrac{(3a^2-2bc)^2}{\sqrt{(3b^2)}}$

27 $\frac{2}{3}\sqrt{(a^2+b^2+d^2)}-\dfrac{\sqrt{(c^2-b^2-d^2)}}{3ac}$.

28 $\dfrac{\sqrt{(3b^2-a^2+9)}}{ad+bc}+2\sqrt{\{(c-b)(a-d)\}}$.

29 If $y=x^2+5x+3$, find the values of y, if x has the values 0, 1, 2, 3, 4, and 5

30 Find the values of the expression $x^2-12x+35$ for the values of x, 0, 1, 2, 5, 7, 11 and 15

31. Shew that $5x^2-32x+12=0$, whether $x=6$ or $=\frac{2}{5}$

32 Shew that $4x^3-15x^2-17x=6$ whether $x=1$, or $=2$, or $=\frac{1}{4}$

CHAPTER II

NEGATIVE QUANTITIES

17 **Positive and Negative Quantities** In Arithmetic $1-7$ is an impossible operation and has *no meaning* But in Algebra such an expression is capable of interpretation and a final result such as -3 can stand alone just as $+3$ does and has an intelligible meaning assigned to it We shall shew by considering a few examples that this is the case

If a person gains R8 by one transaction and then loses R5 by another, then his *gain* on the whole is R3 This is expressed algebraically by

$$+\text{R}8-\text{R}5=+\text{R}3\ ,$$

i e, +R3 denotes that he has now R3 more than when he began

Again if he first gains R8 and then loses R8, his gain being exactly equal to his loss, the effect on his stock is *nothing* This is shewn algebraically thus

$$+\text{R}8-\text{R}8=\text{R}0\ ,$$

i e, R0 indicates that he is neither richer nor poorer by the transactions

But suppose he gains R5 by one transaction and then loses R8 by a second, his loss on the whole is R3 Now if the loss of R8 be considered as made up of one loss of R5 and another of R3, we may express the result of the two transactions by

$$+\text{R}5-\text{R}5-\text{R}3 \text{ or } -\text{R}3\ ,$$

i e, −R3 indicates that he is now poorer by R3

And the result would clearly be the same, if he were to lose R8 first and then gain R5

Also if he loses R8 by a first venture and then again R5 by a second, his total loss is R13 This is expressed thus

$$-\text{R}8-\text{R}5=-\text{R}13\ ,$$

i e, −R13 shews that he is now poorer than formerly by R13

And the result would be the same, if he were to lose R5 first and then again R8

Thus gain and loss are quantities of the *same class*, inasmuch as each is a *sum of money*, but they are of *opposite character* inasmuch as a gain *increases* the stock while a loss *decreases* it

As another example, suppose a man to start from a fixed point O and walk first 6 miles due East and then 4 miles due West

i e, in the *opposite* direction. His position with regard to O is expressed algebraically by

$$+6 \text{ miles} - 4 \text{ miles} = +2 \text{ miles};$$

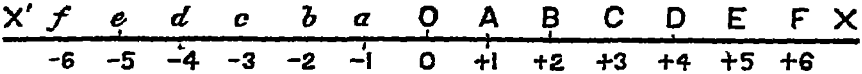
i e, $+2$ miles indicates that he is now 2 miles *East* of O

Next suppose him to start from O and walk first 4 miles due East and then 6 miles due West His position with regard to O is expressed by

$$+4 \text{ miles} - 6 \text{ miles} = -2 \text{ miles},$$

where -2 miles denotes that he is now 2 miles from O but on the *opposite* side of it, *i e*, 2 miles *West* of O

A graphical illustration of what has been said above is given below



Let XOX' be a straight line and let O be the starting point Let each division of XOX' represent one mile Suppose the direction OX (from left to right) to be East and OX' (from right to left) to be West As it is usual to consider distances from *left to right* as positive, and those from *right to left* as negative, mark the distances along OX as $+1$, $+2$, $+3$, $+4$, and those along OX' as -1, -2, -3, -4,

Now to walk 6 miles due East from O is to walk the distance OF, and then to walk 4 miles due West is to walk the distance FB Thus after the two walks the man is at B and his distance from O is OB

Hence $+6 \text{ miles} - 4 \text{ miles} = OB = +2$ miles

Again to walk 4 miles due East from O is to walk the distance OD, and then to walk 6 miles due West is to walk the distance Db Thus after the two walks the man is at b and his distance from O is Ob

Hence $+4 \text{ miles} - 6 \text{ miles} = Ob = -2$ miles

Thus -2 miles denotes a distance equal in magnitude but opposite in direction to that denoted by $+2$ miles, that is, $+2$ miles and -2 miles are quantities of the *same class* inasmuch as each is a distance but *opposite in direction*

As other examples, it will be seen that if $+100$ miles denotes a distance of 100 miles *North* of the Equator, then -100 miles will denote 100 miles *South* of the Equator, if $+20$ years stands for 20 years B C, then -20 years will stand for 20 years A D, if $+5°$ means $5°$ *above* the freezing point in a thermometer, $-5°$ will mean $5°$ *below* the same point, and so on.

Thus we see that many concrete quantities are capable of existing in *two directly opposite* states. These two opposite states or characters of a concrete quantity are called **positive** and **negative** quantities

18 Positive and Negative Quantities distinguished To distinguish between positive and negative quantities, it is convenient to use the signs + and − For from the nature of these signs, it seems clear that whatever quantity we are considering, +5 will always denote what increases that quantity by 5 units, and −5 what decreases that quantity by 5 units Thus, if we are speaking of a man's *gains*, calculated in rupees, +20 will denote an amount that increases his gains by 20 rupees, *i e*, +20 will denote a *gain* of 20 rupees, and −20 will denote an amount that *decreases* his gain by 20 rupees, *i e*, −20 will represent a *loss* of 20 rupees If on the other hand, we are speaking of a man's *losses*, +20 will represent an amount that increases his losses by 20 rupees, *i e.*, +20 will now denote a *loss* of 20 rupees, and −20 will represent an amount that decreases his losses by 20 rupees, *i e*, −20 will now denote a *gain* of 20 rupees And so on Thus + and − are used as *marks* to distinguish between Positive and Negative quantities Hence a quantity preceded by the sign + is called a positive quantity and that preceded by the sign − is called a negative quantity A quantity having no sign prefixed is considered as positive

It thus appears that the signs + and − serve *two distinct* purposes —*First*, they are used to indicate the operation of addition and subtraction, and *secondly*, they are used respectively to mark *positive* and *negative* quantities, and then they are called **positive and negative signs** The positive and negative signs are also called *signs of affection*, as they mark the *quality* of quantities before which they stand

It is thus clear that *$-a$ may stand alone just as $+a$ does*, and *indicates a units of a character opposite to that denoted by $+a$*

The magnitude of a quantity considered without reference to its sign, is called its **Absolute Value** Thus the absolute value of $+3$ and of -3 is 3, of $+a$ and of $-a$ is a and so on

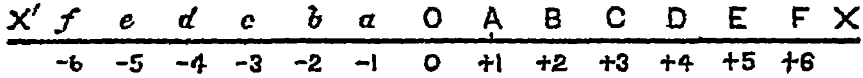
Hence *the sum of a positive and a negative quantity having the same absolute value is* 0, that is, $+a-a=0$, or $-a+a=0$

19 Choice of a Positive Quantity to remain unchanged From the nature of positive and negative quantities, it is plain that we may represent a gain of £20 by +£20 and a loss of £20 by −£20, or we may represent a loss of £20 by +£20 and a gain of £20 by −£20, that is, it is perfectly optional to call any quantity we please *positive*, and then a quantity of *opposite*

character shall be called *negative* Thus in algebraical language, a *loss* of Rs may be spoken of as a *gain* of −Rs, a *rise* of 5° in a thermometer may be called a *fall* of −5°, and so on

But though we are at liberty to call what quantity we like positive, yet, having once made our choice, *we must, throughout the same question, call that quantity positive and the quantity of opposite kind negative*

20 Corresponding positive and negative quantities If each of the divisions of XOX' is a *unit* of length, we see that for



every point A, or B, or C, in the line XOX' to the right of O, there is a corresponding point a, or b, or c, to the left of O, the corresponding pair of points being at the *same* distance from O on either side

Thus *for every positive quantity, there is a corresponding negative quantity, of the same absolute value*

It is also seen from the diagram that further a quantity is to the right, the greater it is in magnitude, and the further it is to the left, the less it is in magnitude Hence a quantity to the right of another is greater than the latter, and one to the left is less

Hence $+5$ is greater than $+4$, or 0, or -6,

and -5 is less than $+1$, or -1, or -3,

Thus a *positive* quantity is greater than any *negative* quantity Also 0 which is neither positive nor negative, is greater than a negative quantity

Moreover to pass from X to X' or from X' to X, we must pass through the point O. Thus *a quantity which changes its value from positive to negative must pass through the value* 0 , *and* vice versa

Examples VIII

Prove graphically the following (use squared paper)

1	$8-5=3$	2	$10-3=7$	3	$5-7=-2$
4	$-9+2=-7$	5	$-1-8=-9$	6	$-7+5=-2$

7 $-3-8=-11$ 8 $-6-2=-8$ 9 $6a-4a=2a$
10 $5a-8a=-3a$ 11 $-a-3a=-4a$ 12 $-10a-3a=-13a$.
13 $-10a+a=-9a$ 14 $-8a+2a=-6a$ 15 $-5a+9a=4a$
16 $3a-11a=-8a$ 17 $-5a+4a=-a$ 18 $-3a+3a=0$
19 $4+3+1=8$ 20 $-1-3-5=-9$ 21. $2a+5a+a=8a$
22 $-a-3a-4a=-8a$

Find graphically the value of

23 $-4-6$ 24 $5-10$ 25 $-3+8$ 26 $-7+5$
27 $-6-7$ 28 $10-13$ 29 $-10x+9x$ 30 $-12x-x$
31 $-a-18a$ 32 $25m-36m$ 33 $-12y+6y$ 34 $20ab-12ab$.
35 $-a^2-15a^2$ 36 $-12x^3+3x^3$ 37 $xy-xy$
38 $-3ab+3ab$ 39 $3x+10x+4x$ 40 $-8a^2-a^2-2a^2$

41 A person first gains R25 and then loses R30, how much does he gain? How much does he lose?

42 A man walks 3 miles towards the north and then 5 miles towards the south, how far north is he? How far south?

43 A and B start from the same point, A goes 8 miles due east and B goes 5 miles due west, how far is each due east and how far due west?

44 A clock, which loses 2 min a day, indicates at noon 3 min past 12, what time will it indicate at noon of the third day?

45 AB is a straight line in which O is a fixed point. Mark off on it the distances -4 1, 0, -3 and 2

CHAPTER III

FUNDAMENTAL LAWS—ADDITION AND SUBTRACTION

Monomials

21 Addition of a Positive Quantity and of a Negative Quantity **Addition** is the process of finding the single quantity which is equal to several quantities put together These several quantities are called **addends** or **summands**, and the single quantity is called their **sum**

Since a positive quantity always produces an *increase, to add a positive quantity is the same as to add its absolute value*

For example, if a person *has* 8 rupees and then he *earns* 3 rupees, he will then have altogether the sum of +8 and +3 rupees with him Thus

$$+8+(+3)=+8+3$$

Hence generally $\qquad +a+(+b)=+a+b \qquad \ldots \ldots$ (A)

Again, since a negative quantity produces a *decrease, to add a negative quantity is the same as to subtract its absolute value*

For example, if a person *has* 8 rupees out of which he *spends* 3 rupees, then he has 8−3 rupees, *i e*, he has then with him the difference of 8 and 3 rupees Hence if we call the *earnings* +8 rupees, and the *expenditure* −3 rupees, the *sum* of +8 and −3 will be the same as the *difference* of +8 and +3 Thus

$$+8+(-3)=+8-3$$

Hence generally $\qquad +a+(-b)=+a-b \qquad \ldots \ldots$ (B)

In establishing (A) and (B), we have supposed $+a$ to be positive, now a little consideration will shew that whatever be the character of the quantity to which $+b$ or $-b$ may be added, *b will always retain its own sign in the result* Hence

$$-a+(+b)=-a+b,$$

and

$$-a+(-b)=-a-b$$

Thus for the addition of any term we have the following **Rule**.—*Place the term with its sign unchanged after the expression to which it is to be added*

The expressions $a+b$, $a-b$, $-a+b$, and $-a-b$ cannot be simplified any more, and hence they are considered as *final results algebraically* But they may have *simple* numerical values when we give numerical values to a and b

Thus if $a=2$ and $b=3$, we have

(1) $a+b=2+3=+5$, $\qquad$ (2) $a-b=2-3=-1$;

(3) $-a+b=-2+3=+1$, $\qquad$ (4) $-a-b=-2-3=-5$

Hence we see that

(i) *To add two positive numbers*, we *add their absolute values and prefix the sign + to the result* (1).

(ii) *To add two negative numbers*, we *add their absolute values and prefix the sign − to the result* (4)

(iii) *To add a positive and a negative number together*, we *subtract the absolute value of the less from the absolute value of the greater, and prefix the sign of the greater to the result* (2 and 3)

22 Algebraic Sum When several quantities are connected by the signs + and −, the result in its simplest form is called their **algebraical sum** Thus the algebraic sum of +8 and +3 is +11, of −8 and −3 is −11, of −8 and +3 is −5, and so on

Thus an algebraic sum may be positive or negative

23 Subtraction of a positive Quantity and of a Negative Quantity

Since subtraction is the inverse of addition, the subtraction of a *positive* quantity must produce a *decrease*, for its addition produces an increase and the subtraction of a *negative* quantity must produce an *increase*, for its addition produces a decrease Therefore to subtract a *positive* quantity, we *subtract* its absolute value and to subtract a *negative* quantity, we *add* its absolute value Thus the result of subtracting +3 from +8 is +8−3, and of subtracting −3 from +8 is +8+3, that is,

$$+8-(+3)=+8-3,$$

$$+8-(-3)=+8+3,$$

Hence generally $+a-(+b)=+a-b$ (C),

and $+a-(-b)=+a+b$ (D)

Thus for the subtraction of any term we have the following **Rule** —*Place the term with its sign changed after the expression from which it is to be subtracted, and proceed as in Addition*

Ex 1 Subtract +5 from +11, −4 from +7, +6 from −8 and −3 from −4

(i) $+11-(+5)=+11-5=+6$

(ii) $+7-(-4)=+7+4=+11$

(iii) $-8-(+6)=-8-6=-14$

(iv) $-4-(-3)=-4+3=-1$

Ex 2 Subtract $-a$ from $-b$, and simplify the difference when $a=8$ and $b=13$

$$-b-(-a)=-b+a=-13+8=-5$$

Definitions The quantity from which another quantity is to be subtracted is called the **minuend**, and this latter is called the **subtrahend**, and the quantity which remains after the operation has been performed is called the **remainder or difference**

24 Law of Signs The results (A) and (B) of Art 21 and (C) and (D) of Art. 23 are very important. We give them below

$$+a+(+b)=+a+b \qquad (i)$$
$$+a+(-b)=+a-b \qquad (ii)$$
$$+a-(+b)=+a-b \qquad (iii)$$
$$+a-(-b)=+a+b \qquad (iv)$$

From (i) and (iv), we see that when the *same* sign is prefixed to a bracket as well as to the term within the bracket, the sign of the term will be + when the bracket is removed, and from (ii) and (iii), that when the signs are *different*, the sign of the term will be −, when the bracket is removed We have thus an important law called the Law of Signs which we briefly enunciate thus —

Like signs give +, and unlike signs give −.

That is, $+(+5)=+5$, $-(-3)=+3$, $+(-4)=-4$, $-(+6)=-6$

Note The student should notice that the word "*signs*" when used alone, *denotes the two signs* + and −, *and no other signs* Hence "*the sign of a term*" means either the sign + or the sign −, which is prefixed to it

Also "*like signs*" means signs *both* of which are + or *both* −, and "*unlike signs*" means signs *one* of which is + and the *other* −

Ex 1. Find the value of $-8-(-7)-(+3)+(-6)$

Required value $=-8+7-3-6=-1-3-6=-10$

Ex 2 Find the value of $a-b-c$, when $a=-3$, $b=-5$ and $c=-8$

$a-b-c=+(-3)-(-5)-(-8)=-3+5+8=+2+8=+10$

Ex 3 Simplify $3a-(-b)+(-c)$, when $a=5$, $b=-11$, $c=12$

$$3a-(-b)+(-c)=3a+b-c=3\times5-11-12$$
$$=15-11-12=4-12=-8$$

Examples IX

1 Find the sum of +7 and −6, of −5 and −9, of +10 and −8 and of −11 and +8

2 Find the sum of a and $-b$, and of $-b$ and $-c$, when $a=-3$, $b=4$ and $c=-5$

3 Simplify the sum of $-x$ and y, when $x=-10$ and $y=-15$

4 Subtract −12 from +3, −15 from −8, and +3 from −1

5 From +18 take +12, from −15 take +10, from +9 take −9, and from −11 take −5

6 From a take $-b$, and find the value of the result when $a=-5$ and $b=-1$.

7 Subtract b from $-a$ and find the value of the remainder when $a=8$ and $b=-6$

8 From $a-b$ subtract $-c$ and simplify the result when $a=1$, $b=2$ and $c=5$

9 Find the difference between the sum of 5, -4 and -6, and the difference of -18 and -3

10 Simplify $-12+3-8-7+20$ and $-3-1+12-4+1$

11. Find the value of $7-(-2)+(-4)$ and $-3-(-2)+(+1)$

12 Simplify

$$4-(+3)-(-2)-(+5) \text{ and } -(-3)+(-4)-(-1)+(-2)$$

If $a=-1$, $b=-2$, $c=-3$, $x=-4$, $y=-5$, $z=6$, find the value of

13 $-a-b+c$

14 $-(-a)+b+(-c)$

15 $a-(-x)-(+b)+(-y)$

16 $-a-(-b)+(-c)-3z$

17 $(-a)-x-(+y)-(-z)$

18 $-(-x)-(-y)-(+2z)-3$

19 $8-(+a)+(-b)-(+c)$

20 $4z-(+x)-(-y)+(-c)$

25 Algebraic Difference In Algebra, $a-b$ always expresses that b is **subtracted from** a, *whatever be the values of* a and b, *positive* or *negative* Thus by the *algebraical difference* between a and b is always meant $a-b$ and not $b-a$ or $a\sim b$

Hence the algebraic difference

of $+6$ and $+2$ is $+6-(+2)=+6-2=+4$,

of $+6$ and -2 is $+6-(-2)=+6+2=+8$,

of -6 and $+2$ is $-6-(+2)=-6-2=-8$,

and of -6 and -2 is $-6-(-2)=-6+2=-4$

26 Definition of "greater than" and of "less than" The introduction into Algebra of negative quantities requires an extended definition of the phrases *greater than* and *less than* We therefore define them as follows —

a is said to be greater than b when their algebraic difference $a-b$ is positive, and less than b, when this difference is negative, a and b being any quantities whatever

Thus $+4>+3$, for $+4-(+3)=+4-3=+1$

$-3>-4$, for $-3-(-4)=-3+4=+1$

$+2>-8$, for $+2-(-8)=+2+8=+10$

$-5<-2$, for $-5-(-2)=-5+2=-3$

$-6<+1$, for $-6-(+1)=-6-1=-7$

Hence the quantities

$-6,\ -5,\ -4,\ -3,\ -2,\ -1,\ 0,\ +1,\ +2,\ +3,\ +4,\ +5,\ +6,$
are in *ascending order of magnitude*, i e, each is *greater than* any one that precedes it

Examples X

1 Find the algebraic difference between 6 and -7, -6 and 7, 6 and 7, -6 and -7

2 Find the algebraic difference between $8m$ and $3m$, $8m$ and $-3m$, $-8m$ and $3m$; $-8m$ and $-3m$

3 Which is the gaeater -5 or -2, -1 or -3, -1 or $+1$?

4 Which is the less -3 or -4, 0 or -1, -100 or 0?

5 Which is the greater -1 or $+1$, and by how much?

6 Which is the greater, the sum of -3 and 2, or the sum of 5 and -7?

7 Is the sum of -3, 2 and -4 less than the difference of -2 and -1? If so, by how much?

8 Which is the less, the difference of -6 and -4, or the sum of 8, -12 and 5, and by how much?

27 Like and Unlike Terms Terms which do not differ at all or differ only in their numerical coefficients, are called **like terms** Thus a, $-3a$, $5a$, $-\frac{1}{2}a$, are like terms, so also are $-2a^2b^3$, a^2b^3, and $\frac{1}{4}a^2b^3$

When this is not the case, they are said to be **unlike terms.** Thus $3a$, $-b$ and $\frac{1}{3}d$ are unlike terms, so *different* powers of the *same* letter are unlike terms, as $5a^2$ and $-2a$

28 Addition of like terms We have seen that to add a term to an expression we place it after the expression, with its sign *unchanged* Thus the sum $4a$ and $3a$ is $4a+3a$, of $-4a$ and $-2a$ is $-4a-2a$, of $5a$ and $-2a$ is $5a-2a$, of $3a$ and $-5a$ is $3a-5a$

In each of these examples, the sum may be further simplified by *collecting the like terms together*, and unless this is done, the result is not considered as *final* Thus we have

$4a+3a=7a$, (just as 4 rupees $+3$ rupees$=7$ rupees) [Art 24], where $+7$, the coefficient of a, is the sum of the coefficients 4 and 3, $-4a-2a=-6a$ [Art 24], where -6 is the sum of the coefficients -4 and -2, $5a-2a=+3a$, where $+3$ is the algebraical sum of the coefficients 5 and -2, $3a-5a=-2a$, where -2 is the algebraical sum of the coefficients 3 and -5

Hence for the addition of Like Terms, we have the following **Rule** :—*Place them successively with their signs, find the algebraic sum of the coefficients, and affix the common letter or letters*

Ex. 1 Find the sum of x, $5x$ and $8x$

Required sum $=x+5x+8x=14x$, (where 14 is the sum of the coefficients)

Ex 2 Required the sum of $-a$, $-3a$, $-4a$, and $-11a$

Sum required $=-a-3a-4a-11a=-19a$, (where -19 is the sum of the coefficients -1, -3, -4 and -11)

Examples XI

Find the sum of

1 $4a, 3a, a, 8a$ 2 $y, 5y, 9y, 2y$

3 $3x, 10x, x, 18x, 24x$ 4 $5a, a, 3a, 7a, 10a$

5 $m, 3m, 8m, 7m, 5m$ 6 $12n, 18n, n, 2n, 4n$

7 $2r, r, 5r, 4r, 6r, 15r$ 8 $y, 6y, 9y, 12y, 8y, 10y$

9 $-5c, -c, -8c, -28c$ 10 $-3x, -5x, -x, -6x, -10x$

11 $-4y, -6y, -2y, -y, -8y$ 12 $-6ab, -7ab, -2ab, -ab, -9ab$

13 $-xy^2, -3xy^2, -7xy^2, -xy^2, -4xy^2$

14 $-max, -3max, -25max, -4max, -19max, -21max$

15 $2ab, \frac{1}{2}ab, 20ab, ab, 15ab$ 16 $\frac{3}{5}xyz, xyz, \frac{1}{2}xyz, \frac{3}{4}xyz, \frac{19}{20}xyz, xyz$

17 $-\frac{3}{4}mn, -mn, -4mn, -\frac{16}{7}mn, -14mn$

18 $\frac{1}{2}pq, \frac{2}{5}pq, \frac{4}{7}pq, \frac{9}{10}pq$ 19 ax, bx, cx, dx 20 $ax, x, 5x, px, 3x$

Note When the addends have *different signs*, we proceed as in the example below

Ex 3 Add together $3a$, $-2a$, $-7a$, $4a$, $-a$, and $6a$

The sum required $=3a-2a-7a+4a-a+6a$

The sum of the coefficients $=3-2-7+4-1+6$

$$=1-7+4-1+6=-6+4-1+6$$
$$=-2-1+6=-3+6=+3$$

Thus $3a-2a-7a+4a-a+6a=+3a$ or $3a$, the required sum

Otherwise —The sum of the coefficients of the positive terms is $+13$ and that of the negative terms is -10, $\therefore$ their algebraic sum is $+3$ Hence required sum $=+3a$ or $3a$

Examples XII

Find the sum of

1 $a, -5a, 9a, -8a$ 2 $3a, -4a, 5a, -a, -2a$

3 $7x, -2x, -x, 4x, -10x$ 4 $8m, 20m, -3m, -15m, m, 17m$

5 $xy,\ -18xy,\ 5xy,\ -26xy,\ -xy,\ 12xy$

[In the following examples the addends are placed in a column]

6	7	8	9
$-abc$	$16rs$	$5axy$	$\frac{9}{4}cz$
$50abc$	$-74rs$	$\frac{3}{5}axy$	$-\frac{1}{2}cz$
$27abc$	$24rs$	$-3axy$	$-\frac{9}{10}cz$
$-76abc$	$31rs$	$-\frac{7}{10}axy$	cz
abc	$3rs$		$-\frac{49}{20}cz$

10	11	12	13
$\frac{3}{4}mn$	$35a$	$8z$	$-6by$
$-\frac{1}{7}mn$	$-17a$	$30z$	$20by$
$\frac{5}{6}mn$	$-a$	$57z$	$-18by$
$-\frac{7}{12}mn$	$20a$	$-108z$	$-42by$
$-\frac{3}{8}mn$	$-45a$	$21z$	$-11by$
		$-25z$	by

14	15	16	17
$\frac{1}{2}ab$	ax^2	$2ax$	$3x^2y$
$\frac{1}{4}ab$	$-bx^2$	$-ax$	$-x^2y$
$-ab$	cx^2	$4px$	ax^2y
$\frac{5}{6}ab$	$-dx^2$	$3ax$	$5x^2y$
$-\frac{1}{3}ab$	fx^2	$-qx$	$3ax^2y$
$-2ab$			

29 Addition of Unlike Terms These terms are added, when we merely place them in a line with their signs unchanged

For example, the sum of $2a$, $-b$, $-5c$ and $3d$ is $2a-b-5c+3d$, and as this sum cannot be further simplified, the result is algebraically complete

Ex Add together $-5x$, $3y$, $-2z$ and xyz

Required sum $= -5x+3y-2z+xyz$

Examples XIII

Add together

1 $5a,\ -3b\ -2c,\ d$ 2 $3a^2,\ -12a,\ b^2,\ -2b$

3 $43ab,\ -5ax,\ 20xy,\ -18pq$

4 $12abc,\ -axy,\ 20pqr,\ -15mr,\ 29df.$

5 $3a^2, 10b^2, -18c^2, 25ab, -bc, 17ca$

6 $5ab, -3xy, 8x^2, -2a^2c, -4abc$

7 $2a^2, b^2, -c^2, 4a, -3d, -5b$

8 $15x^2y, -3xy^2, 3yz, -2y^2z, 5xz, -3xy$

30 Subtraction of Like and Unlike terms Since the minuend may be considered as a negative quantity, the subtraction of any term means the same as the addition of the corresponding negative term

Thus to subtract (i) $5a$ from $-3a$, and (ii) $-8a$ from $-4a$

(i) $-3a-(+5a)=-3a-5a$ [Art. 24] $=-8a$

(ii) $-4a-(-8a)=-4a+8a$ [Art. 24] $=+4a$

Polynomials

31 Addition and Subtraction of Polynomials By definition an *algebraical expression* is a mere collection of its *terms*, thus the expression $a-b+c-d$ may be looked upon as the *algebraical sum* of its terms $+a$, $-b$, $+c$ and $-d$

Therefore to add the *whole* expression $a-b+c-d$ is the same as to *add its terms* $+a$, $-b$, $+c$ and $-d$ in succession Thus if E be an expression to which $a-b+c-d$ is to be added, we have

$$E+(a-b+c-d)=E+(+a)+(-b)+(+c)+(-d)$$
$$=E+a-b+c-d \qquad (A)$$

Hence *to add an expression, affix its terms in succession with their signs unchanged to the expression to which it is to be added*

Similarly to subtract the *whole* expression $a-b+c-d$ is the same as to *subtract its terms* $+a$, $-b$, $+c$ and $-d$ in succession Thus if E be an expression from which $a-b+c-d$ is to be subtracted, we get

$$E-(a-b+c-d)=E-(+a)-(-b)-(+c)-(-d)$$
$$=E-a+b-c+d \qquad (B)$$

Hence *to subtract an expression, affix its terms in succession with their signs changed to the expression from which it is to be subtracted*

Corollary From (A), it follows that

(i) $a+(b+c)=a+b+c$, (ii) $a+(b-c)=a+b-c$,

and from (B), it follows that

(iii) $a-(b+c)=a-b-c$, (iv) $a-(b-c)=a-b+c$

32 Single Brackets From Art 31 (A), we see that the removal of the bracket does not affect the result, and from (B), we see that when the bracket is removed, the sign of every term within the bracket is changed

Hence we have the following **Rules for the removal of Brackets** —

RULE I *If a + sign precedes a bracket, remove the bracket without changing the signs of the included terms*

RULE II. *If a − sign precedes a bracket, remove the bracket and change the sign of each of the included terms*

Ex 1 Simplify $5a+(2a-b)-(a+b)$

The given expression $=5a+2a-b-(a+b)$, by Rule I,

$=5a+2a-b-a-b$, by Rule II,

$=6a-2b$

Ex 2 Simplify $2x-(-x+y)+\overline{3x+2y}-(7x-y)$

The given expression

$=2x+x-y+\overline{3x+2y}-7x+y$ [Rule II]

$=2x+x-y+3x+2y-7x+y$ [Rule I]

$=-x+2y$

Ex 3 Simplify $-3x+(-y+2x)-\overline{8x-3y+z}-(-9x+y)$

The given expression

$=-3x-y+2x-\overline{8x-3y+z}-(-9x+y)$ [Rule I]

$=-3x-y+2x-8x+3y-z+9x-y$ [Rule II]

$=y-z$

Examples XIV

Remove the brackets from

1. $x+(y-z)-(x-z)$
2. $(2a-3x)+(-a+2x)$
3. $(ax+by)-(by-1)$
4. $(a+b+c+f)-(-a+b-c+f)$.
5. $a-(b-c+d)-\overline{e-2a+b}+(d-c)$
6. $x+y+z-(x+y-z)-(-y-z+x)$
7. $2a-3b-(a+2b)-(8b-6a)$
8. $3a-(b-4c)+2d+(a-b)-\overline{2c-d}$

9 $4m-\overline{3m-4}+\overline{m-6}-(2m-8)$

10 $(ax+bx)+(by+cy)+(ax-bx)-(by-cy)$

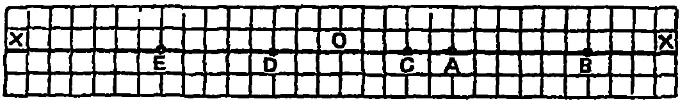
11 $(x^2-2xy+3y^2)-(x^2+3xy-2y^2)-(4y^2-5xy)$

33 The terms of an expression may be written down in any order

This is accepted as self-evident, for $8+3$ gives the same result whether we add 3 to 8 or 8 to 3 Similary to add 3 to 8 and then subtract 6 produces the same result as to subtract first 6 and then add 3, *i e*, $8+3-6=8-6+3$

Generally $a+b-c=a-c+b=-c+a+b=$&c

The following is a **graphical illustration** of the principle



Let the side of a square represent a miles Then with the same convention with regard to the signs as in Art 17, we see that $5a+6a-8a$ indicates that one goes $5a$ miles from O to A, then $6a$ miles from A to B and finally $8a$ miles (in the negative direction) from B to C

Thus $\qquad 5a+6a-8a=OC=3a \qquad$ (i)

Similarly $5a-8a+6a$ shews that he goes $5a$ miles from O to A, then 8 miles (in the negative direction) from A to D, and lastly $6a$ miles from D to C

That is $\qquad 5a-8a+6a=OC=3a \qquad$ (ii)

Again $-8a+5a+6a$ denotes that the man goes $8a$ miles (in the negative direction) from O to E, then $5a$ miles from E to D, and then $6a$ miles from D to C

$$-8a+5a+6a=OC=3a \qquad \text{(iii)}$$

Hence (i), (ii) and (iii) are equal, *i e*,

$$5a+6a-8a=5a-8a+6a=-8a+5a+6a$$

This principle is otherwise enunciated thus —*Additions and subtractions may be made in any order*, and is known as the **Commutative Law** for Addition and Subtraction

Graphical Examples XV

Prove graphically the following (use squared paper)

1 $8-3+2=7$ 2 $2+7-8=1$ 3 $4-12+6=-2$

4 $-5+5+4=4$ 5 $-10+6+1=-3$ 6 $12-8-4=0$

7 $3a-7a+2a=-2a$ 8 $-2a-9a+11a=0$

9 $-8x+5x-x=-4x$. 10 $-2-5+8-1+3=3$

11 $a-3a-4a+6a=0$ 12 $5y-12y-2y+4y=-5y$

34. Addition of Polynomials The following examples will shew how polynomials may be added together

Ex 1 Add together $3a+5b$, $a-2b$ and $4a+b$

$$\begin{aligned}\text{The required sum} &= (3a+5b)+(a-2b)+(4a+b)\\ &= 3a+5b+a-2b+4a+b\\ &= 3a+a+4a+5b-2b+b \text{ [Art 33]}\\ &= 8a+4b\end{aligned}$$

Note It is usual to place the given expressions one under another, so that *like* terms may fall under *like* terms, and then collect them as usual Thus

$$\begin{array}{r} 3a+5b\\ a-2b\\ \underline{4a+b}\\ 8a+4b \end{array}$$ the same result as before

Ex 2 Add $13x+2y-3z$, $5y-18x+4$, $15x+10z-y$, and $z-6x-12y$

$$\begin{aligned}\text{Req sum} &= (13x+2y-3z)+(5y-18x+4)+(15x+10z-y)+(z-6x-12y)\\ &= 13x+2y-3z+5y-18x+4+15x+10z-y+z-6x-12y\\ &= 13x-18x+15x-6x+2y+5y-y-12y-3z+10z+z+4\\ &= 4x-6y+8z+4\,,\end{aligned}$$

or it may be found, thus

$$\begin{array}{rrrr} 13x+ & 2y- & 3z & \\ -18x+ & 5y & & +4\\ 15x- & y+ & 10z & \\ -6x- & 12y+ & z & \\ \hline 4x- & 6y+ & 8z & +4. \end{array}$$

Ex. 3 Add together $6ax+3by-2$, $4by+3cz+5$, and $8cz-2af-2fx$

$$\begin{array}{rrrrrr} 6ax & & +3by & & & -2\\ & & +4by & +3cz & & +5\\ & -2fx & & +8cz & -2af & \\ \hline 6ax & -2fx & +7by & +11cz & -2af & +3. \end{array}$$

Ex 1 From $5a-3b-2c$ take $3a-4b+c$

Remainder $=5a-3b-2c-(3a-4b+c)$

$=5a-3b-2c-3a+4b-c$ [Art 32]

$=5a-3a-3b+4b-2c-c$ [Art 33]

$=2a+b-3c$, collecting like terms

It is usual to place the subtrahend under the minuend, taking care that *like* terms be placed under *like* terms, and then to combine the like terms after *mentally* changing the signs of those in the subtrahend Thus the above example is usually worked as shewn below—

$$\begin{array}{lr} \text{(Minuend)} & 5a-3b-2c \\ \text{(Subtrahend)} & 3a-4b+\ c \\ \hline \text{(Remainder)} & 2a+\ b-3c \end{array}$$

Ex 2

$$\begin{array}{lr} \text{From} & 8x^2-20ab+\ y^2 \\ \text{take} & -3x^2+25ab-2y^2 \\ \hline \text{Rem} & 11x^2-45ab+3y^2 \end{array}$$

Ex 3 From $35x^2+20x-3y$ take $18y-3x+2c-1$,

Place like terms under like terms, thus

$$\begin{array}{lr} & 35x^2+20x-\ 3y \\ & -\ 3x+18y+2c-1 \\ \hline \text{Rem} & 35x^2+23x-21y-2c+1 \end{array}$$

Examples XVII

1 From a take $-b$

2 From $2a$ take $a-x$

3 From $x+y$ take $x-y$

4 From $8x-3y$ take $-3x+2y$

5 From $15a+16$ take $3a+1$

6 From $10a+20$ take $-5a+8$

7 From $a+x-1$ take $-a+x-2$.

8 From $x+\frac{1}{2}y-1$ take $\frac{1}{2}x-y+1$

9 From $3mn-mx+n$ take $4mn-my-n$

10 From $6ax-3by+3$ take $8ax-5by-2$

11 From $2x+3y-4z$ take $x-2y+3z$

12 Take $3x+2y-4z$ from $2x-y$

13. Take $-a+3b-c$ from $4a-2b+1$

14. Take $-4b+5d-9$ from $a-3b+2c$

15 From $3p^2-4pq+q^2+1$ take $3p^2+q^2-5pq$

16 Subtract $a^2-3a-3b$ from $2a^2+4a-b^2$

17 Subtract x^2-2x+1 from x^3-1

18 Subtract $1-m^2+m^3$ from $3-2m^2+m^4$

19 From $\frac{2}{3}ax-\frac{1}{3}xy+\frac{2}{3}$ take $\frac{1}{3}ax+\frac{2}{3}xy-\frac{1}{3}$

20 From $x^3-y^2+2z^3$ take $2x^2-3y^2-z^3$

21 Subtract $2abc+ab-ac+1$ from $5abc-2ab-3ac$

22 Subtract a^2-ax+x^2 from $2a^2-3ax+x^2$

23 Subtract $2ax+x^2+3a^2$ from $4ax-5x^2+a^2$

24 Subtract px^3-qx^2+rx from ax^3-bx^2+x

25 From x^3+5x^2-10 subtract $3x^3-4x-8$

26 Take x^3-3x^2-2x-4 from x^3-1

27 Subtract $\frac{1}{5}-\frac{3}{4}x^2-\frac{2}{5}y^2$ from $-\frac{3}{4}x^2-\frac{1}{8}xy+\frac{1}{10}y^2$

28 From $\frac{1}{2}y-\frac{5}{2}a-\frac{3}{4}x+\frac{1}{3}b$ take $3y+\frac{11}{4}a-\frac{2}{3}x$

29 From $7a^4+2a^3b-3a^2b^2$ subtract $7b^4-3a^2b^2+2ab^3$

30 From $ax^2+2hxy+by^2+c$ take $gx^2-2fxy+hy^2-d$

31 Take $-x^4+2x^3+4x^2-3x-1$ from $x^4-3x^3+4x^2-5x-1$

32 Subtract $x^5-3x^2y^3+3x^4y-5y^5-4xy^4$

from $ax^5+3x^4y-4xy^4-x^2y^3+by^5$

36 Two or more Brackets In Art. 33, we have given rules for the removal of single brackets But sometimes expressions occur, involving more than one pair of brackets The usual method in such cases is to begin with either the *innermost* pair, or the *outermost* pair, and to proceed according to the rules to the *next in order* till all are removed

Ex Simplify $2a-[a-\{a-(x+\overline{a-x})\}]$

$$\begin{aligned}\text{Given expression} &= 2a-[a-\{a-(x+a-x)\}] \text{ [Rule I]}\\ &= 2a-[a-\{a-(+a)\}]\\ &= 2a-[a-\{a-a\}] \text{ [Rule II]}\\ &= 2a-[a]=2a-a=a\end{aligned}$$

Or thus —Given expn $=2a-a+\{a-(x+\overline{a-x})\}$ [Rule II]

$$\begin{aligned}&= a+a-(x+\overline{a-x}) \text{ [Rule I]}\\ &= 2a-x-\overline{a-x} \text{ [Rule II]}\\ &= 2a-x-a+x \text{ [Rule II]}=a.\end{aligned}$$

REMARK It is usually advantageous to begin with the innermost pair

Examples XVIII

Simplify

1 $7-\{5-(4+x)\}$ 2 $x-\{x-(x-y)\}$

3 $3a-\{2b-(a-4b)\}-(a+b)$ 4 $1-[1-\{1-x-(1-\overline{1-x})\}]$

5 $4x-[y-\{x+(y-3a)\}]$ 6 $(a+x)-\{a-x-[a-2x-(2a-x)]\}$.

7 $-[+\{+(-x)\}]-[-\{+(-\overline{-x})\}]$

8 $a+x-[b+y-\{a-x-\overline{b-2y}\}]$ 9 $xy-\{xy-(2xy-\overline{2xy-y^2})\}$.

10 $16a-[3b+\{5a+2b-(3a-\overline{2a-4b})\}]$

11 $17a-4b-[3b+2a-\{5b-6a-(2b-a)\}]$

12 $11x-\{7x-[8x-(9x-\overline{12a-6x})]\}$

13 $ab-\{(3bce-2ab)-(5bce-bcf)+(3ab-3bcf)\}$

14 $2a-\{3b+(2b-c)-4c+[2a-(3b-\overline{c-2b})]\}$

15 $16-x-[(-7x)-\{8-(+9x)-(-6x-\overline{-3})\}]$

16 $a+\{-3b-[-6c+(-3a-\overline{-2b-5c})]\}$

17 $-[15x-\{14y-(15z+12y)-(10x-15z)\}]$

18 $-\{a-[-2a-(-3a-\overline{-4a-5b})]+(-5a-3b)\}$

19 $-5a-\{-4b-[-8c-(-4b+\overline{-6a-7c})]\}$

20 $-m-\{-3m+2n-(-5m-8n)+(-3m-2n-\overline{-3n+m})\}$

37 Rules for the Insertion of Brackets From Art 31 (A), we have $E+(a-b+c-d)=E+a-b+c-d$

Conversely $E+a-b+c-d=E+(a-b+c-d)$

Also from (B), we have $E-(a-b+c-d)=E-a+b-c+d$,

conversely $E-a+b-c+d=E-(a-b+c-d)$

Hence we get the following rules for the *insertion* of brackets

RULE I *If a + sign is to precede a bracket, keep the signs of the included terms unchanged.*

Thus $a-b+c-d=a+(-b+c-d)=a-b+(+c-d)=a-b+c+(-d)$

RULE II *If a − sign is to precede a bracket, change the sign of each of the included terms*

Thus $a-b+c-d=a-(+b-c+d)=a-b-(-c+d)=a-b+c-(+d)$

Ex 1 Enclose in a bracket with a *positive* sign, the last 3 terms of $x-y+z-1$

Given expression $=x+(-y+z-1)$ [Rule I] $=x+(z-y-1)$

Ex. 2 Enclose the same terms in a bracket with a *negative* sign
Given expression $=x-(+y-z+1)$ [Rule II] $=x-(y-z+1)$

Examples XIX.

1 Enclose in a bracket, (1) with a positive, and (2) with a negative sign, the last 4 terms of $a-b+c-d-e+f$

2 Of the expression $a-b+c-d-e+f-g+h$, enclose in a bracket

(1) every 2 terms beginning with the first,
(2) . . . second,
(3) . 3 . first,
(4) 3 . . . second,
(5) the last 5 terms,
(6) the second, third, fourth, and fifth terms,
(7) the fifth, sixth, seventh, and eighth terms,
(8) all the terms except the first

38 The terms of an expression may be grouped in any manner. We have by Art 37,

$$a-b+c-d-e+f=a+(-b+c)+(-d-e)+f$$
$$\text{or } =(a-b)+(c-d-e)+f$$
$$\text{or } =(a-b+c)-d+(-e+f)$$
$$\text{or } =\&c$$

This principle is known as the **Associative Law** for Addition and Subtraction.

Examples for Revision (A)

1 Define "Algebra" To whom is the invention of this science ascribed?

2. Distinguish between 5 6 and 5 6 Find their difference

3 Find the value of $\dfrac{7b+c}{3a+4b}$, when $a=5$, $b=4$, $c=3$

4 Find the sum of $4ax$, $-a^2x$, $8ax^2$, $-2ax$, $-(-3a^2c)$, $-2ax^2$ and $-3ax$, and the value of the result when $x=1$

5 Subtract $-5a+b$ from a^2, and simplify the result when $a=1$, $b=-2$

6 Remove the brackets from $x-\{2y-(2y+3z)-z\}$

7 By how much does $x-y$ exceed $x+y$?

8 How old will a man be in 5 years, if he was x years old 10 years ago ?

---

9 What expression must be subtracted from 0 so as to leave a remainder x^2-2x+1 ?

10 In how many ways can the product of 10, x and y be written ? In how many ways the product of a, b and c ?

11 What must be added to $a-b$ that the result may be c ?

12 Subtract the sum of $3x^2$ and $1-2x$ from unity What is the value of the difference when $x=1$?

13 Simplify $3p-\{q+(2p-q)-(p-q)\}$

14 Add together $4x^2-2xy+y^2$ and $5x^2-2y^2$, and decrease the sum by $8x^2-2xy+y^2$

15 If $x=1$, and $y=5$, how will 45 be represented by these symbols ?

16 I have R60 How much shall I have (i) if I spend m annas and (ii) if I spend Ra and then b annas ?

---

17 Define a *factor*, a *coefficient*, a *power* and an *index* Distinguish between $3a$ and a^3, and find their difference when $a=2$ What are the factors of $3a(b+c)$?

18 Find the remainder when $3x^2-4x+1$ is subtracted from 0, and the value of the remainder when $x=1\frac{1}{3}$

19 Explain what is meant by $-a$

20 Add together

$$3-(2+x),\ x-(3-2x),\ 4x+(8-x),\ 2-\{x+(y-1)\}$$

21 Subtract $4a-(3b-2c)+(2a-c)$ from $3-(2a-3b)+c$

22 Remove the brackets from

$$2a-\{5b+[c-(a+b-2c)]\}-(4b-c)$$

23 Find the quantity which when subtracted from $3x^2-2ax+3xy$ leaves a remainder $2x^2-3ax+3xy+1$

24 A has Ra and B Rb If A gives B Rx, how much has each ? If afterwards B gives A Ry, how much has each then ?

---

25 If $a=6$, $b=4$, $c=3$, $d=0$ and $e=-1$, find the value of

$$5(a-b)(c^2+bd-e)-2(a+e)^2$$

26 What are meant by $2x^2$ and $(2x)^2$, and what is the difference between them when $x=3$?

27 Add together $-b+2a+3c$, $6a-(b+c)$, $4b-(2a-c)$, $3c-(6a+b)$

28 Take $2+\{x-(2y+3z)-5\}$ from $2x-\{4y-(6z-7)\}$

29 Simplify $4x-[3y+\{2-3z-(1-\overline{3x-2})\}]$

30 Arrange the numbers 3, -2, 1, -5, -7, 2 and -1, (i) in *ascending* order of magnitude and (ii) in *descending* order.

31 What is that quantity from which $a^2-3ab-\frac{1}{4}b^2$ being taken, the remainder is $\frac{2}{3}a^2+2ab-\frac{1}{2}b^2$?

32 If a debt of R50 be represented by x, what will $-2x$ represent?

---

33 If $a=8$, $b=0$ and $c=3$, find the value of

$$2a-(3b-c)-(8c-a)-\{4c-(a-b)\}$$

34 By how much does x exceed $5-2x$? Find the value of the difference when $x=\frac{1}{3}$

35 What is the algebraic sum of 5, -9 and 1? What of a, b, $-c$ and -3?

36 Find the sum of

$$5x-(4y+z),\ 2x+(2z-y),\ \text{and}\ 9z-(6x-6y+5z)$$

37 Subtract

$$a-(b-c)+\{2c-(3b-5a)\}\ \text{from}\ a-2b-\{3a-(b-c)-5c\}$$

38 Remove the brackets from

$$4x^2-\{2x^2-(2x-3)\}-\{x^2-(5x-2x^2)\}-\{3-(2x^2-x)\}$$

39 The sum of two expressions is $2x^2+3xy$, and one of them is $3x^2-2xy+2y^2$; find the other.

40 I buy a horse for Rx, if by selling it for Ry I lose R30, express the relation between x and y.

---

41 By how much does 0 exceed $5x-3$?

42 Find the numerical value of

$$17x-(2x-3y)-\{4y+2x-(x-2y)\},\ \text{when}\ x=3\ \text{and}\ y=14$$

43 Add $a-(3b-2c)-d$, $b-\{a-(2c-d)\}$, $c-\{2b-(3d-a)\}$, and $d-\{2c+(4a-b)\}$

44 Find the difference when $a^2-2ab+b^2-3$ is taken from $a^2-3ab-1$

45 Simplify $m-[\{2m-\overline{m-(2m+n)}\}-(3m-2n)]$

46 The difference of two expressions is $2a+3ab-5b$, and the greater is $5a+2ab-3b$, what is the other?

47 Shew graphically (using squared paper) that

(i) $8-5=3$, (ii) $7-5-2=0$, (iii) $7+4-15=-4$

48 A man has x miles to walk, he walks for h hours at c miles an hour, how much has he still to walk?

If $x=25$, $c=3$, $h=5$, what is the answer?

---

49. If $a=10$, $b=2$, $k=4$, and $l=-5$, find the value of

$$2a-(\sqrt{bk+b^3}+b\sqrt{ab-l})$$

50. On a Centigrade thermometer, what is the rise between (i) $-8°$ and $+25°$, (ii) $-15°$ and $-2°$, and (iii) $+20°$ and $-4°$?

51 Add $3x^2-\{ax+(a^2-x^2)\}$, $-3ax+\{x^2-(a^2-2ax)\}$, and $3a^2-(ax-x^2)+2x^2$

52 By how much does $3z-\frac{2y}{3}+\frac{x}{4}-1$ exceed $\frac{x}{2}-\frac{y}{3}+\frac{z}{4}-2$?

53 Simplify $-(a-c)+3b-[3c-(8a+b)+4a]+2a-(2c+b-a)$

54 Shew graphically (using squared paper) that

(i) $2a-6a+3a=-a$, (ii) $a-2a+3a-5a=-3a$

55 The difference of two expressions is $5xy-\frac{5}{2}x^2-\frac{1}{4}y^2$, and the smaller is $3x^2-5xy-\frac{3}{4}y^2$, find the greater

56 If $y=2x^2-3x-7$, tabulate the values of y, when x has the values -3, -1, 0, 2, 3 and 5

---

57. What is left when $5a-3b$ is taken from the sum of

$$3-2a+x \text{ and } 4a-5b-7?$$

58 Express in a simple form

$$3a^2-(4b^2-3c^2)+2c^2-\{(a^2-3b^2)+\tfrac{1}{2}c^2\}$$

59 Shew that $x^3+23x=3(3x^2+5)$ when $x=1, 3$ or 5 Which of the two expressions is the greater when $x=4$?

60 From $a-\{3b-(2a-c)\}$ take $3a-(2b-c)$, and add the remainder to $b+1$

61 Simplify $5x-\{3y-(x-y)\}+8x-(3y+z)-\{z-(x-2y)\}$

62 If $x=b+c-2a$, $y=c+a-2b$, $z=a+b-2c$, shew that

$$x+y+z=0$$

63 A man walks 5 miles due north, then 8 miles due south, and then again 6 miles due north Shew graphically how far he is from the starting point

64 Prove that $a+(b-c)=a+b-c$, and that $a-(b-c)=a-b+c$

---

65 What remains when the difference of $3x^2-2y^2$ and $3xy-5y^2$ is taken from unity?

66. If $x=4$, find the difference between $\sqrt{9}x$ and $\sqrt{9x}$

67 Add together $3a-\frac{b}{2}+\frac{c}{4}$ and $4c-\frac{a}{2}$, and subtract the sum from a

68 Subtract $3x^3-2x^2+4$ from unity, and $3x^2-4x^3$ from 0, and add the two results together

69 Add to the sum of

$$3x-\{2y-(3z-2x)\} \text{ and } x-\{3y-[5z-(x-y)]\}$$

the difference of $5z$ and $3y-\{x+(2y-3z)\}$

70. Find the coefficient of ax and of x in the sum of $5ax$, $4by$ $-6ax$, x and $-3bx$

71 If $X=3a^2-2ab+5b^2$, $Y=7a^2-8ab-5b^2$, $Z=9a^2-5ab+3b^2$ $V=5a^2-ab+3b^2$, find the value of

(i) $Y-(X+Z-V)$; (ii) $Y-\{Z-[X-(Y-V)]\}$.

72 A sum of money was divided between A and B; if A received x rupees and B as much as A and 3 rupees more what was the sum?

---

73 Shew that the sum of $4ax+3bx-2c$, $2cx-2ax+b$ and $-4bx+cx-3a$ is equal to the sum of $2ax-5bx-c$, $3bx+4cx-9a$ and $-2cx+4ax+4b$, if $x=3$

74 Find the coefficient of x^2 in $x^2+3x^2-y^2+1-5x^2$, and of x and y in the sum of $5x-2y+1$, $8y-2x-3$ and $10-4x-6y$

75 What are the like terms in the expression

$$2x^2-3xy+5y^2-x^2+4xy-x^2+6x-2y^2\ ?$$

76 To what expression must $2x^3-x^2+5x-1$ be added so as to give 0?

77. What expression must be added to $3x^2-5x+2y$ so as to give $x-y$ for the sum?

78. If $a=5$, $b=3$, $c=0$, $x=4$, and $y=2$ shew that

(i) $ab(a+b)+3ac(a+c)+2xy(x+y)+17$
$=ax(a+x)+cy(c-y)+ay(a+y)-17$

(ii) $\frac{1}{2}ab+\frac{1}{3}(ax-by)+\frac{1}{2}xy+4\frac{5}{12}$
$=\frac{1}{2}ax+\frac{1}{3}(a+b)(x-y)+\frac{1}{2}by-4\frac{5}{12}$

79 If $y=4x^2-6x+5$, tabulate the values of y, when x has the values -2, -1.5, -1, 0, 1, 1.5 and 2

80 A has as much as B, and C has 2 rupees less than B If A has a rupees, find how much they have altogether.

CHAPTER IV

FUNDAMENTAL LAWS—MULTIPLICATION

Monomials

39 Law of Signs The ordinary definition of Multiplication is not sufficient when the multiplier is a fraction or a negative quantity. We therefore define Multiplication thus —"*To multiply one number by a second is to do to the first what is done to unity to obtain the second*"

From this definition the Law of Signs at once follows

To obtain 4, unity is repeated four times, that is,

$$4=1+1+1+1\ ,$$

therefore $\qquad 7\times 4=7+7+7+7=+28\ ,$

and $\qquad (-7)\times 4=-7-7-7-7=-28$

Again to obtain -4, unity is subtracted four times, that is,

$$-4=-1-1-1-1,$$

therefore $\qquad 7\times(-4)=-7-7-7-7=-28\ ,$

and $\qquad (-7)\times(-4)=-(-7)-(-7)-(-7)-(-7)^*$

$$=+7+7+7+7=+28$$

Thus the definition holds when the multiplier is an *integer*. Similarly it may be shewn to hold when the multiplier is a *fraction*

Hence generally

$$(+a)\times(+b)=+ab \qquad \text{(i)},$$

$$(-a)\times(+b)=-ab \qquad \text{(ii)},$$

$$(+a)\times(-b)=-ab \qquad \text{(iii)},$$

and $\qquad (-a)\times(-b)=+ab \qquad \text{(iv)}$

From (i) and (iv), we see that *if the multiplicand and multiplier have the same sign, the sign of the product is* +, and from (ii) and (iii), that *if they have different signs, the sign of the product is* −

* Here we see that a negative multiplier makes a contrary repetition of the multiplicand, which idea we first meet with in the works of a Hindu algebraist "It has been before observed that negation is contrariety A negative multiplier, therefore, is a contrary one, that is, it makes a contrary repetition of the multiplicand Such being the case, if the multiplicand be positive, (the multiplier being negative) the product will be negative, if the multiplicand be negative, the product will be affirmative"—KRISHNA BHATTA'S *Commentary* on BHASKARA'S *Vijaganita*

This is the law of signs and is briefly stated thus:—**Like signs give +, and unlike signs give −.**

40. The following applications of the Law of Signs should be noticed. We have

$$a \times (-b) = -ab\,,$$
$$a(-b)(-c) = -ab \times (-c) = +abc\,,$$
$$a(-b)(-c)(-d) = +abc \times (-d) = -abcd\,,$$
$$a(-b)(-c)(-d)(-f) = -abcd \times (-f) = +abcdf\,,$$ and so on.

Thus *the product of any number of negative factors is* **positive** *or* **negative** *according as the number of factors is* **even** *or* **odd**.*

Again
$$(-a)^2 = (-a)(-a) = +a^2\,,$$
$$(-a)^3 = (-a)(-a)(-a) = -a^3\,,$$
$$(-a)^4 = (-a)(-a)(-a)(-a) = +a^4\,,$$
$$(-a)^5 = (-a)(-a)(-a)(-a)(-a) = -a^5\,,$$ and so on.

Hence *any* **even** *power of a negative quantity is* **positive**; *and any* **odd** *power of a negative quantity is* **negative**.

Examples XX.

If $a=-2$, $b=5$, $c=-3$, $x=1$, and $y=-1$, find the value of

1.	$2c^2xy^2$	2.	$-a^2x^3y^5$	3.	$4a^2y^2$	4.	$3abc$
5.	$-b^2y^5$	6.	$-x^2y^4$	7.	$2a^2bc$	8.	$-ac^2x$
9.	$-x^3y^2$	10.	$3a^2c^2$	11.	$5ab^2c$	12.	$-2b^3xy^3$
13.	$3bcy^4$	14.	$-2abc^2$	15.	$-a^3x^3y^4$	16.	$4a^3bc^3$
17.	$-6ax^2y^2$	18.	$-c^2a^2y^5$				

If $a=-5$, $x=5$, $y=-3$, $z=-2$, find the value of

19.	a^2-x^2	20.	x^3-y^2	21.	az^2-xy
22.	a^4-xyz	23.	$a^2-(y+z)^2$	24.	$a^3+x^3-y^2$
25.	$(3a-2x)(a+2x)$	26.	$(5x+4y)(2x-3y)$		
27.	$3x^2-5xy+2z^2$	28.	$y^2-3xz-a^2$		
29.	$x^2-y^2-z^2-2yz$	30.	$y^2+z^2+3ayz-a^2$		

31. If $y=x^2-5x+6$, tabulate the values of y, corresponding to the values 0, −1, −2 and −5 of x.

32. If $y=x^3-3x$, tabulate the values of y, when x has the series of values 3, 2, 1, 0, −3 and −1.

33. Simplify $(-1)^2$, $(-1)^3$, $(-1)^4$, $(-1)^5$.

* A number is said to be *even* when it is divisible by 2, as 2, 4, 6, ...; and *odd*, when it is not divisible by 2, as 1, 3, 5, ...

41. To prove that $ab=ba$.

$$\begin{array}{ccccc} * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \end{array}$$

Let a star represent a marble. Thus there are 5 marbles in a row and 4 such rows, therefore the total number of the marbles is 5 repeated 4 times, i.e., 5×4. Again there are 4 marbles in a column and 5 such columns, therefore the total number of the marbles is 4 repeated 5 times, i.e., 4×5.

Hence $$5\times4=4\times5.$$

And we shall come to the same conclusion if we take any number of marbles in a row and any number of such rows. Thus the theorem is proved for positive integers. Similarly the definition of multiplication given in Art. 39, will enable us to prove the theorem when a and b are fractions or negative numbers. Thus generally

$$a\times b=b\times a, \text{ i.e., } ab=ba.$$

42. The factors of a product may be grouped in any manner. Let there be 4 stars in a row and 5 such rows, and let each star represent 3 units.

$$\begin{array}{cccc} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{array}$$

Then clearly the number of units represented by the stars

$$=3\times4\times5.$$

Since each star represents 3 units, therefore in each row there are (3×4) units. And there are 5 rows, thus the *total number of units* the stars represent is (3×4) repeated 5 times, i.e., $(3\times4)\times5$.

Again there are altogether (4×5) stars, and each star represents 3 units. Thus the *total number of units* that the stars represent is 3 repeated (4×5) times, i.e., $3\times(4\times5)$.

Hence $$3\times4\times5=(3\times4)\times5=3\times(4\times5).$$

Now clearly the reasoning will be the same if we take any number of stars in a row and any number of such rows Thus the theorem is proved for positive integers, and with the definition of multiplication in Art 39, we can prove it when the factors are fractions or negative numbers.

Hence generally

$$a\times b\times c=(a\times b)\times c=a\times(b\times c), \text{ or } abc=(ab)c=a(bc)$$

Reasoning similarly we can prove that

$$abcd\ldots=(ab)cd\ldots=(ab)(cd)\ldots=(abc)d\ldots=\&c$$

This is called the **Associative Law for Multiplication**

43 The factors of a product may be taken in any order

This follows as a consequence of the theorems of Arts. 41 and 42

For $\qquad abc=a(bc)=(ab)c,$

and $\qquad a(bc)=(bc)a,\ (ab)c=c(ab)$, and $bc=cb,\ ab=ba$,

$$\therefore\ abc=acb=bca=bac=cab=cba.$$

Similarly it may be shewn that

$$abcd=acbd=bcad=\&c.,$$

and so on for any number of factors

Example. Multiply $2a$ by $3x$, $5x$ by $-3ab$, and $-4ax$ by $-6b$

$$2a\times 3x=2\times 3\times a\times x=6ax$$

$$5x\times(-3ab)=-(5x\times 3ab)=-(5\times 3\times a\times b\times x)=-15abx$$

$$(-4ax)\times(-6b)=+(4ax\times 6b)=4\times 6\times a\times b\times x=24abx$$

This is known as the **Commutative Law for Multiplication**

44. Multiplication of Powers of the same Quantity

Since $a^2=aa$ and $a^3=aaa$, we get $a^2\times a^3=aa\times aaa=aaaaa=a^5=a^{2+3}$;

similarly $\qquad a^6\times a^4=aaaaaa\times aaaa=aaaaaaaaaa=a^{10}=a^{6+4}$;

and so on Thus since *the index of a power indicates the number of factors in that power*, we have

$$a^m=aaaa\ldots \text{ to } m \text{ factors},$$

and $\qquad a^n=aaaa\ldots \text{ to } n \text{ factors},$

$$\begin{aligned}\therefore a^m\times a^n&=(aaa\ldots \text{ to } m \text{ factors})\times(aaa\ldots \text{ to } n \text{ factors})\\&=aaaa\ldots \text{ to } (m+n) \text{ factors},\\&=a^{m+n};\end{aligned}$$

thus $a^m\times a^n=a^{m+n}$, where m and n are *positive integers*

Hence in *multiplying* powers of the same quantity, we **add** the indices of the powers

This law is called the **Index Law**

Examples XXI

Applying the Index Law, find by inspection the product of

1. a^5 and $-a^3$ 2. $-a^4$ and $-a^6$ 3. x^3 and $-x^4$
4. $-a^{2m}$ and a^n 5. a^n and $-a^2$ 6. x^{2m} and $-x^{2n}$
7. $-a$ and a^m 8. x^n and x^{n-1} 9. $-a^{m+1}$ and $-a^{m-1}$
10. a^{2m-1} and $-a^{1-m}$ 11. $-a^{2-m}$ and a^{2m-1}
12. $-2a^2x^3$ and $3ax^{n-2}$

45 By repeated applications of the Index Law, if $m, n, p,$ are positive integers, we have

$$a^m \times a^n \times a^p = (a^m \times a^n) \times a^p = a^{m+n} \times a^p = a^{m+n+p}$$

And *generally* $a^m \times a^n \times a^p \times \ldots = a^{m+n+p+\ldots}$

Ex 1 Multiply $2a^4$ by $3a$, $-4a^2$ by $-ab$, and $7a^3x$ by $-5ax^2$

$$2a^4 \times 3a = 2 \times 3 \times a^4 \times a = 6a^{4+1} = 6a^5$$

$$(-4a^2) \times (-ab) = +(4a^2 \times ab) = 4a^2a \times b = 4a^{2+1}b = 4a^3b$$

$$7a^3x \times (-5ax^2) = -(7a^3x \times 5ax^2) = -35a^{3+1}x^{1+2} = -35a^4x^3$$

Ex 2 Find the value of $(-ax)^3$

Required value $= (-ax)(-ax)(-ax) = -(ax \times ax \times ax)$

$= -(aaa \times xxx) = -a^3x^3$

Ex 3 Multiply together ab, $-2bc$, $-3ac$ and $-4bc^2$

Here the number of negative factors is 3, therefore the product is *negative* Hence

Product $= -(ab \times 2bc \times 3ac \times 4bc^2)$

$= -(2 \times 3 \times 4 \times ab \times bc \times ac \times bc^2)$

$= -(24 \times aa \times bbb \times ccc^2)$

$= -24a^2b^3c^4$

Definition When three or more quantities are multiplied together the result is called their **continued product**

Examples XXII

Multiply

1. $3x$ by ax 2. $5xy$ by $-2y$ 3. $-8ax$ by $13px$
4. $-2abx$ by ac 5. $-7abc$ by $-8bc$ 6. $-pqr$ by $-qrx$

Multiply

7. $25axy$ by $-31bcz$
8. $-312mn$ by $-56pq$
9. $54abz$ by $12cdy$.
10. $-12xyz$ by $3ax$
11. $-16abpq$ by $-5abxy$
12. $20arz$ by $-17brwz$
13. $3a^6$ by $2a$
14. $10x^6$ by $5x^4$
15. $15y^6$ by $-4y$
16. $2a^2$ by $-5a^2$.
17. $-12m^8$ by $-6m^0$.
18. $-7x^{10}$ by $-11x^2$
19. $3ax^2$ by $-a^2x$
20. $2x^3y$ by $-3xy^2$
21. $-a^2bc^2$ by $-2abc$
22. $-m^2n^2x$ by $-4mx^2$
23. $3xyz^3$ by $-2x^2z$
24. $-4ab^2$ by $-6a^2b^3$
25. $2a^2b^2c^2d^4$ by $-abcd$
26. $-4a^2x^3by^4$ by $-3a^2xb^2y$
27. $-7a^2b^2m$ by $3a^2b^4m^2$
28. $-8a^2xy^3z$ by $-20ay^2z^3$
29. Simplify $(-4a)^2$, $(-2ab)^3$, $(-abc)^6$, $(-3x^2)^4$

Multiply together

30. $-ax$, bx and $-acx$
31. $5ab$, $-2cd$ and $-3ef$
32. $-a$, $2b$, $-4c$ and $-15d$
33. $6a$, $-5bc$ and $12px$.
34. $-10ab$, $-2cx$ and $-20yz$
35. $-ab$, $2c$, $-15r$, $-3z$, $-4s$, $6x$ and $-y$
36. $3ab$, $-5a$, $-b$, $2c$, $-4bc$ and $-8ac$
37. $20ab$, $-3abc$, $-4cx$, $5xy$, $-2by$ and $-8c$

Find the continued product of

38. $3x^3$, $-2x$ and $-x^4$.
39. $4c$, $-8c^2$, $-7c$ and $-mc^3$
40. $-a^2$, $3a^3$, $-4a^5$ and $-5a^4$
41. $-ax^2$, byz and $-3a^2xy$
42. $-6ab^2x^3$, $12a^3xy^3$, $-c^2xy$ and $-3a^3cy^2$.
43. $-ax^3$, $2f^2g$, $-4fy^2$ and $-15cxy$
44. a^2x^2, $-b^2y^3$, mc^2z, $-axy^2$ and $5xyz$

Polynomials

Multiplication by a Monomial

46 We have

$$(a+b)n=(a+b)+(a+b)+(a+b)+\dots \text{ repeated } n \text{ times}$$
$$=a+a+a+a+\dots \text{ repeated } n \text{ times}$$
$$+b+b+b+b+\dots \text{ repeated } n \text{ times}$$
$$=an+bn, \text{ where } n \text{ is a } \textit{positive} \text{ integer}.$$

We assume for the present that the result is true whether n be negative or fractional Thus for all values of n,

$$(a+b)n=an+bn$$

Again $(a+b+c)n=(p+c)n$, if $a+b=p$,

$$=pn+cn$$
$$=(a+b)n+cn$$
$$=an+bn+cn$$

And *generally* $(a+b+c+\ \ )n=an+bn+cn+\ .\ $, where any one or more of the quantities a, b, c, may be negative or fractional

Thus *the product of a polynomial by a monomial is the algebraical sum of the partial products of each of its terms multiplied separately by the monomial*

This is called the **Distributive Law** for Multiplication

Ex. 1. Multiply $2a-3b+4c$ by $-5m$

$$\text{Product}=(2a-3b+4c)\times(-5m)$$
$$=2a(-5m)-3b(-5m)+4c(-5m)$$
$$=-10am+15bm-20cm$$

Ex 2 Multiply $-3a^4+2a^3b-8a^2b^3$ by $2a^2b$

$$\text{Product}=(-3a^4+2a^3b-8a^2b^3)\times 2a^2b$$
$$=(-3a^4)2a^2b+(2a^3b)(2a^2b)+(-8a^2b^3)2a^2b$$
$$=-6a^6b+4a^5b^2-16a^4b^4$$

Ex 3 Simplify $8x^2(2x-3)-4x(4x^2-3)+12x(2x-1)$

Given expression

$$=(8x^2\times 2x-8x^2\times 3)-(4x\times 4x^2-4x\times 3)+(12x\times 2x-12x\times 1)$$
$$=(16x^3-24x^2)-(16x^3-12x)+(24x^2-12x)$$
$$=16x^3-24x^2-16x^3+12x+24x^2-12x=0$$

Examples XXIII

Multiply

1. $a-x$ by $3ax$
2. $x-y$ by $-a$
3. $-a-b$ by $-b$.
4. $2ax-by$ by ay
5. $4ab-c^2$ by $3abc^2$
6. $-cb-cd$ by $-5bc$
7. $-x+y-2$ by $2a$
8. $x-3b-ab^3$ by $-bx$
9. $bc+ca-ab$ by $-abc$
10. $2x-y-4z$ by $-3xyz$
11. $-a^2-b^2-c^2$ by $-2a^2b^2c^2$
12. $3a+2b-4c+d$ by ad
13. $2a^2bc-3ab^2c-4abc^2$ by $-abc$

Multiply

[In Exs 14—19, multiply the numerical coefficients as in Arithmetic]

14 $5x-2y+4z$ by $\frac{3}{20}x$

15 $4a-3b-2$ by $-\frac{5}{12}a^2bc$.

16. $-\frac{1}{3}a-\frac{1}{2}b-1$ by $-\frac{6}{7}ab$

17 $\frac{2}{5}a^2x^3-\frac{3}{4}a^3x^2$ by $\frac{5}{6}ax^2$

18. $\frac{1}{3}a^3b-\frac{3}{4}a^2b^2-\frac{2}{3}b^3$ by $-\frac{6}{5}a^3b^3$

19. $-\frac{3}{5}a^2x-\frac{2}{7}by^2+\frac{1}{5}c^2z^3$ by $-\frac{14}{3}b^2cx^3$

20 $-2a^3-3b^3-4c^3$ by $-5a^2b^3c$

21 $2a^3b-8b^2c-5ac^2$ by $3ab^3c^3$.

22 $a^5x^3-2a^2x^5-3a^4x$ by $-4ax^5$

23 $-8x^3y^2z^4+7xyz^3-3x^2y^3z$ by $-2xy^2z^3$

24 $x^3-3x^2y+3xy^2-y^3$ by $8xy$

25 $3a^3b^2c-b^3c^2d-2c^3d^2a+4d^3a^2b$ by $-3abcd$

26. $5a^2b^3c-3ax^3y-abx+cx^2y^4$ by $-4abxy$

Simplify

27 $3a^2(1-a)-4a^3(a-2)-a(3a+5a^2)$

28. $4x^2(x-4)-3x(x^2-5x)+2(x^2-1)$

29. $2ax(a-1)-2ax(x-1)+2ax(a-x)$

30. $3x^3(13a-12b)+3x^3(14b-4a)-3x^3(2a-5b)$

31 $6a(3a-4b+1)-8b(3b-3a+1)-2(3a-4b)$

32 $8x^2(2x-3y+4)-6y^2(4x-2y)-8x(2x^2-3xy-3y^2)$

33 $5x^3y(3x^2y^2-2xy^3-2y^2)-3xy^2(5x^4-2x^3y-4x^2)$
$\qquad +2x^2y^2(2x^2y^2-xy+1)$

34. $a^2(b-c)+a^2(3c+a)-a^2(b-2a)+a^2(a-b)$

35. $ab(a-b)+bc(b-c)-b^2(c-a)+b(c^2-a^2)$

36. $a^3(b-c)+b^3(c-a)+c(a^3-b^3)-ab(a^2-b^2)$

37. $x^3(y^2z-yz^2)+z^3(x^2y-xy^2)-x^2z^2(yz-xy)-xz(x^2y^2-y^2z^2)$

47 **Degree of a Term.** Each of the letters in a term is called a **dimension** of the term, and the *number* of the letters in it, denotes its **degree**

Thus abc has *three* letters, it has therefore *three* dimensions, or is of the *third* degree, $3a^2xy^3$ has *six* letters $aaxyyy$, it has therefore *six* dimensions, of is of the *sixth* degree, and so on. The numerical coefficient is not counted

Hence *the degree of a term is determined by adding the indices of the letters involved in it* thus $2ax^2y^3z^4$ is of $1+2+3+4$, or 10 dimensions

48 Degree of an Expression The degree of a polynomial is the degree of the term of *highest* dimensions in it Thus the polynomial x^2+3x-1 is of the *second* degree, $2x+3x^3+4$ is of the *third* degree, and so on

An expression of the second degree is often called a **quadratic expression**, or simply a **quadratic**

It is sometimes useful to consider the degree of a given polynomial *with respect to a particular symbol* involved in it, say x The polynomial is then said to be an **expression in** x and the symbol x is said to be the **symbol of reference** Thus $x^3+2ax^2+3a^2x+2a^4$ is an expression of the *third* degree when x is the symbol of reference, and of the *fourth* degree when a is the symbol of reference

49 Homogeneous Expression When all the terms of a polynomial are of the *same* degree, the polynomial is said to be **homogeneous** Thus $a^3-5a^2b+2abc$ is a homogeneous expression of the *third* degree

50 Arrangement of an expression according to powers of a letter An expression is said to be arranged according to the **ascending** or **descending** powers of a letter, when the power of that letter *increases* or *decreases* in each successive term beginning with the first Thus the expression

$$1+x+x^2+x^3+x^4+x^5$$

is arranged according to the *ascending* powers of x Also, the same expression when written

$$x^5+x^4+x^3+x^2+x+1$$

is arranged according to the *descending* powers of x

Again an expression containing two or more letters, may be arranged *differently* Thus for example, the expression

$$a^3(b-c)+b^3(c-a)+c^3(a-b),$$

when arranged according to the descending powers of a, b and c will be written respectively

$$(b-c)a^3-(b^3-c^3)a+(b^3c-bc^3),$$
$$(c-a)b^3-(c^3-a^3)b+(c^3a-ca^3),$$
and $$(a-b)c^3-(a^3-b^3)c+(a^3b-ab^3)$$

Cor Hence in an expression arranged in *descending* powers, the *first* term is the *highest* term and the *last* is the *lowest* term in it

MULTIPLICATION BY BINOMIAL

51 We have $(a+b)(c+d)=n(c+d)$, where $n=a+b$,

$$=nc+nd \text{ [Art 46]}$$

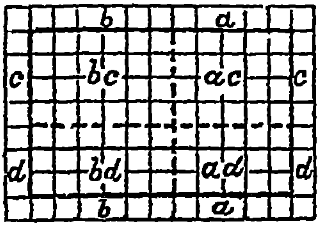
$$=(a+b)c+(a+b)d, \text{ replacing } n \text{ by } a+b,$$

$$=ac+bc+ad+bd$$

Thus when one expression is multiplied by another, we have the following rule for finding the product

Rule *Multiply each term of the multiplicand by each term of the multiplier, the algebraic sum of the partial products thus found is the product required*

The following is a graphical illustration of this important result —



The whole rectangle contains $(a+b)(c+d)$ units of area, and it is equal to the sum of the 4 smaller rectangles whose areas are ac, bc, ad and bd sq units

That is, $(a+b)(c+d)=ac+bc+ad+bd$

Ex 1 Multiply $x+2$ by $x+3$

Required product $=(x+2)(x+3)=(x+2)x+(x+2)3$

$$=x^2+2x+3x+6=x^2+5x+6$$

The work is usually arranged thus

$$\begin{array}{l} x+2 \\ \underline{x+3} \\ x^2+2x \\ \quad\;\; \underline{+3x+6} \\ x^2+5x+6 \end{array}$$

The second partial product is shifted one place to the right to bring like terms under like terms

Ex 2 Multiply $5a^2-2bc$ by $3a^2-8bc$

$$\begin{array}{r} 5a^2-2bc \\ 3a^2-8bc \\ \hline 15a^4-6a^2bc \\ -40a^2bc+16b^2c^2 \\ \hline 15a^4-46a^2bc+16b^2c^2 \end{array}$$

[Verify the work by taking $a=1$, $b=2$ and $c=3$]

Ex 3 Multiply $\frac{1}{2}x+3y$

by $2x-\frac{1}{3}y$

$$\begin{array}{l} \frac{1}{2}x+3y \\ 2x-\frac{1}{3}y \\ \hline x^2+6xy \\ \qquad -\frac{1}{6}xy-y^2 \\ \hline x^2+\frac{35}{6}xy-y^2 \end{array}$$

The fractional coefficients are multiplied as in Arithmetic

Examples XXIV

Multiply

1 $x+1$ by $x+5$ 2 $x+5$ by $x-3$ 3 $x-7$ by $x+8$.

4 $x-11$ by $x-9$ 5 $-a+6$ by $a-2$ 6 $x+a$ by $x+b$

7 $x+a$ by $x-b$ 8 $x-a$ by $x+b$ 9 $x-a$ by $x-b$

10 $-x-7$ by $-x-9$ 11 $a+b$ by $a+b$ 12 $a-b$ by $a-b$.

13 $a+b$ by $a-b$ 14 $a-3b$ by $4a-b$

15 $1-8x$ by $1+12x$ 16 $1-3x^2y$ by $2-4x^2y$

17 $5a^2+b$ by $3a^2-2b^2$ 18 $6a^2b-c$ by $4ab^2+3c$

19 $3a^2b-5c^2$ by $2ab-4c$ 20 $7ab^2-c^3$ by $5a^2-3c^3$

21 $2x^2-5y$ by $4x^2-3y$ 22 $4a^3+3b^2c$ by $6a^3-7b^2c$

Simplify

23 $(a+b)(c-d)$ 24 $(a-b)(c-d)$

25 $(a-b)(c+d)$ 26 $(2m-9)(m+13)$

27 $(2x-3y)(3x-2y)$ 28 $(ax-by)(ax+by)$

29 $(\frac{2}{3}x-a)(\frac{3}{4}x-\frac{1}{2}a)$ 30. $(\frac{5}{6}ax-1)(\frac{3}{10}bx+1)$

31 $(2x-1)(3x-2)-(2x-1)(5x-2)$

32 $(5q-8r)(3q-14r)-(3q-7r)(5q-16r)$

33 $2x(x-1)+(x-1)(x+2)-2(x-1)(x+3)$

34 $a^2(a-1)+a(a+1)(a-1)-2a(a+2)(a-2)$

52 In the following examples, the multiplicand consists of more than two terms

Ex 1. Multiply a^2-ab+b^2
by $a+b$

$$\begin{array}{rl} & a^3-a^2b+ab^2 \\ & \quad\;\; +a^2b-ab^2+b^3 \\ \hline \text{Product} & a^3 \qquad\qquad\quad +b^3 \end{array}$$

Ex 2. Multiply a^2+ab+b^2
by $a-b$

$$\begin{array}{rl} & a^3+a^2b+ab^2 \\ & \quad\;\; -a^2b-ab^2-b^3 \\ \hline \text{Product} & a^3 \qquad\qquad\quad -b^3 \end{array}$$

Note These two results are very important and should be committed to memory.

Ex 3 Multiply $x^4+x^3y+x^2y^2+xy^3+y^4$
by $x-y$

$$\begin{array}{rl} & x^5+x^4y+x^3y^2+x^2y^3+xy^4 \\ & \quad\;\; -x^4y-x^3y^2-x^2y^3-xy^4-y^5 \\ \hline \text{Product} & x^5 \qquad\qquad\qquad\qquad\qquad -y^5 \end{array}$$

Ex 4 Multiply $x^2+2xy-3y^2$
by $4x-5y$

$$\begin{array}{rl} & 4x^3+8x^2y-12xy^2 \\ & \quad\;\; -5x^2y-10xy^2+15y^3 \\ \hline \text{Product} & 4x^3+3x^2y-22xy^2+15y^3 \end{array}$$

Observe that multiplicand and multiplier are both *homogeneous* and of the second and first degree respectively also the product is *homogeneous* and of the $(2+1)$ or third degree.

Examples XXV

Multiply

1 a^2+2a+7 by $a+5$

2 x^2-2x+4 by $x+2$

3 x^2+3x+9 by $x-3$

4 x^3+x^2+x+1 by $x-1$

5 $1-5x^2-12x^3$ by $1-5x$

6 $2y^2-y-5$ by $4+3y$

7 $p^2+4pq+4q^2$ by $p+2q$

8 $3x^2-5xy+7y^2$ by $2x-3y$

9 $25x^2-20xy+4y^2$ by $5x-2y$

10 a^3-2a+3 by a^2+2

11 $x^2-2xy+y^2$ by $-x-3y$.

12 $2c^2+cd-3d^2$ by $-2c-d$

13 $x^5+2x^3y-y^3$ by x^2-2y

14 $5a^3-7a^2-3a+8$ by $7a^2+8$

15 $25a^2+10ab+4b^2$ by $5a-2b$

16 $25x^2-40xy+64y^2$ by $5x+8y$

17 $a^4-a^3b+a^2b^2-ab^3+b^4$ by $a+b$

18 x^2+ax-b by $ax-b$

19 Multiply $4x^2-3(x-2a)$ by $3a-5(a-2x^2)$

20 Find the continued product of $c+3n$, $c-3n$, and $c+3n$

21 Find the continued product of $x-a$, $a+x$ and a^2+x^2

22 Simplify $(2x-1)^2-4(2x+1)(x-2)+3(x-1)(4x-3)$

23 If $A=2ax^2-ax+3$ and $B=4+bx-2bx^2$, find the value of $Ab+Ba$

53 Square of a Binomial We have

$$(a+b)^2=(a+b)(a+b)=a^2+ab+ab+b^2=a^2+2ab+b^2,$$

whatever be the values of a and b Hence the equality

$$(a+b)^2=a^2+2ab+b^2 \qquad \text{(i)}$$

is regarded as a *general result* from which particular cases will follow if any *numbers* or *expressions* are substituted for a and b

Thus if $a=2x$ and $b=3$, $(2x+3)^2=(2x)^2+2\cdot 2x\cdot 3+3^2=4x^2+12x+9$

We enunciate the result thus —*The square of the* **sum** *of two quantities is equal to the sum of their squares* **plus** *twice their product*

Again $$(a-b)^2=a^2-2ab+b^2 \qquad \text{(ii)}$$

This may follow from (i) by the substitution of $-b$ for b, or by multiplication as follows —

$$(a-b)^2=(a-b)(a-b)=a^2-ab-ab+b^2=a^2-2ab+b^2$$

This is also a general result and may be stated in words thus — *The square of the* **difference** *of two quantities is equal to the sum of their squares* **minus** *twice their product*

These two important results are commonly called *formulæ*

Any general result expressed by means of symbols is called a **formula**

The following are **graphical illustrations** of the two formulæ given above —

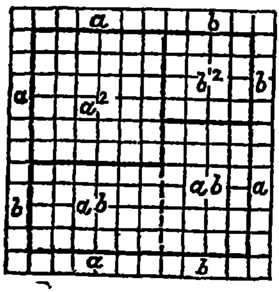


Fig 1

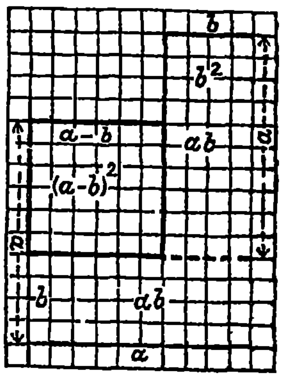


Fig 2

Fig 1 illustrates formula (i) The whole figure is the square on $a+b$, and it is equal to the two smaller squares a^2 and b^2 and the two rectangles ab and ab

Thus, $$(a+b)^2=a^2+2ab+b^2$$

Fig 2 illustrates formula (ii) The whole figure is equal to the square $(a-b)^2$ and the two rectangles ab and ab

Again the same figure is equal to the large square which is the square on a and the small square which is the square on b

Thus $\qquad (a-b)^2+2ab=a^2+b^2$,

or $\qquad (a-b)^2=a^2+b^2-2ab$, taking $2ab$ from both sides

Examples XXVI

Write down the square of

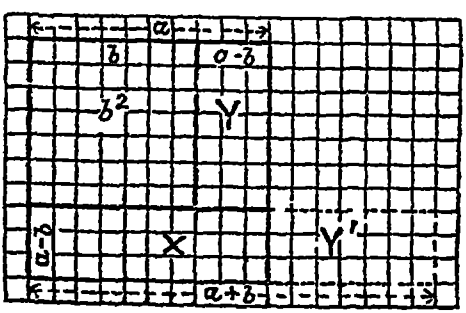
1	$x+3$	2	$a+4$	3	$1+2x$	4	$2x+7$
5	$a+2b$	6	$3x+a$	7	$2x+3a$	8	$ax+1$
9	$3+2xy$	10	$x-y$	11	$3x-4.$	12	$1-5x$
13	$2x-3y$	14	$3x-8a$	15	$x-5y$	16	$3a-4b$
17.	$8-3ax$	18	$1-abc$	19	$ab-xy$	20	l^2+mn
21	$2pq+r^2$	22	$ab+2c^2$	23	c^2-2ab	24	$3pq-4r^2$
25	$-1-a$	26	$-3a-1$	27	a^3+x^3	28.	$2ax+3by$
29	x^3-a^3	30	a^4-b^4	31	a^2x+ax^2	32	a^3-3a^2b
33	$m^2n^2-l^4$	34	$2a^3+b^3$	35	x^3+3yz	36.	$3mx^2-3m^2x$

54 Difference of two squares By actual multiplication we have $\qquad (a+b)(a-b)=a^2-b^2 \qquad$ (iii)

This is another important formula and is stated in words thus —

The product of the **sum** *and* **difference** *of two quantities is equal to the* **difference** *of their squares*

The following is a **graphical illustration** —



The large square is the square on a, and this square minus the small square, viz, b^2 is equal to $X+Y$ or $X+Y'$, which is the rectangle $(a+b)(a-b)$

That is, $\qquad (a+b)(a-b)=a^2-b^2$

Ex 1 $(1-2x)(1+2x)=1^2-(2x)^2=1-4x^2$

Ex 2 $(-x+3y)(-x-3y)=(-x)^2-(3y)^2=x^2-9y^2$

Ex 3 $(2x^2+5y^2)(2x^2-5y^2)=(2x^2)^2-(5y^2)^2=4x^4-25y^4$

Examples XXVII

Write down the value of

1 $(x+2)(x-2)$ 2 $(x+5)(x-5)$ 3 $(1+x)(1-x)$

4 $(a-3)(a+3)$ 5 $(2x+7)(2x-7)$ 6 $(3-4x)(3+4x)$

7 $(7-10x)(7+10x)$ 8 $(11+5x)(11-5x)$ 9 $(2x-y)(2x+y)$

10 $(a+bx)(a-bx)$ 11. $(x-ay)(x+ay)$ 12 $(-x+m)(-x-m)$

13 $(-2a-y)(-2a+y)$ 14 $(1+ax^2)(1-ax^2)$ 15 $(ax-3)(ax+3)$

16 $(3x+4y)(3x-4y)$ 17 $(2ax+3y)(2ax-3y)$ 18 $(x^2+2y)(x^2-2y)$.

19 $(2a^2-3bc)(2a^2+3bc)$. 20 $(a^2+x^2)(a^2-x^2)$

21. $(a^2-2b^2)(a^2+2b^2)$ 22 $(2ab+c^2)(2ab-c^2)$

23 $(-5x^2-3ab)(-5x^2+3ab)$ 24 $(4c^2-7ab)(4c^2+7ab)$

25 $(x^3+1)(x^3-1)$ 26 $(2a^3-3b^2)(2a^3+3b^2)$

27 $(3x^4-5a^3)(3x^4+5a^3)$

55 The product of the two binomial factors $x+a$ and $x+b$ should be noticed

$$\begin{array}{l} x+a \\ \underline{x+b} \\ x^2+ax \\ \underline{\quad +bx+ab} \\ x^2+ax+bx+ab \\ =x^2+(a+b)x+ab \end{array} \qquad (1),$$

where a and b may have any value, positive or negative.

This result shews that (i) the *first term* of the product x^2 is the *square* of x the *first term* of each factor, (ii) the *coefficient of x* is the *sum* of the *second terms* of the two factors, and (iii) the *last term* is the product of the *last terms* of the factors Bearing this in mind, we can write down at sight the product of any two binomials, the *first term of each of which is the same*

Thus $(x-3)(x+4)=x^2+x-12$, for the coefficient of x is $-3+4=1$ and the last term is $-3\times4=-12$

It should be noticed that by changing the signs of a and b we can get two other equalities, thus —

$$(x-a)(x-b)=x^2-(a+b)x+ab \quad \text{(ii)},$$
$$(x+a)(x-b)=x^2+(a-b)x-ab \quad \text{(iii)}$$

Examples XXVIII

Write down the value of

1	$(x+2)(x-1)$	2	$(x-1)(x+3)$	3	$(a-4)(a-9)$
4	$(a-8)(a+5)$	5	$(a+10)(a-11)$	6	$(x-2)(x-3)$
7	$(x+3)(x+7)$	8	$(x+4)(x-5)$	9	$(z-5)(z-10)$
10	$(x+8)(x+11)$	11	$(m+6)(m-9)$	12	$(y-8)(y+3)$
13	$(y-4a)(y-7a)$	14	$(x+3a)(x+5a)$	15	$(2m+3)(2m-1)$
16	$(2a-3)(2a-7)$	17	$(7a-3)(7a+4)$	18	$(3x+2)(3x+4)$.
19	$(am-11)(am+5)$	20	$(5x-6)(5x-11)$	21	$(4x+a)(4x-b)$.
22	$(4x+1)(4x+10)$	23	$(mx+3)(mx+4)$	24	$(mx-8)(mx-10)$
25	$(2x-a)(2x+b)$	26	$(x^2+c)(x^2+d)$	27	$(x^2+a^2)(x^2-b^2)$
28	$(2x^3+3)(2x^3+9)$	29	$(2x^3-3)(2x^3-9)$	30	$(3p^3-4)(3p^3+1)$

CHAPTER V

FUNDAMENTAL LAWS—DIVISION

Monomials

56 Division is defined as the inverse operation to that of multiplication

Thus if $a \div b=c$, the quotient c is such that

$$c\times b=a$$

Hence $\quad (a\div b)\times b=a$,

i.e., $\quad a\div b\times b=a$

In $a\div b=c$, a (the quantity divided) is called the **dividend**, b (the quantity by which a is divided) is called the **divisor**, and c (the result) is called the **quotient**

57 Since Division is the inverse of Multiplication, it is clear that the fundamental Laws of Association, Commutation and Distri-

bution which have been proved for Multiplication are also true for Division

Thus $a \div b \div c \div d = a \div c \div b \div d = a \div d \div c \div b = \&c$,

$$a \times b \div c \times d = a \div c \times b \times d = a \times d \div c \times b = a \times b \times d \div c = \&c \ ,$$

and so on

Again $(a \div b \div c) \times bc = a \div b \div c \times b \times c$

$$= a \div b \times b \div c \times c = a \ ,$$

$$a \div b \div c = a \div bc$$

Similarly $\quad a \div b \div c \div d \div \quad\quad = a \div bcd$

Thus *to divide a quantity by two or more quantities successively is the same as to divide it at once by their product*

Conversely we have $a \div bcd \quad = a \div b \div c \div d \div$

58 From the definition of Division, we have

$$a \times \frac{1}{d} \times d = a \times 1 \div d \times d = a \times 1 = a \ ,$$

$$a \times \frac{1}{d} = a \div d$$

Hence *to divide a quantity by another is the same as to multiply the former by the* **reciprocal** *of the latter*

Def Quantities are said to be **reciprocal** to one another when their product is unity Thus 2 and $\frac{1}{2}$, $\frac{5}{3}$ and $\frac{3}{5}$, a and $\frac{1}{a}$, &c, are reciprocal quantities

59 Law of Signs Since

$(+a) \times (+b) = +ab$, we have $+ab \div (+a) = +b$,

$(-a) \times (+b) = -ab$, we have $-ab \div (-a) = +b$,

$(+a) \times (-b) = -ab$, we have $-ab \div (+a) = -b$,

$(-a) \times (-b) = +ab$, we have $+ab \div (-a) = -b$

Hence *when the dividend and the divisor both have the same sign, the quotient has the sign* $+$, *and when they have different signs, the quotient has the sign* $-$

Thus we have the same Law of Signs for Division as before which is simply stated thus —**Like signs give + and unlike signs give −**

60 Division of Monomials Rule —*Indicate the operation in the form of a fraction, strike out the common factors as in Arithmetic the remaining factors will be the quotient required*

Ex. 1. Divide $20abcd$ by $5ac$

Required quotient $=\dfrac{4\cdot 5ac\cdot bd}{5ac}=4bd$

Ex. 2. Divide $156abxyz$ by $-13axz$

Required quotient $=\dfrac{12\cdot 13axz\cdot by}{-13axz}=-12by.$

Ex. 3. Divide $-2macxy$ by $-mxy$

Required quotient $=\dfrac{-2\cdot mxy\cdot ac}{-mxy}=2ac$

Ex. 4. Divide $13abc$ by $2ad$

Required quotient $=\dfrac{a\cdot 13bc}{a\cdot 2d}=\dfrac{13bc}{2d}$

Ex. 5. Divide $24xyz$ by $-28axyz$

Required quotient $=\dfrac{6\cdot 4xyz}{-7a\cdot 4xyz}=-\dfrac{6}{7a}$

Examples XXIX

Divide

1. $abxyz$ by ax
2. $-120acdxy$ by $15cxy$.
3. $391mnpxy$ by $-23npy$
4. $-15abcx$ by $6aby$
5. $-420axyz$ by $-70axz$.
6. $-10abqr$ by $15axpr$
7. $13bcrs$ by $-13abrs$
8. $-2mnxy$ by $-3amyz$

61. Division of Powers of the same quantity. By Art. 44

$$a^{m-n}\times a^{n}=a^{m-n+n}=a^{m},$$

where m and n are positive integers and $m>n$. Hence by the definition of Division

$$a^{m-n}=a^{m}\div a^{n},$$

m and n being both positive integers and $m>n$.

Thus in *dividing* one power of a quantity by a lower power, we **subtract the index of the divisor from that of the dividend**.

This is the **Index Law** for Division.

Cor. Meaning of a^{0}. When $m=n$, we have

$$a^{m-n}=a^{m}\div a^{n}=a^{m}\div a^{m}=1\ ,$$

and then $m-n=0$, and therefore $a^{m-n}=a^{0}$. Thus $a^{0}=1$

Ex 1. Divide $20a^5$ by $5a^2$

Required quotient$=20a^5\div 5a^2=20\times a^5\div 5\div a^2$ [Art 57]

$=20\div 5\times a^5\div a^2=4\times a^{5-2}=4a^3$

Or more conveniently thus —

Required quotient$=\dfrac{20a^5}{5a^2}=\frac{20}{5}a^{5-2}=4a^3$

Ex 2 Divide $150a^5x^4$ by $30a^2x^3$

Required quotient$=\dfrac{150a^5x^4}{30a^2x^3}=\dfrac{150}{30}\times a^{5-2}x^{4-3}=5a^3x$

Ex 3 Divide $20a^2x^3y$ by $-4ax^2y$

Required quotient$=\dfrac{20a^2x^3y}{-4ax^2y}=\dfrac{20}{-4}\times a^{2-1}x^{3-2}y^0=-5ax$

Ex 4 Divide $-144x^3y^{10}$ by $-30xy^3$

Required quotient$=\dfrac{-144x^3y^{10}}{-30xy^3}=\dfrac{-144}{-30}\times x^{3-1}y^{10-3}=\dfrac{24}{5}x^2y^7$

Examples XXX.

Divide

1 a^4x^3 by $-a^2x$

2 $3a^2bc$ by ac

3 $16a^3x^3y^2$ by $-4a^2xy$

4 $-180a^5b^4c^3d^2$ by $45b^3c^2d$

5 $-129x^4y^2z^3$ by $-3x^3y^2z$

6 $-3x^3yz^4$ by $2ax^2yz^3$

7 $-16a^4x^3y^2$ by $-24a^3x^5y$

8 $-20x^3y^6z^9$ by $5x^2y^5$

9 $-50a^6b^3m^7$ by $-75a^5bm^4$

10 $4a^2x^3y^5$ by $5xy^4$

Polynomials

DIVISION BY A MONOMIAL

62 We have $(a+b)\div n=(a+b)\times\dfrac{1}{n}=a\times\dfrac{1}{n}+b\times\dfrac{1}{n}=a\div n+b\div n$

Similarly,

$$(a+b+c+d)\div n=(a+b+c+d)\times\frac{1}{n}=a\div n+b\div n+c\div n+d\div n$$

And generally $(a+b+c+\ldots)\div n=a\div n+b\div n+c\div n+\ldots$

Thus *if a polynomial be divided by a monomial, the quotient shall be equal to the sum of the partial quotients obtained by dividing each of its terms separately by the monomial*

REMARK It is to be noted that the Law of Distribution has *full* application to Multiplication, but only a *partial* application to Division In Multiplication *both* the Multiplicand and Multiplier may be distributed, whereas in Division, the *Dividend only* can be distributed but not the Divisor Thus it is true that $(a+b)\div c = a\div c+b\div c$, but it is not true that $a\div(b+c)=a\div b+a\div c$

Ex 1. Divide $ax+bx$ by x

Required quotient$=(ax+bx)\div x=ax\div x+bx\div x=a+b$

Otherwise more conveniently thus —

$$\text{Required quotient}=\frac{ax+bx}{x}=\frac{ax}{x}+\frac{bx}{x}=a+b \quad [\text{Art } 60]$$

Ex 2 Divide $3ax^2-2abx+a^2cx$ by $-ax$

$$\text{Required quotient}=\frac{3ax^2-2abx+a^2cx}{-ax}=\frac{3ax^2}{-ax}-\frac{2abx}{-ax}+\frac{a^2cx}{-ax}$$
$$=-3x+2b-ac$$

Ex 3 Divide $-12a^2x^3+5abx^2-10bx$ by $-2x$

$$\text{Required quotient}=\frac{12a^2x^3}{2x}-\frac{5abx^2}{2x}+\frac{10bx}{2x}=6a^2x^2-\tfrac{5}{2}abx+5b$$

Examples XXXI

Divide

1 $ab+bc$ by b

2 $2ax-bx$ by $-x$

3 $3a^2x-2abx$ by ax

4. $-15x^3+5x^2$ by $-5x$

5 $16a^2b^2-8a^3b^2$ by $-4a^2b$

6 $-20x^4-6x^3$ by $-2x^3$

7 $5ab^2-abc+abd$ by ab

8 $-ax^3+a^2x^2-a^3x$ by $-ax$.

9 $2x^5-x^4+3x^3$ by $-x^2$

10. $2x^3+6x^2y-9xy^2$ by $3x$

11 $3a^2bc-9ab^2c-6abc^2$ by $-3abc$

12 $-24a^3x^2y-3axy+6x^2y^2$ by $-3xy$

13 $-144a^3-108a^2b+96ab^2$ by $-12a$

14 $5a^3bx^3y^2-3a^2x^2y^3+2a^4xy^3z^3$ by $-a^2xy^2$

15 $6a^4x^2-30a^3x^4+24a^2x^3$ by $6a^2x^2$.

16 $18a^4b^7c^3-24a^3b^3c^2+30a^2b^4c^3$ by $6a^2b^2c^2$

17 $16a^3xy-14a^2x^2+4a^2x^3$ by $4a^2x$

18 $30a^3b^2c-24a^8c^3+12a^3c^2x-6a^3c$ by $6a^3c$

19. $3p^4q-9p^3q^2+3p^2q^3-2pq^4$ by $-3pq$

20 $-12x^6y^8+4x^5y^7-6x^4y^6-3x^3y^5$ by $-3x^3y^5$

DIVISION BY A BINOMIAL

63 To divide $2x^2+13x+15$ by $x+5$

We have $2x^2+13x+15=(2x^2+10x)+(3x+15)$

$$=(x+5)2x+(x+5)3 \ldots \quad \text{(i)}$$

$$(2x^2+13x+15)\div(x+5)=\frac{(x+5)2x+(x+5)3}{x+5}$$

$$=\frac{(x+5)2x}{x+5}+\frac{(x+5)3}{x+5} \text{ distributing dividend}$$

$$=2x+3 \quad \ldots \quad \text{(ii)}$$

Dividend and Divisor are arranged in *descending* powers of x, and the Quotient is in the same order [see (ii)], therefore the first term in each is the *highest* term in it Also from (i) we see that Dividend = Divisor × *first* term of Quotient + Divisor × *second* term of Quotient Hence *first* term of Quotient = first term of Dividend ÷ first term of Divisor, *i e*, first term of Quotient $=2x^2\div x=2x$

Now multiply the whole Divisor by $2x$, and we get $(x+5)2x$

If $(x+5)2x$ is subtracted from dividend, we see from (i) that the remainder is $(x+5)3$ or $3x+15$ The first term $3x$ of this remainder is the product of 3, the *second* term of quotient and x the first term of divisor Thus the second term of quotient $=3x\div x=3$

Multiply the whole divisor by 3 and subtract the result and there is *no remainder* Thus the division terminates and the complete quotient $2x+3$ is the sum of the two partial quotients

The process explained above is usually shewn thus—

$$\begin{array}{r|l}
x+5\ ) & 2x^2+13x+15\ (\ 2x+3 \\
 & \underline{2x^2+10x} \\
 & 3x+15 \\
 & \underline{3x+15}
\end{array}$$

Hence to divide one polynomial by another, we have the following

Rule —*Arrange both dividend and divisor according to the* DESCENDING *or* ASCENDING *powers of some common letter*

Divide the first term of the dividend by the first term of the divisor to obtain the FIRST TERM *of the quotient, multiply the whole divisor by this term and subtract the product from the dividend and put down the remainder.*

Consider the remainder as a new dividend and repeat the above process thus the SECOND TERM *of the quotient is obtained*

Continue the same operation with the successive remainders to obtain the OTHER TERMS *of the quotient till there is no remainder left*

The sum of these partial quotients is the complete quotient required

Ex 1 Divide $a^2-2ab+b^2$ by $a-b$

Here the dividend and the divisor are arranged according to the *descending* powers of a

$$
\begin{array}{r|l|l}
a-b & a^2-2ab+b^2 & a-b \\
 & \underline{a^2-ab} & \\
 & -ab+b^2 & \\
 & \underline{-ab+b^2} &
\end{array}
$$

REMARK We may arrange the dividend and the divisor according to the *ascending* powers of a and obtain the same result, thus

$$
\begin{array}{r|l|l}
-b+a & b^2-2ab+a^2 & -b+a, \text{ the same result as before} \\
 & \underline{b^2-ab} & \\
 & -ab+a^2 & \\
 & \underline{-ab+a^2} &
\end{array}
$$

Ex 2 Divide $11x^2+2x^3+5+17x$ by $2x+5$

Here the divisor is arranged according to the *descending* powers of x, but not the dividend, arrange it then according to the *descending* powers of x.

$$
\begin{array}{r|l|l}
2x+5 & 2x^3+11x^2+17x+5 & x^2+3x+1 \\
 & \underline{2x^3+5x^2} & \\
 & 6x^2+17x & \\
 & \underline{6x^2+15x} & \\
 & 2x+5 & \\
 & \underline{2x+5} &
\end{array}
$$

[Work this example by arranging according to the *ascending* powers of x]

Ex 3. Divide $ax^3-(a^2+b)x^2+b^2$ by $ax-b$

$$
\begin{array}{r|l|l}
ax-b & ax^3-a^2x^2-bx^2+b^2 & x^2-ax-b \\
 & \underline{ax^3-bx^2} & \\
 & -a^2x^2 & \\
 & \underline{-a^2x^2+abx} & \\
 & -abx+b^2 & \\
 & \underline{-abx+b^2} &
\end{array}
$$

Ex 4 Divide $81x^4-1$ by $3x+1$

$$
\begin{array}{r|l}
3x+1 & 81x^4-1 \quad (27x^3-9x^2+3x-1 \\
 & \underline{81x^4+27x^3} \\
 & \quad -27x^3-1 \\
 & \quad \underline{-27x^3-9x^2} \\
 & \qquad 9x^2-1 \\
 & \qquad \underline{9x^2+3x} \\
 & \qquad\quad -3x-1 \\
 & \qquad\quad \underline{-3x-1}
\end{array}
$$

Examples XXII

Divide, by arranging (1) according to the *descending* and (2) according to the *ascending*, powers of x

1 x^2+4x+3 by $x+1$

2 $x^2+7x+12$ by $x+4$

3 $x^2-10x+21$ by $x-3$

4 y^2+4y-5 by $y-1$

5 $2x^2+5x+2$ by $x+2$.

6. $4x^2+23x+15$ by $4x+3$

7 $2m^2-13m-24$ by $m-8$

8 $10-13a+4a^2$ by $2-a$

9 $12+x-6x^2$ by $4+3x$

10 $25-40a+16a^2$ by $5-4a$

11. $14x^2+5x-1$ by $7x-1$

12 $x^2-5xy+6y^2$ by $x-2y$

13 $4x^2-9$ by $2x-3$

14 $27x^3-1$ by $3x-1$

15 $4a^3-3a-1$ by $a-1$

16 $4x^3-2x^2+1$ by $2x+1$

17 $6x^3-5x^2+6x+8$ by $3x+2$

18 $m^3-6m^2+11m-6$ by $m-2$.

19 $24x^3-22x^2+17x-5$ by $12x-5$

20 x^3+a^3 by $x+a$

21 $2ax^2-7cx-8ax+28c$ by $2ax-7c$

22 $5ax^3-2apqx-5bx^2+2bpq$ by $5x^2-2pq$

23 $ax^3-5x+5ax^2-25$ by ax^2-5

24. $6a^3b^2-12b^4-ab^3$ by $3ab+4b^2$

25 $81-16x^4$ by $2x+3$

26 $ax^3+(2-a^2)x^2-ax+2$ by $ax+2$

27 $x^3-a(a+1)x+a^2$ by $x-a$

28 $a^2x^3-b(a^2+b)x+ab^2$ by $ax-b$.

29. $3x^4+6x^3-7x^2-8x+4$ by $3x^2-4$

30. $6x^4-2x^3-23x^2+5x+20$ by $2x^2-5$.

31 Divide the sum of $x(x-5)$ and $7(5-x)$ by $x-7$

32 Divide the product of $3x-9$ and $3x-12$ by $x-3$

33 Divide the product of $2y-8a$ and $2y-4a$ by $y-4a$

34 Subtract $3(x-4)$ from $10x(x-2)$, and divide the difference by $2x-3$

35 Simplify $(2x+6)(4x-20)-(x-5)(x+3)$

36 Simplify $\{(x+2y)(x-2y)-2y(2x-2y)+4y^2\}-(x-2y)$

Examples for Revision (B)

1 Add together $3(a^2-xy+c^2)$, $(a-x)(c-y)$ and $cx+ay-ac$

2 Subtract $ax-by$ from $(a+b)(x-y)$

3 Multiply x^2+ax-a by $x-a$

4 Divide $2x^2+5ax-3a^2$ by $2x-a$, and check the result by multiplication

5 Shew that $x(x^2+1)=4x^2-6$, if $x=-1$ or 2 or 3

6 If $x=a+2b+3c$, $y=b+2c+3a$, $z=c+2a+3b$, shew that

$$x+y+z=6(a+b+c)$$

7 Find the number, which is 5 times the number that exceeds x by 10.

---

8 Add together $(2x-1)(x-2y)$, $2(1-x)(x-y)$ and $x(2y-1)$

9 Subtract $(a-b)x-(c-d)y$ from $(a+b)x+(c-d)y$.

10 Simplify $(a-2)^2-(a-3)(a+3)+(a-1)(a+8)$

11 Multiply $3(x^2-8)-4x(x-5)$ by $5(x-1)+x-6$

12 Divide the difference of x^3+3x and $3x^2+1$ by $x-1$

13 If $A=b^2+c^2-a^2$, $B=c^2+a^2-b^2$, $C=a^2+b^2-c^2$, $D=a^2+b^2+c^2$, shew that $A+B+C+D=2(a^2+b^2+c^2)$

14 A's age is x years and B's age is 3 times of what A's age was 8 years ago; what is the age of B?

---

15 Explain the terms *dimension* and *degree* How is the degree of a term determined? State the degree of each of the terms —

$3a^4b^2c$, $5x^6$, $-a^3bcx^3$ and $-2abc$

16 Add $5(x^2+y^2)$ to the difference of $(x-3y)(2y-x)$ and $5x(y-x)$

17. Take $2(a+x)+10-14(x+y)$ from $5(a+x)-10-8(x+y)$

18 Without actual multiplication, find the coefficient of x^2 in the product of $4x^2-8x+3$ and $5x-3$

19 Divide $a^2x^3-(a^2+b^2)x-ab$ by $ax+b$

20 Shew that $\{(x^2-x-6)-(x-3)\}+\{(y^2-6y+8)-(y-4)\}=x+y$

21. There are $(a+b)$ mangoes in one heap, twice as many in another, and 5 times as many in a third; what is the total number of mangoes? If $(7a-b)$ mangoes be sold out of them, what number remains?

---

22. How is the degree of an expression ascertained? State the degree of the following expressions:—

(1) ax^3-bx^2+cx-d,

(2) $3x^5-2a^2x^4+4ax^3+1$,

(3) $a^4x^2+3a^3b^3x^2-2ab^2x+a^5$,

(1) when the symbol of reference is x, and (2) when it is a.

23. Add together $3\{x-4(y-3z)\}$ and $5y-2\{z-4(3x-y)\}$.

24. Take $3x-8$ from the sum of $(x+1)(x-3)$ and $(4-x)(x-2)$.

25. Simplify $\{x(4+a)-a(x-1)-3x\}\times(x-a)$.

26. The product of two expressions is $ax^3-(2a-3)x^2-7x+2$ and one of them is $x-2$; find the other.

27. If $a=x(1+y)$, $b=y(1+z)$, $c=xy+z(1+y)$,

shew that $a+b-c=x+y-z$.

28. A person leaves x rupees and a property of the same value, to be divided among his 4 sons; find the value of each share.

---

29. What is a *homogeneous* expression? State which of the following expressions are homogeneous and which not, and also their degrees:—(1) $2bc+ca-4ab$, (2) $3x^4-2x^2y+y^4$, (3) $ax^2+2hxy+by^2$.

30. Subtract $(x-1)(x-7)$ from $(x-2)(x-3)$, and add the result to $(x-1)(x-2)$.

31. Find the value of $(x^2-1)(x+1)^2$.

32. Divide $4(x+2)(x-2)+5(x-1)$ by $(2x-3)+2(x-2)$.

33. Express symbolically the quotient when the difference of x and y is divided by their product. Find also the quotient in a simple form.

34. If $A=ax+by$ and $B=bx-ay$, find the value of $(Aa+Bb)\div(a^2+b^2)$.

35. I paid x pies for each of a articles and y annas for each of 4 articles, and had then 2 rupees left; how many rupees had I with me at first?

---

36. Arrange $x^3-2x^2+3x^4-1+4x$ (1) according to the ascending and (2) according to the descending, powers of x.

37. What must be added to $2a^3-6a^2x+3ax^2-1$ in order that the sum may be a^3-ax^2-1?

38 Simplify $3x-4(2y-x)-2\{x-(4y-3x)\}$ and $2x-3(x-2y)-4\{y-2(3x-2y)\}$, and multiply the results together

39 If D represent the dividend, d the divisor and Q the quotient, express the relation between D, d and Q

40 Without actual multiplication, find the coefficient of x^2 in

$$(x^3-6x^2+3x-1)(2x+1)$$

41 Shew that $(a+bx)(b-ax)+(ax+b)(bx-a)+2x(a+b)(a-b)=0$

42 From a piece of wire $(x+2a)$ yards long, 3 pieces each $2(x-5a)$ feet long are cut off, how many yards are then left?

---

43 Simplify $a^2-3\{b^2-2c(a-3b)-2b(b+3c)\}$ and find its numerical value when $a=-2$, $b=3$, $c=-4$

44 Multiply $2x+6y$ by $2x-5y$, and divide the result by $x+3y$.

45 If $x=b-c$, $y=c-a$, $z=a-b$, find the values of

$$ax+by+cz \text{ and } (b+c)x+(c+a)y+(a+b)z$$

46. Simplify the product $(x-a)(x+b)$ and from the result deduce the value of $(x+1)^2$.

47 Shew that $(ax+b)(cx+d)-(a+bx)(c+dx)+(ac-bd)(1-x^2)=0$.

48. A person having $(4x^2-a^2)$ rupees spends $5a(x+a)$ rupees out of it, and then divides the remainder among $(x-2a)$ boys, what does each boy receive?

49. If x has the values -3, -1, 0, 2, 5, tabulate the values of $2x^2-3x+1$

---

50 Find the value of $x^2+y^2+2xy-1$, when $x=4.6$ and $y=5.4$

51. Subtract $2x^4-3x^3y+4x^2y^2$, from $2y^4-3y^3x+4y^2x^2$, and arrange the result according to the descending powers of x.

52 Multiply a^2-x^2 by $a+x$, and divide the product by $a-x$

53 Find the value of $2x^3-3x+1$, (1) when x is changed into $x+1$ and (2) when x is changed into $x-1$

54 Shew that $(ax+b)(cx+d)-(cx+b)(ax+d)+(a-c)(b-d)x=0$

55 If $A=4x-3a$ and $B=3x-4a$, find the value of

$$(A-2B)(2A-B)$$

56 If x and y are respectively the tens' and units' digits of a number, shew that the sum of the number and the number formed by reversing the digits is $11(x+y)$ Check the result by taking the number 57.

---

57 If $a=-1$, $b=-2$, $c=-3$, find the value of

$$\{a-(b-c)\}^2+\{b-(c-a)\}^3+\{c-(a-b)\}^2.$$

58 From $2x^2-5a^2$, subtract the product of $x+2a$ and $x-2a$, and multiply the difference by a^2+x^2

59 What are the coefficients of x, x^2 and y^3 in the expression
$$ax^3-4bx^2y+2cxy^2-y^3\,?$$

60 If $A=2x-3$ and $B=3x-2$, find $(A+B)^2$ and $(A-B)^2$

61 Simplify $(x-1)(x+3)+5(x-4)^2-2(x-2)(x+2)$

62 100 rupees was divided among A, B and C, so that A received $3x$ rupees, B $(2x+5)$ rupees and C $(80-4x)$ rupees, what is the value of x?

63 If $y=x^3+2x-1$, tabulate the values of y, for the values
$-4,\ -2,\ -1,\ 0,\ 3$ and 4 of x

---

64 Simplify $(x+a)^2-(x-a)(x+a)-(x-2a)^2$

65 Find the continued product of $y+3x$, $y-3x$ and $y+3x$

66 Divide the sum of $4-3x+10x^2$, $5x-9x^2+3$ and $2x^2-5-7x$ by $3x-2$, and find the value of the quotient when $x=-1$

67 If $X=2x-a$ and $Y=x-2a$, find the product of $X-Y$ and $X+2Y$

68 Divide x^3+y^3 by $x+y$ and from the result deduce the quotient when $(a+b)^3+c^3$ is divided by $a+b+c$

69 A man walks $(3p-q)$ miles an hour, in how many hours will he walk $3p^2+5pq-2q^2$ miles?

70 Tabulate the values of the expression $2x^3-1$, corresponding to the values $-3,\ -2,\ -1,\ 0,\ 2$ and 3 of x

CHAPTER VI

EASY FORMULÆ AND FACTORS

Formulæ

64 We shall further illustrate the Formulæ given in Art 53

$$\left.\begin{array}{ll}\textbf{Formula I} & (a+b)^2=a^2+2ab+b^2 \\ \textbf{Formula II} & (a-b)^2=a^2-2ab+b^2\end{array}\right\}$$

Ex 1 Expand $(a+b+c)^2$

Here are 3 terms, but enclosing $b+c$ in a bracket, we may consider these two terms as one Hence

$$\begin{aligned}(a+b+c)^2&=\{a+(b+c)\}^2\\ &=a^2+2a(b+c)+(b+c)^2\\ &=a^2+(2ab+2ac)+(b^2+2bc+c^2)\\ &=a^2+b^2+c^2+2bc+2ca+2ab\end{aligned}$$

Ex 2. Find the square of $a+b+c+d$

Here are 4 terms, but enclosing $a+b$ and $c+d$ in brackets, we may consider them as single terms. Hence

$$(a+b+c+d)^2=\{(a+b)+(c+d)\}^2$$
$$=(a+b)^2+2(a+b)(c+d)+(c+d)^2$$
$$=(a^2+2ab+b^2)+2(a+b)c+2(a+b)d+(c^2+2cd+d^2)$$
$$=(a^2+2ab+b^2)+(2ac+2bc)+(2ad+2bd)+(c^2+2cd+d^2)$$
$$=a^2+b^2+c^2+d^2+2ab+2ac+2ad+2bc+2bd+2cd$$

Note Hence generally *the square of a polynomial = the sum of the squares of each term + twice the product of every two of them.*

Ex 3 Multiply x^2+ax-b by $ax-b$

Required product $=(x^2+ax-b)(ax-b)$
$$=\{x^2+(ax-b)\}(ax-b)$$
$$=x^2(ax-b)+(ax-b)^2$$
$$=ax^3-bx^2+a^2x^2-2abx+b^2$$
$$=ax^3+(a^2-b)x^2-2abx+b^2.$$

Ex 4 Find the value of $25a^2+10ab+b^2$ when $a=-2$, $b=7$

Given expression $=(5a)^2+2(5a)b+b^2$
$$=(5a+b)^2=\{5(-2)+7\}^2$$
$$=(-10+7)^2=(-3)^2=9$$

Ex. 5 Simplify $(x-y+2z)^2-2(x+y-2z)(x-y+2z)+(x+y-2z)^2$

Let $a=x-y+2z$ and $b=x+y-2z$, thus given expression
$$=a^2-2ab+b^2=(a-b)^2$$
$$=\{(x-y+2z)-(x+y-2z)\}^2=(-2y+4z)^2$$
$$=4y^2-16yz+16z^2$$

Examples XXXIII

Expand

1 $(3x-2y-3z)^2$	2 $(2x+6y+z)^2$
3 $(3a-5b+4c)^2$	4 $(ax-by+cz)^2$
5 $(ab+xy-c^2)^2$	6 $(a^2+ab+b^2)^2$
7 $(1-2x+3x^2)^2$	8 $(a^2+b^2+c^2)^2$
9 $(a-b-c+d)^2$	10 $(4a+5b-c-2d)^2$

Find the value of

11 $(2x+3y+4z)(3y+4z)$ 12 $(3x^3-4ax^2-a^3)(3x^3-a^3)$

13 $(x^2+xy+y^2)y(x+y)$ 14 $(4x^2-5ax+3a^2)a(5x-3a)$

15 $4a^2+9b^2$, when $2a=p+1$, $3b=q+1$

16 $16x^2-25y^2$, when $4x=2p-1$, $5y=3q-1$

17 $25x^2-40xy+16y^2$, when $x=3$ and $y=10$

18 $36p^2+60pq+25q^2$, when $p=5$ and $q=-6$

19 $16m^2-48mn+36n^2$, when $m=-18$, $n=-12$

20 $9x^2+12xy+4y^2$, when $x=a-2$ and $y=a+3$

Simplify

21 $(x-y)^2+2(x-y)(x+y)+(x+y)^2$

22 $(x-1)^2-2(x+1)(x-1)+(x+1)^2$

23 $(2x-1)^2-2(2x-1)(x-1)+(x-1)^2$

24 $(x-2y+z)^2+2(x-2y+z)(x+2y-z)+(x+2y-z)^2$

25 $(2a+b-1)^2-2(2a+b-1)(2a-b+1)+(2a-b+1)^2$

65 The following are other applications of **Formula III** $(a+b)(a-b)=a^2-b^2$ [Art 54]

Ex 1 Multiply a^2+ab+b^2 by a^2-ab+b^2

Required product $=\{(a^2+b^2)+ab\}\{(a^2+b^2)-ab\}$

$$=(a^2+b^2)^2-(ab)^2$$
$$=a^4+2a^2b^2+b^4-a^2b^2$$
$$=a^4+a^2b^2+b^4$$

Ex 2 Find the continued product of $1-x$, $1+x$, $1+x^2$ and $1+x^4$

Required product $=(1-x)(1+x)(1+x^2)(1+x^4)$

$$=(1-x^2)(1+x^2)(1+x^4)$$
$$=(1-x^4)(1+x^4)=1-x^8$$

Examples XXXIV

Find the product of

1 $a+b+c$ and $a+b-c$ 2 $a+b-c$ and $a-b+c$

3 $a-b-c$ and $a+b-c$ 4 $a+b+c$ and $a-b+c$

5 $x+2y+3z$ and $x-2y-3z$ 6 $x-2y+3z$ and $x-2y-3z$

7 $x+2y-3z$ and $x-2y+3z$ 8 $a^2+2ab+b^2$ and $a^2-2ab+b^2$

9 x^2-ax+a^2 and x^2+ax+a^2 10 x^2+ab-b^2 and x^2+ab+b^2

11 $a-b-c+d$ and $a-b+c-d$ 12 $x+y-z+1$ and $x-y+z+1$

Find the continued product of

13 $a+x$, $a-x$ and a^2+x^2. 14 x^4+y^4, x^2+y^2, $x+y$ and $x-y$

15 x^2-x+1, x^2+x+1 and x^4-x^2+1

Simplify

16 $(a-2b)(a+2b)+(2b+3c)(2b-3c)+(3c-d)(3c+d)$

17 $(a-b+c)^2+2(a-b+c)(a+b-c)+(a+b-c)^2$

18 $(2a+b+c)^2-2(2a-b-c)(2a+b+c)+(2a-b-c)^2$

19 Find the value of $(1-2x)(1+2x)$, when $x=ab-1$

20 Find the value of x^2-xy+y^2, when $x=ab+cd$ and $y=ab-cd$.

66 Formula IV. $(a+b)^3=a^3+3a^2b+3ab^2+b^3$

$$=a^3+b^3+3ab(a+b)$$

$$\begin{aligned}(a+b)^3&=(a+b)^2(a+b)=(a^2+2ab+b^2)(a+b)\\&=(a^2+2ab+b^2)a+(a^2+2ab+b^2)b\\&=a^3+2a^2b+ab^2+a^2b+2ab^2+b^3\\&=a^3+3a^2b+3ab^2+b^3\end{aligned}$$

That is, *the cube of the* **sum** *of two quantities = the* **sum** *of their cubes* **plus** *three times their product into their* **sum**

Formula V $(a-b)^3=a^3-3a^2b+3ab^2-b^3$

$$=a^3-b^3-3ab(a-b)$$

That is, *the cube of the* **difference** *of two quantities = the* **difference** *of their cubes* **minus** *three times their product into their* **difference**

This formula may be obtained by direct multiplication as above, or by changing $+b$ into $-b$ in Formula IV

Ex 1 Multiply $p^2+4pq+4q^2$ by $p+2q$

Required product $=(p+2q)^2(p+2q)=(p+2q)^3$

$$\begin{aligned}&=p^3+3p^2(2q)+3p(2q)^2+(2q)^3\\&=p^3+6p^2q+12pq^2+8q^3\end{aligned}$$

Ex 2 Find the value of $(3x-4y)^3$

Required value $=(3x)^3-3(3x)^2(4y)+3(3x)(4y)^2-(4y)^3$

$$=27x^3-108x^2y+144xy^2-64y^3$$

Ex 3 Simplify $(a+b)^3+3(a+b)^2(a-b)+3(a-b)^2(a+b)+(a-b)^3$.

Let $m=a+b$ and $n=a-b$, thus the given expression

$$=m^3+3m^2n+3mn^2+n^3=(m+n)^3$$
$$=\{(a+b)+(a-b)\}^3=(2a)^3=8a^3$$

Ex 4 Find the value of $m^3-6m^2n+12mn^2+8n^3$, when $m=7$ and $n=2$

Given expn $=\{m^3-3m^2(2n)+3m(2n)^2-8n^3\}+16n^3$
$$=(m-2n)^3+16n^3=(7-4)^3+16(2^3)=3^3+128=155$$

Ex 5 If $x-y=3$, shew that $x^3-9xy-y^3=27$

Since $x-y=3$, we have $(x-y)^3=3^3$,

thus $x^3-3xy(x-y)-y^3=27$,

or $x^3-3xy\times3-y^3=27$,

i e, $x^3-9xy-y^3=27$

Examples XXXV

Multiply

1 x^2-4x+4 by $x-2$ 2 $4x^2+12xy+9y^2$ by $2x+3y$

3 $25m^2+20mn+4n^2$ by $5m+2n$ 4 $25x^2-30xy+9y^2$ by $5x-3y$

Find the value of

5 $(3a+2b)^3$ 6 $(x-1)^3$ 7 $(1+2x)^3$ 8 $(2x-3y)^3$ 9 $(2x-4)^3$

10 $(1-4x)^3$ 11 $(1+x^2)^3$ 12 $(2-x^2)^3$ 13 $(ax-y^2)^3$

Simplify

14 $(a+b)^3-3(a+b)^2(a-b)+3(a+b)(a-b)^2-(a-b)^3$

15 $(x+2)^3+3(x+2)^2(x-2)+3(x+2)(x-2)^2+(x-2)^3$

16 $(2x-1)^3-3(2x-1)^2(2x+3)+3(2x-1)(2x+3)^2-(2x+3)^3$

17 $(2a-b)^3+3(2a-b)^2(a+b)+3(2a-b)(a+b)^2+(a+b)^3$

18 $(x-4)^3+(2x+7)^3+9(x-4)(2x+7)(x+1)$

19 $(a+b+c)^3+6a\{a^2-(b+c)^2\}+(a-b-c)^3$

20 $(2a-b+3c)^3-(2a+b-3c)^3-6(3c-b)(2a-b+3c)(2a+b-3c)$

Find the value of

21 $8a^3+36a^2+54a+8$, when $a=-4$

22 $x^3-9x^2y+27xy^2-27y^3$, when $x=-3$ and $y=-1$

23 $8x^3-60x^2+150x-136$, when $x=-2$

24 $x^3+6x^2y+12xy^2-8y^3$, when $x=5$ and $y=-1$

25 $27k^3+108k^2+144k-72$, when $k=-2$

26 $2a^3-3ab(a-b)+b^3$, when $a=3$ and $b=5$

27 If $x+y=1$, shew that $x^3+3xy+y^3=1$

28 If $x-y=2$, shew that $x^3-6xy-y^3=8$

29 If $x+y=a$, shew that $x^3+3axy+y^3=a^3$

30 If $x-y=a$, shew that $x^3-3axy-y^3=a^3$

31 If $2x+3y=4$, find the value of $8x^3+72xy+27y^3$

32 If $ax-by=c$, find the value of $a^3x^3-3abcxy-b^3y^3$

33 If $x^2+y^2=1$, find the value of $x^6+3x^2y^2+y^6$

34 If $x^2-y^2=3$, shew that $x^6-9x^2y^2-y^6=27$

67 **Formula VI.** $(a+b)(a^2-ab+b^2)=a^3+b^3$ [Art 52, Ex 1]
That is, *the* **sum** *of two quantities into the sum of their squares diminished by their product = the* **sum** *of their cubes*

Formula VII. $(a-b)(a^2+ab+b^2)=a^3-b^3$ [Art 52, Ex 2]
That is, *the* **difference** *of two quantities into the sum of their squares increased by their product = the* **difference** *of their cubes*

REMARK. The student should notice that this Formula is obtained from Formula VI, by putting $-b$ for $+b$

Ex 1 Multiply $4x^2-6xy+9y^2$ by $2x+3y$

Since $4x^2=(2x)^2$, $6xy=(2x)(3y)$ and $9y^2=(3y)^2$, the required product is of the form $(a^2-ab+b^2)(a+b)$ Hence product required

$$=(2x)^3+(3y)^3=8x^3+27y^3$$

Ex 2 Multiply $25a^2+10ab+4b^2$ by $5a-2b$

Required product $=\{(5a)^2+(5a)(2b)+(2b)^2\}(5a-2b)$ which is of the form $(a^2+ab+b^2)(a-b)$,

$$=(5a)^3-(2b)^3=125a^3-8b^3$$

Examples XXXVI

Write down the value of

1 $(x^2-2x+4)(x+2)$

2 $(x^2+3x+9)(x-3)$

3 $(x^2-xy+y^2)(x+y)$

4 $(x^2+xy+y^2)(x-y)$

5 $(a^2+2ax+4x^2)(a-2x)$

6 $(a^2-2ax+4x^2)(a+2x)$

7 $(4a^2-6ab+9b^2)(2a+3b)$

8 $(4a^2+6ab+9b^2)(2a-3b)$

Write down the value of

9 $(a^2+a+1)(a-1)$

10 $(x^2-5x+25)(x+5)$

11 $(4a^2-2a+1)(2a+1)$

12 $(1+2x^2+4x^4)(1-2x^2)$

13 $(1-4x+16x^2)(4x+1)$

14 $(x^4-x^2+1)(1+x^2)$

15 $(4x^2+6x+9)(2x-3)$

16 $(9x^4+6x^2yz+4y^2z^2)(3x^2-2yz)$

17 $(16a^4-20a^2x+25x^2)(4a^2+5x)$

18 $(4m^4+18m^2n^2+81n^4)(2m^2-9n^2)$

Simplify

19 $(x^2-xy+y^2+1)(x+y)-(x^3+y^3)$

20 $(x^2+2xy+4y^2+x+2y)(x-2y)-(x^3-4y^2)$

21 $(a^2-ax+x^2)(a-x)+a^2(a+2x)$

68 The following examples [see Art 55] will further illustrate

Formula VIII $(x+a)(x+b)=x^2+(a+b)x+ab$ (i)

The following are two other forms of this formula

$$(x-a)(x-b)=x^2-(a+b)x+ab \quad \text{(ii)},$$

$$(x+a)(x-b)=x^2+(a-b)x-ab \quad \text{(iii)}$$

Ex Multiply x^2+2x-3 by x^2+2x+4

Put $a=x^2+2x$, thus required product

$$=(a-3)(a+4)=a^2+(4-3)a-12$$

$$=a^2+a-12=(x^2+2x)^2+(x^2+2x)-12$$

$$=x^4+4x^3+5x^2+2x-12$$

Examples XXXVII

Multiply

1 $x+a+1$ by $x+a+2$

2 $a+b-2$ by $a+b+5$

3 $x+y+1$ by $x+y-6$

4 $x+2y-1$ by $x+2y-3$

5 $ax-b+3$ by $ax-b-7$

6 x^2-xy+4 by x^2-xy+6

7 $2a-b-4$ by $2a-b-7$

8 $2x-3y+5$ by $2x-3y-6$

9 $3x^2-4x-6$ by $3x^2-4x-8$

10 $2a^2-ab-3$ by $2a^2-ab+8$

11 $4x-5y-9$ by $4x-5y+6$

12 $3x+5y-8$ by $3x-4y-8$

13 a^2-c^2+3ab by a^2-d^2+3ab

14 $x^2-3y^2-4z^2$ by $x^2-2y^2-4z^2$

Factors

69 Monomial Factors An expression* of the form $an+bn+cn+\ldots$, in which a factor is common to each term, is resolved *by taking out the common factor and enclosing the rest in a bracket*

Thus $an+bn+cn+\ldots=n(a+b+c+\ldots)$

Ex 1 Resolve $2ax+3xy$ into factors

Given expression $=x(2a+3y)$,

required factors are x and $2a+3y$

Ex 2 Resolve $xy+yz-y^2$ into factors

Given expression $=y(x+z-y)$, the factors are y and $x+z-y$

Ex 3 Resolve $6a^3x^2-9a^2x^2-15abxy$ into factors

Given expression $=3ax(2a^2x-3ax+5by)$, &c

Examples XXXVIII

Resolve into factors

1 $ab+ax$ 2 $m+mx$ 3 $2x-x^2$ 4 a^2-ac

5 m^2+2mx 6 $3r^3-15rs$ 7 a^2b+ab^2 8 $ab-bc+abc$

9 $8a^3-6ab-4ac$ 10 $20pq-15p^2r+55rp$ 11 $3a^3b-3ab^4$

12 $2a^2b^2+3ab$ 13 $12p^2q-3pq^3$ 14 $8x^2-10mx^5$

15 $18m^8-6m^5n$ 16 $3x^2y+xy+2xy^2$

17 $4a^3b-16a^2b-20ab^2$ 18 $2x^3y^3-3rx^2y+2rxy^2$

19 $6x^3y-9x^2y^2+12x^3y^3$ 20 $81x^6y^4+63x^4y^8$

21 $24m^4x^6z^3-42m^2x^3y^5$ 22 $7a^3xy^2+14a^2x^2y^2-21a^2xy^3$

70 When a *common factor* other than a monomial occurs in every term, the method is the same as above

Ex 1 Resolve $a(x+1)+2(x+1)$ into factors

Here $x+1$ is a common factor, put it $=m$, thus

given expression $=ma+2m=m(a+2)=(x+1)(a+2)$

* Here "expression" means a "*rational* and *integral* expression", that is, one in which the symbol of reference is free from the radical sign and does not occur in the denominator of any term

Ex. 2 Resolve $a^3x+ab^2x+a^2by+b^3y$ into factors

Given expression $=(a^3x+ab^2x)+(a^2by+b^3y)$

$$=ax(a^2+b^2)+by(a^2+b^2)=(a^2+b^2)(ax+by)$$

The same result also may be obtained by a *suitable rearrangement and grouping* of the terms [see Art 73], thus

given expression $=a^3x+a^2by+ab^2x+b^3y=(a^3x+a^2by)+(ab^2x+b^3y)$

$$=a^2(ax+by)+b^2(ax+by)=(ax+by)(a^2+b^2)$$

Ex 3 Resolve $4m^4n-6m^3n^2+4m^2n^3-6mn^4$ into factors

Given expression $=2mn(2m^3-3m^2n+2mn^2-3n^3)$,

and the expression within the bracket

$$=(2m^3-3m^2n)+(2mn^2-3n^3)$$
$$=m^2(2m-3n)+n^2(2m-3n)$$
$$=(2m-3n)(m^2+n^2)$$

given expression $=2mn(2m-3n)(m^2+n^2)$

Examples XXXIX

Resolve into factors

1 $ab+bc+cd+ad$

2 $m^2+m+n+mn$

3 $ax+a+x+1$

4 $2ax-4ay-bx+2by$

5 $1-x-y+xy$

6 $a^2-ab-ac+bc$

7 $x^2-mx-my+xy$

8 a^3+a^2+a+1

9 $1-x+x^2-x^3$

10 $f^2g-cfh+fgh-ch^2$

11 $ab+b^2c+a^2c+abc^2$

12 $acx^2-bcx-adx+bd$

13 $abq-bcq^2+acq-a^2$

14 $xyz^2-x^3z-y^3z+x^2y^2$

15 $(a+b)x-(a+b)y+(a+b)^2$

16 $5(p-q)mp+10(p-q)mq-25(p-q)m^2$

17 $xyz+bxz+cxy+bcx$

18 $a^2b+3ab-4a^2-12a$

19 $m^3xy-m^2y^2-m^2x^2+mxy$

20 $8m^2n^2-24m^2np-16mn^3+48mn^2p$

21 $4a^3gh-2a^2cfg-8a^2bfh+4abcf^2$

74 Factors of Expressions of the form a^2-b^2 From Art 54, we have the identity

$$a^2-b^2=(a+b)(a-b)$$

Thus an expression which is the **difference of two squares** can be separated into factors

Ex 1 Resolve $4x^2-9y^2$ into factors

$$4x^2-9y^2=(2x)^2-(3y)^2=(2x+3y)(2x-3y)$$

$\therefore$ required factors are $2x+3y$ and $2x-3y$

Ex 2 Resolve $9(a+b)^2-4c^2$ into factors

$$\text{Given expression}=\{3(a+b)\}^2-(2c)^2$$
$$=(3a+3b+2c)(3a+3b-2c)$$

Ex 3 Find the value of $571\times571-429\times429$

$$\text{Reqd value}=(571)^2-(429)^2=(571+429)(571-429)$$
$$=1000\times142=142000$$

Ex 4 Find the square of 839

Now $$(839)^2=(839)^2-(39)^2+(39)^2$$
$$=(839+39)(839-39)+(39)^2$$
$$=878\times800+(39)^2=702400+(39)^2$$

and $$(39)^2=(39)^2-1+1=(39+1)(39-1)+1$$
$$=40\times38+1=1521$$

$\therefore$ required square $=702400+1521=703921$

Ex 5 Find the square of 875

Now $$(875)^2=(875)^2-(25)^2+(25)^2$$
$$=(875+25)(875-25)+625$$
$$=900\times850+625=765625.$$

Examples XL

Resolve into factors

1	$1-x^2$	2	m^2-16	3	$64-q^2$
4	$1-81z^2$	5	$25y^2-1$	6	$16a^2-9b^2$
7	$25a^2x^2-49b^2$	8	$144x^2-169y^2$	9	$81q^2-64r^2$
10	$625a^2x^4-121$	11	$81p^2-100q^2$	12	$144x^2-121y^2$
13	$49a^2c^2-81d^4$	14	$l^2m^2-n^2q^2$.	15	$9z^4-16x^2y^2$
16	$25a^2x^2-16c^2y^2$	17	a^4-x^4	18	$16x^4-25a^6$
19	$4a^4-b^4$	20	$36a^6-x^4$	21	$25a^8-9b^{10}$
22	m^4-16	23	$16a^4-1$	24	$1-x^8$
25	a^8-x^8	26	$x^{16}-a^{16}$	27	x^3-4xy^2
28	$50a^3-2a$	29	$3-48a^2x^2$	30	$32m^2n^2-2$

Resolve into factors

31	$3a^3-3ax^2$	32	$20a^3x-5ab^2x$	33	$50x^3-32xy^2$
34	$3x-12x^3$	35	$50a^4b^2x^2-8a^2x^4y^2$	36	$81x^5-16xy^4$
37	$3x^3-\frac{1}{3}xy^2$	38	$2a^5-\frac{1}{8}ax^4$	39	$162a^6b^2-32a^2b^6$
40	$(x-y)^2-z^2$	41	$m^2-(2n+q)^2$	42	$9y^2-4(2b-3x)^2$
43	$4(3x-4y)^2-1$	44	$4a^2-9(b-3c)^2$	45	$16-9(5z+6a)^2$
46	$25p^2-4(3q-1)^2$	47	$(5x-2y)^2-9z^2$	48	$4(lx-my)^2-81x^2y^2$

49	$(5r-6s)^2-25t^2$	50	$(a+b)^2-(c+d)^2$
51	$(3x-2y)^2-(1+2z)^2$	52	$(3x+8)^2-(3x-8)^2$
53	$(3a-5b)^2-(3a+5b)^2$	54	$(7a-6b)^2-(6a-7b)^2$
55	$(3ax+by)^2-(3by-ax)^2$	56	$(a^2-ax+x^2)^2-(a^2+ax+x^2)^2$
57	$(2a+b-3c)^2-(a-3b+2c)^2$	58	$(a-2b-4c)^2-(3a+2b+6c)^2$
59	$(1+xy)^2-(x+y)^2$	60	$(x^2+y^2)^2-x^2(x-y)^2$
61	$(mx+y)^2-(x+my)^2$	62	$(pr-qs)^2-(ps-qr)^2$
63	$(x+a)^4-(x-a)^4$		

Find the value of

64	$(354)^2-(254)^2$	65	$(879)^2-(121)^2$	66	$(487)^2-(394)^2$
67	$(9728)^2-(9727)^2$	68	$(5218)^2-(4882)^2$	69	$(64)^2$
70	$(85)^2$	71	$(145)^2$	72	$(193)^2$
73	$(416)^2$	74	$(989)^2$	75	$(9999)^2$

76 Shew that $(x^2+2xy+3y^2)^2-(x^2-2xy+3y^2)^2=8xy(x^2+3y^2)$

77 Shew that $(a^2+ab-b^2)^2-(a^2-ab+b^2)^2=4a^2b(a-b)$

78 Shew that $(a^2+ab)^2-(ab+b^2)^2=(a+b)^3(a-b)$

72 Factors of Expressions of the form a^3+b^3 or a^3-b^3

From the Formulæ of Art 67, we have

$$a^3+b^3=(a+b)(a^2-ab+b^2),$$

and
$$a^3-b^3=(a-b)(a^2+ab+b^2)$$

Hence any expression which is the sum or difference of two cubes can be easily resolved into factors

Ex 1 $8x^3+y^3=(2x)^3+y^3$

$$=(2x+y)\{(2x)^2-(2x)y+y^2\}$$
$$=(2x+y)(4x^2-2xy+y^2)$$

Ex 2 $27x^3-1=(3x)^3-1^3$

$$=(3x-1)\{(3x)^2+(3x)\,1+1^2\}$$
$$=(3x-1)(9x^2+3x+1)$$

Note With a little practice the student may omit the second step and write down the work thus —

$$27x^3-1=(3x)^3-1=(3x-1)(9x^2+3x+1)$$

Ex. 3 $3a^5x+24a^2x^4=3a^2x(a^3+8x^3)$

$$=3a^2x(a+2x)\{a^2-a(2x)+(2x)^2\}$$
$$=3a^2x(a+2x)(a^2-2ax+4x^2)$$

Ex 4 $8(a+b)^3+1 \;=8x^3+1$, if $x=a+b$,

$$=(2x+1)(4x^2-2x+1)$$
$$=\{2(a+b)+1\}\{4(a+b)^2-2(a+b)+1\}$$
$$=(2a+2b+1)(4a^2+8ab+4b^2-2a-2b+1)$$

Ex 5 $(a^2+bc)^3-8b^3c^3=(a^2+bc)^3-(2bc)^3$

$$=x^3-y^3,\text{ if } x=a^2+bc,\ y=2bc,$$
$$=(x-y)(x^2+xy+y^2)$$
$$=(a^2-bc)\{(a^2+bc)^2+(a^2+bc)(2bc)+4b^2c^2\}$$
$$=(a^2-bc)(a^4+4a^2bc+7b^2c^2)$$

Examples XLI

Resolve into factors

1.	a^3+64	2	x^3-8y^3	3	$27a^3-b^3$
4	$27x^3+a^3$	5	$8a^3+1$	6	$64k^3-1$
7.	$1+27k^3$	8	$512x^3-27$	9	$125a^6+27x^3$
10	$54m^3-2$	11	x^3+64y^3	12	$a^3x^3-64y^3$
13	$64a^3+125b^3$	14	$8a^3c^3-b^3$	15	$343x^3+8$
16	x^6+y^3	17	$216a^3-8x^6$	18	$729+216c^3$
19	$x^3+\frac{1}{8}y^3$	20	$\frac{1}{4}a^3-54x^6$	21	x^4y+8xy^4
22	$192p^5q-375p^2q^4$	23	$x^3-8(y+z)^3$	24.	$64(a+b)^3+27c^3$.
25	$(a+b)^3x^3-8c^3y^3$	26	$(x-2y)^3+y^3$	27.	$(a^2-bc)^3+8b^3c^3$
28	$x^6+(x^2-2a^2)^3$			29	$8x^3-(x^2+1)^3$

Write down the following quotients

30 $(27x^3+1000)\div(3x+10)$ 31 $(1-64c^3)\div(1+4c+16c^2)$

32 $(729a^3+125b^3)\div(81a^2-45ab+25b^2)$

73 Factors found by suitable rearrangement and grouping of terms

Ex Factorize $x^3+ax^2-a^2x-a^3$

(i) Take the terms as they are, thus

$$x^3+ax^2-a^2x-a^3=(x^3+ax^2)-(a^2x+a^3)$$
$$=x^2(x+a)-a^2(x+a)$$
$$=(x+a)(x^2-a^2)=(x+a)(x+a)(x-a)$$

(ii) Arrange the given expression as follows, thus we have

$$x^3-a^2x+ax^2-a^3=(x^3-a^2x)+(ax^2-a^3)$$
$$=x(x^2-a^2)+a(x^2-a^2)$$
$$=(x^2-a^2)(x+a)=(x+a)(x-a)(x+a)$$

(iii) Arrange the terms as below, and we have

$$x^3-a^3+ax^2-a^2x=(x^3-a^3)+(ax^2-a^2x)$$
$$=(x-a)(x^2+ax+a^2)+ax(x-a)$$
$$=(x-a)(x^2+2ax+a^2)$$
$$=(x-a)(x+a)^2=(x-a)(x+a)(x+a)$$

Examples XLII

Factorize (arranging where possible in more than one way)

1 $1+ab-a^2+a^3b$

2 $a^2-a^3b-b^3+ab^2$

3 a^4+a-a^3-1

4 x^4-x^3-x+1

5 x^3-ax^2-x+a

6 $a^2-a^3x-x^2+ax^3$

7 $1+2x-x^2-2x^3$

8 $a^2-a^2b-ab^2-b^2$

9 $a^2x^2-b^2x^2-a^2+b^2$

10 $x(x+a)-y(y+a)$

11 $x^2(x+a)-y^2(y+a)$

12 $1+a+a^2x-a^2x^2$

13 $a^3-4ax^2+a^2x-4x^3$

14. $a^5-a^3-a^2+1$

15 $a^2x^3+a^2-x^3-1$

16 ax^3-a-x^3+1

17 $a^2x^2-a^2-x^2+1$

18 $a^2x+a-x-1$

19. $1+cx+2cx^3-4x^4$

20 $a^4+a^3b+ab^3+b^4$

21 $a^4+a^3x-ax^3-x^4$

22 $a^5-a^3b^2+a^2b^3-b^5$.

74 Factors of Quadratics of the form x^2+px+q, found by inspection The general form of quadratic expressions in which the coefficient of x^2 is *unity*, is x^2+px+q, where p or q, or both may be positive or negative. Thus *a quadratic in its general form*

consists of 3 terms only, viz., the first term which contains x^2, the second term which contains x, and the third or *last* term which does not contain x at all. By comparing the identities of Art. 55, *viz.*,

$$x^2+(a+b)x+ab=(x+a)(x+b) \quad \ldots (1),$$

$$x^2-(a+b)x+ab=(x-a)(x-b) \quad \ldots (2),$$

and

$$x^2+(a-b)x-ab=(x+a)(x-b) \quad \ldots (3),$$

with x^2+px+q, we see that (1) corresponds to the form x^2+px+q, (2) to x^2-px+q, and (3) to x^2+px-q or x^2-px-q according as a is greater or less than b. Hence it follows from (1) and (2), that when q (the last term) is *positive*, a and b (the second terms of the required factors) *have both the same sign*, and *that sign is the sign of* p (the coefficient of x), and from (3), that when q is *negative*, a and b have *opposite signs and the greater of the two has the sign of* p.

Moreover we see that p is the algebraic sum of a and b, and q their product. Hence in resolving expressions of the form x^2+px+q, we have only to see whether q is the *product of two quantities such that their algebraical sum* is p.

Ex. 1. Resolve x^2+5x+6 into factors.

Here the two factors of 6 are positive for the coefficient of x is positive and their sum is 5. Now pairs of numbers, of which the product is 6, are (i) 1 and 6, (ii) 2 and 3. We reject the first pair and take the second for $2+3=5$. Hence the second terms (*i.e.*, the a and the b) of the required factors are $+2$ and $+3$,

$$\therefore x^2+5x+6=(x+2)(x+3).$$

Ex. 2. Resolve $x^2-15x+36$ into factors.

Of the pairs of numbers whose product is 36, we take 3 and 12 for their sum is 15, and as 36 is *positive*, 3 and 12 must each have the *same sign*, which is here $-$, as the coefficient of x has the $-$ sign. Thus the second terms of the required factors are -3 and -12,

$$\therefore x^2-15x+36=(x-3)(x-12).$$

Ex. 3. Resolve $x^2+8x-48$ into factors.

The two factors of 48 whose algebraic sum (or difference) is 8, are 4 and 12; and 48 being *negative*, 4 and 12 must have *opposite* signs and 12 the greater must be positive as the co-efficient of x is positive. Hence the second terms of the required factors are $+12$ and -4.

$$\therefore x^2+8x-48=(x+12)(x-4).$$

Ex. 4. Resolve $x^2-13x-48$ into factors.

Here we take the factors 3 and 16, for their difference is 13, and 48 being *negative*, 3 and 16 must have *opposite* signs, and 16 must be

negative as the co efficient of x is *negative*. Thus the second terms of the required factors are $+3$ and -16

$$x^2-13x-48=(x+3)(x-16)$$

Ex 5 Resolve $x^2-(a+b)x+(a+1)(b-1)$ into factors

Given expression $=x^2-\{(a+1)+(b-1)\}x+(a+1)(b-1)$

$$=\{x-(a+1)\}\{x-(b-1)\}=(x-a-1)(x-b+1)$$

Note It is easy to see that if $x+a$ and $x+b$ be the factors of

$$x^2+px+q \qquad \text{(i)},$$

those of

$$x^2+pxy+qy^2 \qquad \text{(ii)}$$

will be $x+ay$ and $x+by$, that is, *the factors of* (ii) *can be written down from those of* (i) *by writing* ay *for* a *and* by *for* b

Thus from Ex 1, $x^2+5xy+6y^2=(x+2y)(x+3y)$,

from Ex 4, $x^2-13xy-48y^2=(x+3y)(x-16y)$, and so on

Examples XLIII.

Resolve into factors

1	$x^2+7x+12$	2	$a^2+7a+10$	3	$x^2+10x+21$
4	$p^2+24p+80$	5	$z^2+17z+42$	6	$x^2+9x+20$
7	a^2-4a+3	8	$a^2-9a+20$	9	x^2-6x+8
10	$a^2-7a+12$	11	x^2-5x+6	12	$x^2-27x+170$
13	$x^2+4x-32$	14	$a^2-4a-21$	15	a^2+a-72
16	$x^2-4x-32$	17	$x^2+6x-40$	18	$x^2+19x-42$
19	x^2+x-42	20	$x^2-3x-54$	21	$x^2-15x-54$
22	$m^2+4m-96$	23	$m^2-29m-96$	24	$m^2-20m-96$
25	$a^2+2a-120$	26	$a^2-19a-120$	27	$p^2+7p-144$
28	$x^2+8x-48$	29	$x^2+2x-48$	30	$x^2-13x-48$
31	$l^2+19l-20$	32	$l^2-23l-78$	33	$l^2-7l-78$
34.	$x^2+x-156$	35	$x^2-20x-156$	36	$a^2b^2+15ab+36$
37	$a^2b^2-20ab+36$	38	$x^2y^2-3xy-18$	39	$p^2q^2+14pq-32$
40	$m^2n^2-6mn-16$	41	$a^2x^2+5ax-36$	42	$a^2-3ab-54b^2$
43	$x^2-15xy-100y^2$	44	$x^2+2xy-24y^2$	45	$x^2-10xy-24y^2$
46	$a^2+4ab-60b^2$	47	$l^2+18lm+45m^2$	48	$a^2+17ab-60b^2$
49	$m^2+8mn-128n^2$	50	$m^2-28mn-128n^2$	51	$p^2+pq-380q^2$
52	$36+24x-5x^2$	53	$64-16a-195a^2$	54	$1+2m-24m^2$

Resolve into factors

55 $l^2-l-420$ 56 $x^2+x-650$ 57 $q^2+105q+2000$

58 $1+3x-18x^2$ 59 $1-7x-18x^2$ 60 $9+12x-32x^2$

61 $9-42x-32x^2$ 62 $9-48x+28x^2$ 63 $25-90m+72m^2$.

64 $25+30x-16x^2$ 65 $25-75x-16x^2$ 66 $m^2-5m-300$

67 $m^2-5mn-50n^2$ 68 $x^2+2ax-80a^2$ 69 $1+3xy-4x^2y^2$

70 $1+55xy+750x^2y^2$ 71 $l^2+4l-192$ 72 $x^2-6xy-40y^2$

73 $a^2+9a-486$ 74 $x^2-2x-360$ 75 $x^2+26x-560$

76 $24x-x^2-128$ 77. $72+6x-x^2$ 78 $80-16x-x^2$

79. $x^2+\frac{4}{3}x-\frac{5}{9}$ 80 $x^2-\frac{10}{3}x+1$ 81 $x^2-\frac{1}{2}x-5$

82. $a^2+(z+2)a+2z$ 83 $x^2+(1+a)xy+ay^2$

84 $y^2+(2x-1)y+x(x-1)$ 85 $x^2-(2a+1)x+a^2+a-6$

86 $x^2-2(y-2)x+(y-1)(y-3)$ 87 $x^2+(y+1)x-(y-2)(2y-1)$

88 $x^2-4abx-(a^2-b^2)^2$ 89 $x^2+2ax+(a^2-1)$

75 Factors of Quadratics of the form ax^2+bx+c, found by inspection An expression of this form can be resolved as in the last article by reducing it to the form x^2+px+q, *which is done by multiplying it by the coefficient of* x^2

Ex 1 Resolve $6x^2+23x+20$ into factors

Multiply the given expression by 6, thus

$$
\begin{aligned}
6\times\text{given expn} &= 6\times 6x^2+6\times 23x+6\times 20 \\
&= (6x)^2+23(6x)+120 \\
&= X^2+23X+120, \text{ where } X=6x, \\
&= (X+15)(X+8) \\
&= (6x+15)(6x+8), \text{ replacing } X \text{ by } 6x, \quad . \ (A) \\
\text{given expn} &= (6x+15)(6x+8)\div 6 \ldots \quad . \quad . \quad . \ (B) \\
&= (2x+5)(3x+4)
\end{aligned}
$$

Note We see from (A) that 15 and 8 are the factors of 120 [$=6$, (coefficient of x^2)$\times 20$ (last term)], that 23 (coefficient of x)$=15+8$, and that the factors obtained are $(6x+15)(6x+8)$, the first term in each of which is $6x$ Also from (B) we see that the *required factors* are these factors divided by 6, the coefficient of x^2.

Hence the Rule —*Multiply the last term by the coefficient of x^2, resolve this product into two factors, whose algebraic sum is equal to the coefficient of x*, then divide the product of the factors thus found by the coefficient of x^2 to obtain the required factors

Ex 2 Resolve $3x^2-14x+8$ into factors

We may proceed as in Ex. 1, or according to the above rule thus, $3\times 8=24$, which is *positive*, and the two factors, which make up 24 and whose *sum* is 14, are 12 and 2,

$$3x^2-14x+8=(3x-12)(3x-2)\div 3=(x-4)(3x-2)$$

Ex 3 Resolve $2x^2+x-6$ into factors

Proceed as in Ex 1 or thus —

$2\times(-6)=-12$, which is *negative*, and the two factors, which make up 12 and whose *difference* is 1, are 4 and 3,

$$2x^2+x-6=(2x+4)(2x-3)\div 2=(x+2)(2x-3)$$

Examples XLIV.

Resolve into component factors

1 $4x^2+11x+6$. 2 $5a^2+19a-4$ 3 $6x^2+x-12$

4 $6x^2-35x+36$ 5 $3x^2-10x-25$ 6 $4x^2+23x-72$

7 $8y^2-6y-35$ 8 $15x^2+4x-96$ 9 $12x^2-17x-5$

10 $10x^2+3x-4$ 11 $9m^2+9m-28$ 12 $8x^2+61x-24$

13 $15x^2-26x-21$ 14 $8x^2+14x-15$ 15. $12x^2-17x-40$

16 $8x^2+10x-7$ 17 $6x^2+11x-10$ 18 $4x^2-13xy-12y^2$.

19 $10-3x-27x^2$ 20 $12x^2-x-6$ 21 $12x^2-32x+5$

22 $4x^2-5x-21$ 23 $24p^2-62pq+35q^2$.

24 $3-13x-10x^2$ 25 $9x^2y^2+52xy-12$.

26 $8x^2+29ax-12a^2$ 27 $6-11x-10x^2$

28 $10a^2-31ab+15b^2$ 29 $12x^2+31xy+20y^2$

30 $12a^2-28ab-5b^2$ 31. $18m^2+9mn-35n^2$

32 $12x^2-41ax+24a^2$ 33 $42x^2-41x-20$

34 $56y^2+99yz-45z^2$ 35 $24x^2-37xy-72y^2$

CHAPTER VII

THE USE OF SQUARED PAPER

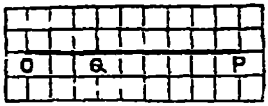
76 We have already seen how Theorems in Algebra may be graphically illustrated by means of "squared paper" In the present Chapter we shall give some more examples of its various uses The beginner is recommended to use the "tenth-inch"

squared paper (i e, paper ruled into squares, the side of each of which is $\frac{1}{10}$th of an inch) as we have done

77 To prove graphically that $a+(-b)=+(a-b)$, where a and b are both positive and $a>b$

Let a side of square represent the unit.

[We shall generally employ this unit]



Since a is positive, let $OP(=a$ units) be drawn to the right, and because b is negative, let $PQ(=b$ units) be drawn from P to the left

Thus $$OQ=a+(-b)$$

And b being $<a$, Q is to the right of O, and therefore OQ is positive

Also $$OQ=OP-PQ=a-b$$

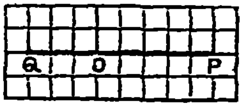
$$a+(-b)=+(a-b)$$

The reasoning will clearly be the same whether a or b or both, are fractions

78 To prove graphically that $a+(-b)=-(b-a)$, where $a<b$

Take the same unit as in Art 77

Draw $OP(=a$ units) to the right



Then because $b>a$, Q falls to the left of O, and therefore OQ is negative and is equal to $b-a$

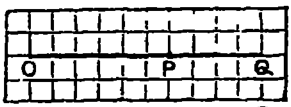
But $$OQ=a+(-b)$$

$$a+(-b)=-(b-a),$$

whether a and b are integers or fractions

79 To prove graphically that $a-(-b)=a+b$

Draw $OP(=a$ units) to the right



Now to add $-b$, we take b units to the left from P, therefore to subtract $-b$, we must take b units in the contrary direction Thus we draw $PQ(=b$ units) to the right from P

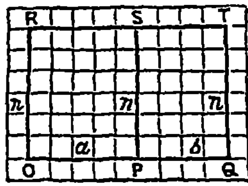
Hence $$OQ=a-(-b)$$

Also $$OQ=OP+PQ=a+b$$

$$a-(-b)=a+b$$

80 To prove graphically that $n(a+b)=na+nb$

Let $OP=a$ units, and $PQ=b$ units



Draw $OR(=n$ units) at right angles to OQ, and complete the rects OT, OS and PT

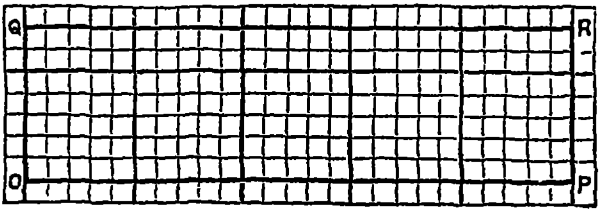
Then rect $OT=n(a+b)$

Also rect $OT=$rect $OS+$rect PT

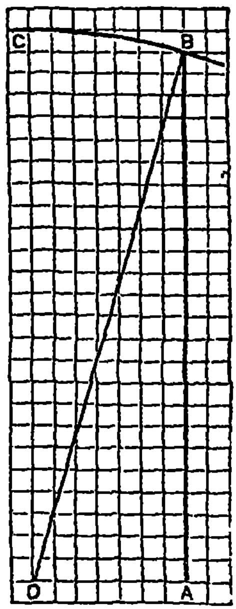
$=na+nb$

$n(a+b)=na+nb$

81 To multiply graphically 2·5 by 7



Take 10 sides of a square for unit, so that the unit of area is 100 times the area of a square



Thus $OP=2·5$ units of length and $OQ=7$ units of length

Now $OP\cdot OQ=$area of rect $OPRQ=175$ squares (by counting)

$=175$ hundredths$=1·75$ units of area

$2·5\times 7=1·75$

82 Ex 1 A rope 25 ft long is attached to the top of a tower, and when stretched tight just reaches the ground at a point 7 ft from the foot of the tower Find the height of the tower

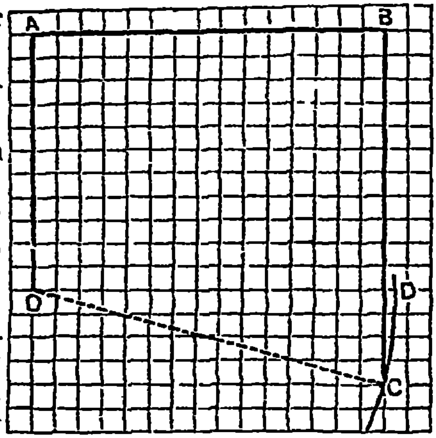
Take a side of a square to represent 1 foot

Let A be the foot of the tower, and O the point where the rope touches the ground

With centre O and radius OC (which is 25 ft), draw a circle to cut the vertical through A at B Then B is the top of the tower

From the diagram we see that AB is 24 ft

Ex 2 A ship sails 11 miles due north, then 15 miles due east and finally 15 miles due south How far is she now from her starting port?



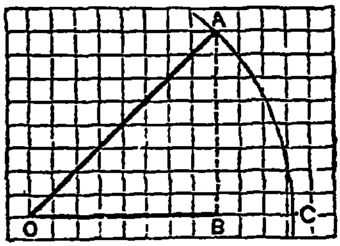
Take a side of a square to represent 1 mile

Let O be the port from which the ship starts

Now 11 miles north will bring her to A, then 15 miles east to B, and lastly 15 miles south to C

With centre O and radius OC, draw a circle, cutting the horizontal line through O at D Thus OD is the required distance, and from the diagram, it is seen to be $15\frac{1}{2}$ miles nearly

Ex. 3 A boy cycling a certain distance due north-east finds that he is 8 miles due east of his starting point, what distance has he cycled?

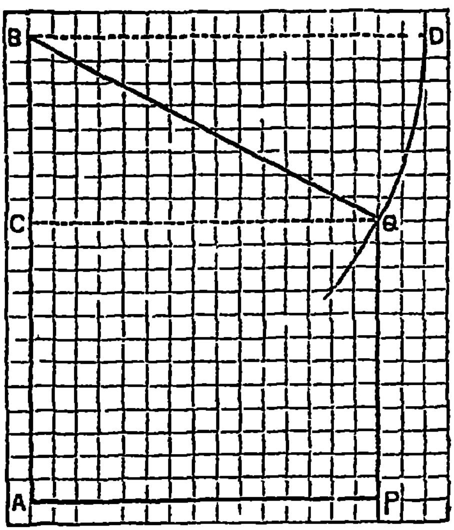


Take a side of a square to represent 1 mile Let $OB=8$ miles, and let the vertical through B cut OA, the line of the boy's path at A Then OA is the required distance

Let the circle with centre O and radius OA cut OB produced at C

Thus $OA=OC=11\frac{1}{3}$ miles nearly

Ex 4 The distance between the tops of two upright posts, 20 ft. and 12 ft in height, is 17 ft Find the distance between them



Take a side of a square to represent 1 foot.

Let AB be a post 20 ft in height

Now if $BC=8$ ft., then $AC=12$ ft, so that the top of the other post must be on a horizontal through C

With centre B and radius $BD(=17$ ft) draw a circle cutting the horizontal through C at Q Thus PQ, the vertical through Q, which is 12 ft, represents the other post

From the diagram we see that the required distance $AP=15$ ft

Examples XLV

[*Work the following examples graphically by using squared paper*]

1 Prove that $(-a)+(-b)=-(a+b)$

2 Prove that $-a-(-b)=-a+b$, (i) when $a>b$, and (ii) when $a<b$

3 What must be added to 12 to make -16?

4 What must be taken from -8 to give -14?

5 What must be added to -13 to give 15?

6 What must be subtracted from 10 to make 18?

7 ,, ,, 17 ,, -9?

8 ,, ,, -5 ,, 20?

9 A is 11 years old, 4 years ago B was twice as old as A was 3 years back, what is B's age?

10 A boy has 18 rupees with him, he spends half his money on a book and one-third on a pair of shoes, how much has he left?

11 Prove that $(a+b)+(a-b)=2a$

12 Prove that $(a+b)-(a-b)=2b$

13 Prove that $a(a+b)=a^2+ab$

14 Prove that $n(a-b)=na-nb$

15 Prove that $(x+3)^2=x^2+6x+9$

16 Prove that $(x-4)^2=x^2-8x+16$

17 Prove that $(x+2)(x-2)=x^2-4$

18 Prove that $(x+3)(x-4)=x^2-x-12$

19 Multiply (i) 2 3 by 8, (ii) 3 4 by 4 2, (iii) 5 6 by 3 5

20 The sides of a right-angled triangle are 3 1 ft and 4 6 ft, find the hypotenuse to the nearest half-foot

21 A room is 18 ft long and 15 ft wide, find approximately the distance between two opposite corners

22 On a base of 4 inches, construct a triangle whose sides shall be $4\frac{1}{2}$ and 5 inches respectively Find the altitude of the triangle to the nearest tenth of an inch

23 Find to the nearest inch the altitude of an equilateral triangle whose side is 2 5 inches

24 A ladder 24 ft long rests against a vertical wall, if its foot is at a distance of 8 ft from the wall, how far up the wall does the ladder reach?

25 A ladder with its foot at a distance of 25 ft from the wall of a house, just reaches a window of the house, 34 ft high, find to the nearest foot the length of the ladder

26 The distance between two vertical posts, 30 ft. and 21 ft high, is 12 ft., find the distance between their tops

27 A man goes 15 yards west, then 20 yards north and lastly 35 yards east, find approximately how far he is from the starting point

28 A ship sails 17 miles south, then 21 miles east, then 35 miles north, then again 31 miles west, next 33 miles south, and lastly 13 miles east, find to the nearest mile her distance from the starting point

29 A cyclist runs 12 miles south, then 18 miles west, and then again 32 miles north, what is his approximate distance from the place of starting?

30 A and B start at the same time from the same place, A goes due east at 15 miles an hour, and B goes due north-east at 18 miles an hour, what is the distance between them at the end of two hours?

31 A horse is tethered to a post by a rope 34 ft long, if the shortest distance of a straight hedge from the post is 16 ft, over what length of the hedge can he graze?

CHAPTER VIII

SIMPLE EQUATIONS IN ONE VARIABLE

83 Identity and Equation A statement of equality of two algebraical expressions is called an **equation**

Thus $2(x-1)=2x-2$ and $x-1=3$ are equations

In an equation, the two expressions connected by the sign of equality are called **sides** or **members** of the equation, the expression on the left being called the *left side* and that on the right being called the *right side*

When two algebraical expressions are equal for all values of the letters involved in them, the equation is termed an **identical equation**, or briefly an **identity**

Thus $10+5=6\times2+3$, $2(x-1)=2x-2$, &c, are identities

From these examples, it is seen that the right side of an identity is the same as the left side *put in a different form*

When two algebraical expressions are equal for some particular value or values of one or more letters involved, the equation is termed an **equation of condition** or briefly an **equation**

Thus $x-1=3$ is an equation of condition, for the value 4 only of x makes $x-1$ equal to 3, so is $x^2-5x+6=0$, for the values 2 and 3 only, and no other values, of x make x^2-5x+6 equal to 0, and so on

Ordinarily an equation of condition will be termed an "equation"

84 The sign, $\equiv$, is sometimes used to denote an identity, thus $(a+1)x \equiv ax+x$ We shall however continue to use the sign $=$ to denote an identity

85 The symbol, whose value we want to find, is called the *Unknown Quantity*, or briefly the **variable** The last letters of the alphabet, x, y, z, u, &c, are usually employed to denote variables *

An equation is said to be **satisfied** by a value of the variable when that value makes the *two sides equal* Thus 4 *satisfies* the equation $2x+1=13-x$, because 4 standing for x, makes $2x+1=9$, also $13-x=9$

The value of the variable which *satisfies* an equation is called its **root or solution**

To **solve** an equation is to find its *root* or *roots*

86 **Simple Equation** When the *first* power only, and no higher power, of the variable occurs in an equation, it is said to be a SIMPLE EQUATION It is also called an Equation of the FIRST DEGREE or a LINEAR EQUATION * Thus $ax+b=0$, $ax+by=c$, are Simple or Linear Equations, in one and two variables respectively

87 **Axioms** The following axioms are necessary for the solution of Simple Equations —

I **If equal quantities are added to equal quantities, the sums are equal**

Thus if $a=b$ and $x=y$, then (i) $a+m=b+m$, (ii) $a+x=b+y$

II **If equal quantities are subtracted from equal quantities, the remainders are equal**

Thus if $a=b$ and $x=y$ then (i) $a-m=b-m$, (ii) $a-x=b-y$

*A term adapted from *Analytical Geometry*

III If equal quantities are multiplied by equal quantities, the products are equal

Thus if $a=b$ and $x=y$, then (i) $am=bm$, (ii) $ax=by$

IV If equal quantities are divided by equal quantities, the quotients are equal

Thus if $a=b$ and $x=y$, then (i) $a-m=b-m$, (ii) $a-x=b-y$

Examples XLVI (*Oral*)

Find the value of x which satisfies the equation

1	$2x=8$	2	$3x=15$	3	$4x=20$	4	$5x=-15$
5	$6x=-18$	6	$-3x=21$	7	$9x=27$	8	$-11x=22$.
9	$8x=0$	10	$-8x=0$	11	$\frac{x}{2}=3$	12	$\frac{x}{5}=2$
13	$-\frac{x}{3}=4$	14	$\frac{3x}{4}=-6$	15	$-\frac{2x}{5}=4$	16	$\frac{x}{3}=0$
17	$\frac{2x}{3}=0$	18	$\frac{4x}{5}=\frac{8}{10}$	19	$3x=2$	20	$5x=1$
21	$\frac{3x}{4}=\frac{2}{3}$	22	$\frac{5x}{6}=\frac{3}{4}$	23	$x+3=5$	24	$x-1=6$

25 $3x+8=20$ 26 $8x-5=11$. 27. $x+5x=12$

28 $2x+3x=15$ 29 $5x-8x=6$ 30 $-x+4x=-18$

31 $5x-2x+x=8$ 32 $8x-3x-x=-7-5$ 33 $7x+2x-3x=-42$

34 $-2x-4x-7x=-44+5$ 35 $-5x-x-6x=-1-7-4$

88 Transposition of Terms Suppose $x-a=y$, add a to both sides, thus

$$x-a+a=y+a \text{ (Ax I)}, \text{ or } x=y+a,$$

that is, $-a$ is *changed* into $+a$, when it is taken from the *left* side and placed in the *right*

Again if we subtract y from both sides (Ax II), we have

$$x-y=y+a-y, \text{ or } x-y=a,$$

which shews that $+y$ is *changed* into $-y$, when it is taken from the *right* side and put in the *left*

These two examples are sufficient to shew that—**Any term may be transposed from one side of an equation to the other side if its sign be changed**

Cor 1 We may write *all the terms* of an equation on one side of the sign of equality and zero on the other side

Thus from $2x=x+3$, we have $2x-x-3=0$

Cor 2 If the *same term with the same sign* occurs on both sides of an equation, it may be *removed* without the equality being affected

Thus from $x+a=2+a$, we have $x=2$

Cor 3 If the *sign of every term* on *both* sides of an equation be *changed*, the equality still holds

Thus from $x-a=b-c$, we have $-x+a=-b+c$

Ex 1 Solve the equation $3x-2=2x+7$

$$3x-2=2x+7$$

Add 2 to both sides, thus

$$3x-2+2=2x+7+2 \text{ (Ax I)}$$

$$i\,e, \quad 3x=2x+9$$

Subtract $2x$ from both sides, thus

$$3x-2x=2x+9-2x \text{ (Ax II)},$$

$$i\,e, \quad x=9$$

[Verification The beginner to satisfy himself should always *verify* the solution, *i e*, prove its correctness, by substituting the value of the variable in both sides of the equation

Thus if $x=9$, $3x-2=27-2=25$

Also if $x=9$, $2x+7=18+7=25$

Since these results are equal, the solution is correct]

Ex 2 Solve the equation $5x+21=20x-24$

$$5x+21=20x-24$$

Subtract $20x$ from both sides (Ax II), thus

$$5x+21-20x=20x-24-20x,$$

$$i\,e, \quad 21-15x=-24$$

Take 21 from both sides (Ax II), thus

$$21-15x-21=-24-21,$$

$$i\,e, \quad -15x=-45$$

Divide both sides by -15 (Ax IV), thus

$$x=\frac{-45}{-15}=3$$

In the beginning the student is recommended to work a good many examples in details, for then the reason for each step will be impressed on his mind But after a little practice, the solution may be put in a shorter form thus —

$$5x+21=20x-24$$

Transpose, thus $\quad 5x-20x=-24-21,$

or $\quad -15x=-45$

Change the signs, thus $\quad 15x=45$

Divide by 15, thus $\quad x=3$

Ex 3 Solve the equation $8x-5+2x=13+2x-7x$

$$8x-5+2x=13+2x-7x$$

Remove $2x$ from both sides (Art 88, Cor 2), thus

$$8x-5=13-7x$$

Transpose, thus $\quad 8x+7x=13+5,$

or $\quad 15x=18$

Divide by 15, thus $\quad x=\frac{18}{15}=1\frac{1}{5}$

Ex 4 Solve $4x-3=4(2x-1)+13$

$$4x-3=4(2x-1)+13$$

Remove the bracket, thus

$$4x-3=8x-4+13=8x+9$$

Transpose, thus $4x-8x=9+3,$

i e, $\quad -4x=12$

$$x=\frac{12}{-4}=-3$$

Ex 5 Solve $3x-10(2x-3)+21=0$

Remove the bracket, thus

$$3x-20x+30+21=0,$$

i e, $\quad -17x+51=0$

Transpose, thus $\quad -17x=-51.$

$$x=\frac{-51}{-17}=3.$$

Ex 6 Solve $(x-5)^2+(2x-3)^2=5(x-2)^2$

Simplify, thus

$$x^2-10x+25+4x^2-12x+9=5(x^2-4x+4),$$

or $$5x^2-22x+34=5x^2-20x+20$$

Remove $5x^2$ from both sides, thus

$$-22x+34=-20x+20$$

Transpose, thus $$-22x+20x=20-34,$$

or $$-2x=-14,$$

$$2x=14,$$

divide by 2, thus $$x=7$$

Examples XLVII

Solve the following equations, verifying each of them and giving the solutions of the first ten in details

1 $6x+7=55$ 2 $3x-5=19$ 3 $x=4-15x$

4 $5x-8=16-3x$ 5 $53+30x=3-20x$

6 $3+x-15x=10x-102-3x$ 7 $10x-3(x-1)+3=90$

8 $5-3(4-x)+4(3-2x)=0$ 9 $6x+2(11-x)=3(19-x)$

10 $52x+4(3x-2)=376$ 11 $(3x-5)20-(4x-3)12=0$

12 $13x-21(x-3)=10-21(3-x)$

13 $11(x-8)+13(4x-1)=15(2x-1)+13$

14 $2x-1-2(3x-2)+3(4x-3)=4(5x-4)$

15 $x^2+3(5x-18)+3x^2=4x^2-3x$

16 $2x(x-3)+x(x-4)=x(3x-2)-80$

17 $15(x^2-3)+20x=x(15x+17)$

18 $(x-3)(x-2)=(x-4)(x+5)$

19 $(x-2)(2x-1)=2(x+1)(x+3)$

20 $(8-3x)(5-x)=(3x-4)(x-6)-13$

21 $(x-1)^2-(x-2)^2=5$ 22 $(2x+7)^2+3x(x-10)=7(x^2+9)$

23 $(3x-4)^2+(2x-5)^2=13(x-6)^2$

24 $(20x+3)^2-(15x-8)^2=5(2-5x)(3-7x)+30$

89 If denominators occur in an equation, multiply both sides by the least common multiple of the denominators

Thus to solve $\frac{3}{10}(x-2)-\frac{1}{3}(2x-3)=\frac{4}{5}(7-x)$

To clear the equation of fractions, multiply both sides by 30, the L C M of the denominators

Thus $$9(x-2)-10(2x-3)=24(7-x),$$
or $$9x-18-20x+30=168-24x$$
Transpose, thus $$9x-20x+24x=168+18-30,$$
whence $$13x=156$$
$$\therefore x=\frac{156}{13}=12$$

Note Since the line between the denominator and numerator of a fraction serves the purpose of a vinculum, we may write the given equation thus

$$\frac{3(x-2)}{10}-\frac{2x-3}{3}=\frac{4(7-x)}{5}, \text{ just as } \tfrac{1}{2}x=\frac{x}{2},\ \tfrac{3}{4}y=\frac{3y}{4}, \text{ \&c}$$

Examples XLVIII

Solve the following equations

1 $\frac{x}{12}-\frac{x}{6}=2-\frac{x}{4}$

2 $\frac{3x-8}{10}=\frac{15-2x}{3}$

3 $\frac{4x+3}{7}+\frac{4-3x}{4}=0$

4 $2x-\frac{1}{3}(x+14)=12$

5 $\frac{5x}{9}-\frac{2x-1}{3}=\frac{4}{15}$

6 $\frac{3x}{4}+\frac{7x}{15}+\frac{11x}{6}=366$

7 $\frac{x-2}{4}-\frac{3-x}{6}-3\frac{1}{6}=0$

8 $\frac{5x}{2}+\frac{2x}{3}-17=\frac{3x}{5}+60$

9 $\frac{1}{x}+\frac{3}{2}+\frac{8}{x}=1$

10 $3-\frac{4}{x}=\frac{5}{x}-1$

11 $\frac{1}{2}-\frac{x-1}{5x}=\frac{5}{2x}$

12 $\frac{4-x}{3x}-1=\frac{7}{4x}$

13 $\frac{3-2x}{4}=1-\frac{4x-5}{6}$

14 $\frac{x+6}{4}-\frac{16-3x}{12}=\frac{25}{6}$

15 $\frac{12-3x}{4}-1=\frac{3x-11}{3}$

16 $1-\frac{x-2}{5}=\frac{x+2}{4}$

17 $7x+15=23-\frac{1-9x}{2}$

18 $\frac{x}{3}+2x\left(\frac{3}{2}-\frac{1}{6}\right)-1=2\frac{1}{2}$

19. $3-\frac{x-2}{3}=\frac{4x+7}{5}$

20 $\frac{x}{4}-\frac{x-4}{6}=\frac{4}{3}+\frac{24-x}{12}$

Solve the following equations

21 $\frac{x+1}{2}+\frac{x+2}{3}=14+\frac{5-x}{4}$

22 $x+\frac{11-x}{3}=\frac{19-x}{2}$

23 $\frac{x}{2}-\frac{5x+4}{3}=\frac{4x-9}{3}$

24 $x-\frac{x-7}{3}+\frac{3x-1}{5}=\frac{2x}{7}+9$

25 $\frac{x+6}{24}+\frac{16-3x}{12}=3x-4\frac{5}{6}$

26 $\frac{3x+5}{8}-\frac{21+x}{3}=39-5x$

27 $\frac{3x+4}{5}-\frac{7x-3}{2}+\frac{16-x}{4}=0$

28 $\frac{3x+1}{7}-\frac{x+8}{5}+\frac{x+1}{3}=0$

29 $\frac{x+4}{3}-\frac{x-4}{5}=2+\frac{3x+5}{15}$

30 $\frac{x+1}{2}+\frac{x+2}{3}+\frac{x+3}{4}=16$

31 $\frac{4-x}{4}-\frac{5-x}{5}+\frac{6-x}{6}=1$

32 $\frac{7x}{2}-1-\frac{3x+1}{2}=\frac{4x+51}{14}$

33 $\frac{3(x+1)}{16}+\frac{7x+11}{15}-\frac{7x+1}{20}=3$

34 $\frac{3x+5}{7}-\frac{2x+7}{3}+10-\frac{3x}{5}=0$

35 $\frac{1}{2}(x-3)-\frac{1}{3}(x-4)+\frac{1}{4}(x-5)=0$

36 $\frac{1}{7}(5x+3)+\frac{2}{9}(13x+8)-\frac{1}{5}(7x-11)=0$

37 $\frac{5}{7}(2x-11)-\frac{3}{4}(x-5)=\frac{x}{3}-(10-x)$

38 $\frac{1}{6}(8-x)+x-1\frac{2}{3}=\frac{1}{2}(x+6)-\frac{x}{3}$

39 $\frac{3}{2}(3x-2)-\frac{2}{3}(2x-3)=\frac{5}{6}(x+3)+3\frac{1}{2}$

40 $\frac{1}{3}(5x-2)+\frac{1}{4}(3x-5)=\frac{1}{2}(3x+1)+\frac{1}{6}(5x+1)$

41 $\frac{1}{6}(\frac{2}{3}x+4)+\frac{2x-15}{18}=\frac{x}{6}\left(\frac{6}{x}-1\right)$

42 $\frac{9x+7}{2}-\left(x-\frac{x-2}{7}\right)=36$

43 $\frac{x-7}{11}-\frac{3x-5}{7}+\frac{125}{77}=2x-17$

44 $x-\frac{x-2}{3}=5\frac{3}{4}-\frac{10+x}{5}+\frac{x}{4}$

45 $\frac{x-2}{5}-\frac{10-x}{3}=\frac{2x-3}{6}-1\frac{13}{30}$

46 $\frac{x-1}{7}+\frac{23-x}{5}=7-\frac{4+x}{4}$.

47 $\frac{x}{8}-\frac{x-1}{2\frac{1}{2}}=\frac{3x-4}{15}+\frac{x}{12}$

48 $\frac{5x-1}{2}-\frac{7x-2}{10}=6\frac{3}{5}-\frac{x}{2}$

49 $\frac{3x+7}{14}-\frac{2x-7}{21}+2\frac{3}{4}=\frac{x-4}{4}$

50 $\frac{2x+1}{29}-\frac{402-3x}{12}=9-\frac{471-6x}{2}$

Solve the following equations

51 $\frac{4x-34}{17}-\frac{238-5x}{3}=\frac{69-x}{2}$

52 $\frac{4x-21}{7}+7\frac{5}{6}+\frac{7x-28}{3}=x+3\frac{3}{4}-\frac{9-7x}{8}+\frac{1}{12}$

53 $\frac{11x-13}{25}+\frac{19x+3}{7}-\frac{5x-25\frac{1}{3}}{4}=28\frac{1}{4}-\frac{17x+4}{21}$

90 Equations involving decimals can ordinarily be solved by working in decimals But sometimes by multiplying or dividing by a suitable power of 10, they may be easily solved

Ex Solve $1{\cdot}2x-\frac{{\cdot}18x-{\cdot}05}{5}={\cdot}4x+9{\cdot}178$

Multiply by 5, $6x-{\cdot}18x+{\cdot}05=2x+45{\cdot}89,$

or $4x-{\cdot}18x=45{\cdot}89-{\cdot}05,$

or $3{\cdot}82x=45{\cdot}84,$

$$x=\frac{45{\cdot}84}{3{\cdot}82}=12$$

Examples XLIX

Solve the following equations

1 $3{\cdot}1x-{\cdot}4={\cdot}1x-{\cdot}19x$ 2 $3{\cdot}5x-4{\cdot}7=6{\cdot}5-2{\cdot}1x$

3 ${\cdot}4x-{\cdot}35={\cdot}34x+{\cdot}01$ 4 $1{\cdot}32x+{\cdot}02x=1{\cdot}17-x$

5 $1{\cdot}5x+{\cdot}2-{\cdot}875x={\cdot}0625x-1{\cdot}375$

6 $11{\cdot}1x-3(2x-5)=7(1{\cdot}8x-3)-3{\cdot}9$

7 $x-\frac{2x-3}{7}=\frac{5-x}{35}.$

8 $1{\cdot}2x-\frac{{\cdot}18x-{\cdot}05}{5}={\cdot}4x+8{\cdot}9$

9 ${\cdot}07x-{\cdot}53=\frac{{\cdot}18x-{\cdot}95}{4}+{\cdot}1x$

10 $\frac{{\cdot}5x-{\cdot}4}{3}+\frac{1{\cdot}3-3x}{2}=\frac{1{\cdot}6-8x}{1{\cdot}2}$

11 $\frac{5{\cdot}2x}{13}-\frac{1-{\cdot}2x}{5}\left(\frac{3}{5}-{\cdot}1\right)={\cdot}1x+{\cdot}028-\frac{{\cdot}5x-{\cdot}9}{4}$

12 $5\left(x-\frac{5{\cdot}1}{2{\cdot}6}\right)+\frac{{\cdot}02}{13}(3x-1)=x-\frac{{\cdot}01}{39}\left(5x-\frac{1-3x}{{\cdot}4}\right)$

CHAPTER IX

SYMBOLICAL EXPRESSIONS—SUBSTITUTIONS—PROBLEMS

Symbolical Expressions

91 In this Chapter we shall shew how to solve Problems **Problems** are statements in words of the relations that exist between certain quantities, one or more of which we want to find These relations are called *conditions* of the problem and when rendered into the *symbolical language* of Algebra lead to equations, on the solution of which that of the problem depends Such a rendering is known as *symbolical expression*

To enable the student to acquire facility to express verbal statements readily in algebraical language, we give some more examples in addition to those given in Chapter I

Just as 7×2 is double of 7, so $x \times 2$ or $2x$ is double of x Similarly $3x$ is treble of x, $4x$ is quadruple of x, &c

If the product of two numbers is 48 and one of them is 6, then the other is $48 \div 6$ Similarly if the product of two numbers is a and one of them is x, then the other is $\frac{a}{x}$

The price of 15 apples at 2 annas is 15×2 annas, just in the same way the price of x apples at m annas is mx annas $= \frac{mx}{16}$ rupees $= 12mx$ pies

If 15 mangoes cost 11 annas, then one costs $\frac{11}{15}$ annas In the same way when x mangoes cost y annas, then one costs $\frac{y}{x}$ annas, and 10 cost $\frac{10y}{x}$ annas

If a man walks 35 miles in 8 hrs, then in 1 hr. he walks $\frac{35}{8}$ miles, similarly when a man walks x miles in 12 hr, in 1 hr he walks $\frac{x}{12}$ miles and in y hr, $\frac{x}{12} \times y$ miles or $\frac{xy}{12}$ miles

As a train running at the rate of 30 miles per hr performs a journey of 137 miles in $\frac{137}{30}$ hrs, so a train which runs at the rate of x miles an hour performs y miles in $\frac{y}{x}$ hrs

Just as a man who is 35 years old will be (35+8) years old in 8 years and was (35−12) years old 12 years ago, so a man who is x years old now will be $(x+y)$ years old in y years and was $(x-a)$ years old a years ago

Write down 3 consecutive numbers of which x is the least

Since the difference between 2 consecutive numbers is unity, the number next after x is $x+1$ and the next number is $x+1+1$ or $x+2$

Thus the required numbers are x, $x+1$ and $x+2$

Question What is the consecutive number next before x?

Are the numbers $x+1$, x and $x-1$ consecutive numbers?

Definition Two numbers are said to be **consecutive** when they differ by unity. Thus 4 and 5, 28 and 27, &c, are consecutive numbers

The **natural numbers** 1, 2, 3, 4, 5, are consecutive numbers

We shall now shew how verbal statements can be symbolically expressed in the form of an equation

Ex 1 The excess of x over 15 is y Express this statement in the form of an equation

The excess of 10 over 6 is $10-6$; so the excess of x over 15 is $x-15$ And this excess is y by the question

$$y=x-15$$

Ex 2 Express symbolically the statement that x is greater than a by 12

The number which is greater than 8 by 3 is $8+3$, so the number greater than a by 12 is $a+12$ And this is x by the question

$$x=a+12$$

Ex. 3 Eighteen years hence A's age will be 3 times his present age which is x years Express this statement by an equation.

18 years hence A's age will be $(x+18)$ years, and this age by the question is 3 times x years

$$3x=x+18$$

Ex 4 A has x rupees and B has y rupees, after giving a rupees to B, A finds that he has $\frac{2}{3}$ of what B then has Express this statement in the form of an equation

When A gives B a rupees, then A has $(x-a)$ rupees and B has $(y+a)$ rupees

By the question A's money now is $\frac{2}{3}$ of B's

$$x-a=\frac{2}{3}(y+a)$$

Examples L (*Oral*)

1 What number is greater than x by a?

2 What number is greater than 20 by x?

3 What number is less than y by 8?

4 What number is less than 50 by m?

5 What number exceeds 12 by x?

6 What number exceeds y by x?

7 By how much does 21 exceed x?

8 By how much does y exceed x?

9 By how much does $x+1$ exceed $x-4$?

10 One part of 15 is x, what is the other part?

11 One part of y is 15, what is the other part?

12 How often is x contained in 27?

13 How often is x contained in y?

14 What number divided by m will give 53?

15 What number multiplied by x will give 53?

16 If m is divided into x equal parts, what is the value of each part?

17 The sum of two numbers is a and one of them is x, find the other

18 The difference of two numbers is x and the greater is y find the other

19 The sum of two numbers is $x+3$ and one of them is $x-3$, find the other

20 What is the value of x mangoes at 4 pies each?

21 What is the value of x mangoes at 6 annas a score?

22 If x oranges cost 8 annas, what is the price of one?

23 If 50 oranges cost x pies, what is the price of one?

24 If y oranges cost 8 pies, what is the value of one?

25 Write down 3 consecutive numbers of which the middle is x

26 Write down 3 consecutive numbers of which the greatest is x

27 If 57 contains x three times, what is the value of x?

28 A man walks 40 miles in x hrs, what is his rate per hour?

29 A train goes at the rate of 30 miles per hour, in what time will it go x miles?

30 If y is less than m by c, how do you express this statement symbolically?

31 A man walks x miles in a hours at the rate of b miles an hour. Express this statement in the form of an equation.

Examples LI

1 Find the number which is 3 times x, diminished by a.

2 The sum of two numbers is 31; if one of them is x, find the other.

3 The difference of two numbers is 45; if one of them is y, find the other.

4 Twenty is divided into 2 parts, one of which is x; find the other.

5 The product of two numbers is 63 and one of them is x; what is the other?

6 The quotient when a number is divided by $3x$ is 18; what is the number?

7 A father is 35 years older than his son whose age is $(x-1)$ years; what is the age of the father?

8 A walks 3 miles an hour to go to P, a distance of 50 miles; find the distance between him and P at the end of x hours.

9 A walks $1\frac{1}{2}$ miles an hour faster than B who walks $(x-1)$ miles an hour; if they start together, find the respective distances walked by each at the end of 6 hours.

10 Four persons equally contributed to make up the sum £$(x+4)$; how many shillings did each give?

11 A post whose length is $12x$ has a third coloured red and a fourth coloured black, and the rest white; what is the length of this portion?

12 The age of a person is x years; how old was he 6 years back and what will be his age 6 years hence?

13 A performs twice as much work as B in 1 hour; if W represent the work which A performs in 2 hours, how would you represent the work B does in 1 hour?

14 If sound travels at the rate of m feet per second, what is its velocity in miles per hour?

15 How many square yards of matting will be required for a room whose length is a yards and breadth b feet?

16 If x mangoes cost a rupee, what is the cost in annas of a mangoes and how many can be bought for m pies?

17 If a person travels 84 miles a week, what distance will he travel in x days?

18 A performs a journey of 25 miles in x days; what is his rate of travelling per hour?

19 In what time will a man walk 30 miles at the rate of x miles per hour?

20 If a person travels x miles in a hours, what distance does he travel in 3 hours?

21 A person gives $(x-3)$ rupees more to A than to B; if A get y rupees, what does B get?

22 A picture and its frame together cost x rupees; if the value of the picture be $(20-y)$ rupees, find that of the frame

23 A's age is x years and B's age is 5 times what A's age will be 3 years hence; what is B's age?

24 A railway train performed a distance of a miles in x hours; it had to stop a quarter of an hour at an intermediate station; at what rate did it run?

25 A and B have each x rupees; if A pays y rupees out of his money to B, what has each then?

26 The amount in a bag is £m; if it consist entirely of half-sovereigns, find their number If there be 6 half sovereigns and the rest half crowns, what is the number of the latter?

27 Express 38 and 83 symbolically, when x stands for 3 and y for 8

28 If x stands for 1, y for 2, and z for 3, express 231 and 312 symbolically

29 If x represent the units' figure and y the tens' figure of a number, write down the number formed by reversing the digits.

30 The sum of a and 15 is x; express the statement symbolically

31 If the excess of 5 over x be denoted by y, express the statement in symbols

32 If the same number be less than x by 3 and greater than y by 4, how do you express the statement algebraically?

33 The denominator x of a fraction exceeds the numerator by 3 If the fraction is $\frac{2}{3}$, express this statement in the form of an equation, and find x from the equation

34 Of the two parts into which 45 is divided, one is 8 times the other If x represent the smaller part, express this statement by an equation, and find x from the equation

35 The sum of 3 consecutive numbers of which x is the least, is 54 Express this statement symbolically, and hence find x

36 The sum of 3 consecutive numbers of which x is the middle one, is 81. Express the statement in symbols

37 A father's age is 40 years and his son's age is 10 years, x years ago, the father was 7 times as old as his son Express this statement by an equation, and find x from the equation.

38 A and B play with x rupees each, after A wins R15, he has 3 times as much as B Express the statement in the form of an equation and find x from the equation

39 The price of a horse is x rupees and that of a saddle is m rupees less If the price of the horse is 4 times that of the saddle, express the statement in the form of an equation

40 A boat which can row a miles in still water, goes in h hours 18 miles up a river which flows x miles per hour Express the statement by an equation

Substitutions

92 The following **Geometrical and other Formulæ** should be carefully remembered as they are often required in calculations From these the value of any *one* of the involved quantities can be found by *substituting* the values of the rest which are supposed to be given

Examples LII

If A be the area of a rectangle whose length is l and breadth is b, then A is given by the formula

I $$A = l \times b$$

(1) Find the area of the floor of a room whose length is 15 ft 9 in and breadth 124 inches

(2) If the area of a room 16 ft long is 180 sq ft, what is its breadth?

If A be the area of the four *walls of a room of length l and breadth b, then A is known from the formula*

II $$A = 2h(l+b)$$

(1) Find the area of the 4 walls of a room whose length is 18 ft, breadth 12 ft and height 10 ft

(2) The area of the walls of a room is 756 sq ft. and its height is 10 ft 6 in, find the perimeter of the floor

(3) The perimeter of the floor of a room is 30 ft 9 in and the area of the walls is 369 sq ft, find its height

(4) The length and breadth of a room are 16 ft 3 in and 12 ft 6 in, respectively, and the area of the walls is 575 sq ft., find the height of the room

(5) The length of a room is 5 ft. more than its breadth and its height is 11 ft.; if the area of the walls is 682 sq. ft., find the length of the room.

If Δ *represents the area of a triangle whose base is* b *and height* h, *then* Δ *is given by the formula*

III $$\Delta = \tfrac{1}{2}bh$$

(1) Find the area of a triangle whose base is 7 ft. 8 in. and height 5 ft. 4 in.

(2) The area of a triangle is $17\frac{1}{2}$ sq. ft. and its height is 6 ft. 8 in.; find its base.

(3) The area of a triangle whose base is 5 yds. 2 ft., is $277\frac{2}{3}$ sq. ft.; what is its height?

If the area of a parallelogram is P, *its base* b, *and height* h, *then* P *is found from the formula*

IV $$P = bh$$

(1) The area of a parallelogram is 4464 sq. in. and its base is 6 ft.; find its height.

(2) The base and height of a parallelogram are respectively 5 ft. 3 in. and 7 ft.; find its area.

If T *represent the area of a trapezium in which the parallel sides are* a *and* b *respectively and* h *the distance between them, then* T *is given by the formula*

V $$T = \tfrac{1}{2}(a+b)h$$

(1) Find the area of a trapezium whose parallel sides are 8.3 cm., 6.5 cm. and in which the distance between the parallel sides is 4.6 cm.

(2) The area of a trapezium is 125 sq. cm., and the parallel sides are 15.4 cm. and 18.6 cm.; find the distance between them.

(3) The distance between the parallel sides of a trapezium is 8.4 cm. and its area is 114.45 sq. cm.; if one of the parallel sides is 14.5 cm., find the other.

If C *be the circumference of a circle whose radius is* r, *then* C *and* r *are connected by the formula*

VI $$C = 2\pi r,$$

where $\pi = \frac{22}{7}$ *roughly.*

The value of π cannot be determined exactly. A nearer approximation to the value of π is 3.1416.

(1) Find the circumference of a circle whose radius is (i) $1\frac{3}{4}$ ft and (ii) 8 cm

(2) Find the radius of a circle whose circumference is $7\frac{6}{7}$ yds

(3) Find the radius of a circle whose circumference is 16 in

If A represent the area of a circle whose radius is r, then A is given by the formula

VII $$A = \pi r^2,$$

where π has the same value as in VI

(1) Find the area of a circle (i) whose radius is 15 cm and (ii) whose diameter is 8 inches

(2) Find the diameter of a circle whose area is $28\frac{2}{7}$ sq cm

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If A be the area of the ring formed by two concentric circles whose radii are R and r, then A is obtained from the formula

VIII $$A = \pi R^2 - \pi r^2 = \pi (R^2 - r^2)$$

(1) Find the area of the ring formed by the concentric circles of radii 15 and 13 inches respectively.

(2) The area of the ring is $113\frac{1}{7}$ sq cm and the radius of the outer circle is 10 cm , find the radius of the inner circle

(3) The area of the ring is 154 sq. in. and the radius of the inner circle is 9 in ; find the radius of the outer circle

(4) The area of the ring is 550 sq cm and the sum of the radii of the inner and outer circles is 35 cm , find their radii

[From the formula we have $A = \pi(R+r)(R-r)$, thus $R-r$ can be found]

If V represent the volume of a cylinder, h its height and a the area of its base, then V, h and a are connected by the equation

IX. $$V = ah$$

Also if V, a, h, represent the volume, area of base, and height of a right circular cone, then

X $$V = \tfrac{1}{3}ah$$

(1) Find the volume of a pyramid whose height is 5 yd 2 ft, and which stands on a square base whose side is 25 ft.

(2) A cone has a base of radius 10 5 cm , if its volume is 924 cubic cm , find its height.

The volume V *of a sphere whose radius is* r *is given by the formula*

XI $$V=\frac{4}{3}\pi r^3$$

(1) Find the volume of a sphere of diameter 30 cm

(2) The volume of a sphere is 310464 c inches, find its diameter

---

Just as 19 = 3 × 5 + 4, *where* 19 *is the dividend,* 3 *the divisor,* 5 *the quotient and* 4 *the remainder, so*

XII $$D=dQ+R,$$

where D *is the dividend,* d *the divisor,* Q *the quotient and* R *the remainder*

Find (1) the dividend, when $d=8$, $Q=54$ and $R=7$, (2) the divisor, when $Q=27$, $R=11$ and $D=470$, (3) the remainder, when $D=816$, $d=67$ and $Q=12$, (4) the quotient, when $D=3457$, $d=231$, $R=223$

In a right-angled triangle, if c *is the hypotenuse,* a *and* b *the sides containing the right angle, then the relation between* a, b *and* c *is given by the formula*

XIII $$c^2=a^2+b^2,$$

$$\text{or} \quad a^2=c^2-b^2=(c+b)(c-b)$$

In a right-angled triangle, find

(1) the hypotenuse, when the sides are 12 and 35

(2) one of the sides, when the other side = 12 and hypotenuse = 13

(3) the sides, when their sum = 46 and hypotenuse = 34

(4) the sides, when their difference = 17 and hypotenuse = 25

If a body, starting from rest, passes over a space s *in time* t *with velocity* v, *then* s *is obtained from the formula*

XIV $$s=vt \qquad \text{(i)}$$

Thus if v be the number of miles passed over in 1 hour and t the number of hours, then $s=vt$ miles, if v be the number of miles passed over in 1 hour and t the number of minutes, then $s=\frac{vt}{60}$ miles and so on

The following two equations obtained from (i) are important

$$v=\frac{s}{t} \qquad \text{(ii)}$$

$$t=\frac{s}{v}. \qquad \text{(iii)}$$

(1) If a man walk at the rate of $4\frac{1}{2}$ miles per hour, what distance will he walk in 3 hours 20 minutes?

(2) A cyclist scored a distance of $40\frac{1}{2}$ miles at the rate of 297 yds. per minute, in what time did he perform the distance?

(3) A railway train ran a distance of 30 miles in 40 minutes, what was its velocity in miles per hour?

If a body falling freely under the action of gravity describes s *feet in* t *seconds, then* s *is given by the formula*

XV $$s=\frac{1}{2}gt^2,$$

where g (acceleration of gravity) = 32 ft. roughly

For example, if a stone falling from a certain height takes 10 seconds to reach the ground, then

the height from which it fell $=\frac{1}{2}\times 32\times(10)^2$ ft. $=1600$ ft

(1) A cocoanut falling from the tree reaches the ground in $1\frac{3}{4}$ secs Find the height of the cocoanut tree

(2) What time does a stone, dropped from a height of 112 yds, take to reach the ground?

If I *is the interest of* P *rupees for* n *years and* r *is the interest of 1 rupee for 1 year, then* I *is obtained from the formula*

XVI $$I=Pnr$$

Hence $P=\frac{I}{nr}$, $n=\frac{I}{Pr}$, $r=\frac{I}{Pn}$

Also if M *is the amount, then* $I=M-P$, *and*

$$M=P+Pnr,\ P=\frac{M}{1+nr}$$

N B *If the rate of interest be 5 per cent, then* $r=\frac{5}{100}=.05$

(1) Find the interest on R725 for $3\frac{1}{2}$ years at 6 p c per annum

(2) What sum will amount to R738 in 4 years at 5 p c?

(3) In what time will R450 amount to R513 at 4 p c per annum?

(4) At what rate per cent. per annum will R825 amount to R1014 12*a* in $5\frac{3}{4}$ years?

Problems

93 We give now a collection of easy problems for exercise In solving a problem, we generally *represent the unknown quantity* (*i e*, the quantity to be found out) *by x and then express the verbal*

statement of the problem symbolically after the manner described in Art 91 This symbolical expression will be in the form of an equation from which the quantity sought is found [*Vide* Art 91, Exs 33—40 Each of these examples might have been a *problem*, if x were taken as the quantity sought]

Ex 1 *If 20 be added to a number, the sum becomes 3 times the number, what is the number?*

Let $\qquad x=$ the number,*

then $\qquad x+20=$ the sum of x and 20

Also by the condition of the problem,

$$3x=\text{the sum of } x \text{ and } 20$$

Hence $\qquad x+20=3x$,

an equation from which x can be found.

Transpose, thus $\qquad 3x-x=20,$

or $\qquad 2x=20,$

$$x=10$$

Thus the required number is 10

Ex 2 *What number is that to which if 23 be added and the sum divided by 5, the quotient will be 13 diminished by half the number?*

Let $\qquad x=$ required number,

then $\qquad \frac{x+23}{5}=$ quotient of the sum of x and 23 divided by 5,

and $\qquad 13-\frac{1}{2}x=$ this quotient, by the condition of the problem,

$$\frac{x+23}{5}=13-\tfrac{1}{2}x, \text{ whence } x=12$$

Thus the number required is 12

Verification $(12+23)-5=7$, and $13-\frac{1}{2}$ of $12=7$

Hence the solution is correct

N B The student will do well to verify the solution of each problem that he solves

Examples LIII

1 Find the number, the double of which diminished by 5 gives that number increased by 5

2 What is that number a third, a fourth, and a fifth part of which taken together amount to 94?

* The sign of equality $=$, is here used to mean '*Denote*', '*Represent*', or '*Stand for*'

3 A certain number is equal to 570 diminished by four times the number, what is the number?

4 Find the number, five times which exceeds the double by the number which is the difference between 284 and the number

5 What is that number, 3 times which taken from 120, will leave a remainder which is equal to that number increased by 8?

6 If from 558 you subtract 8 times a certain number, the remainder will be 10 times that number, what is that number?

7 The sum of a certain number and 15 is multiplied by 12, if you now subtract twice the number from the product, the remainder will be 240 diminished by 5 times that number Find the number

8 If from two-thirds of a number you subtract 51, the remainder is 84 more than half the number, what is the number?

9 What is that number, to which if you add 3, the sum will be the same as if you take 93 from 5 times the number?

10 Find a number, such that if its double and treble together be subtracted from 75, there will be a remainder 30

11 Three-fourths of a number diminished by 1 is divided by half that number increased by 1, and the result is $\frac{5}{4}$; what is that number?

Ex 3 *Find the numbers whose sum is 31 and difference is 15*

Let x=the smaller number,

then, ∵ the difference between the numbers is 15,

$x+15$=the greater number

And by the first condition of the problem, their sum is 31

$$x+(x+15)=31,$$

or

$$2x+15=31,$$

whence

$$x=8;$$

∴ one of the numbers is 8, and the other is 8+15, or 23

Ex 4 *The difference between two numbers is 20, and four-fifths of the smaller number exceeds half the larger number by 14 Find the numbers*

Let x=greater number,

then $x-20$=smaller number,

∴ by the condition of the problem

$$\tfrac{4}{5}(x-20)-\tfrac{1}{2}x=14$$

Multiply by 10 , thus

$$8(x-20)-5x=140,$$

or

$$8x-160-5x=140,$$

$$3x=140+160=300,$$

or

$$x=100$$

Thus greater number $=100$ and smaller number $=80$

Examples LIII (*Continued.*)

12 Find two numbers, such that their sum may be 183, and their difference may be 13

13 Find two numbers whose difference is 15, such that if 40 be taken from the greater, the remainder is 51 diminished by the less

14 Find two numbers whose sum is 26, such that 3 times the greater diminished by twice the less, will be equal to their difference increased by 22

15 Three fifths of the greater of two numbers is equal to the excess of 54 over a fourth of the less , if the numbers differ by 5, find them

16 The sum of two numbers is 64, and if 4 times the greater be added to 5 times the smaller, the sum will be 285 , what are the numbers ?

17 The difference between two numbers is 8 and that between their squares is 288 Find the numbers

18 The sum of 3 consecutive numbers is 1620 , what are they ?

[Represent the numbers by $x-1$, x and $x+1$]

19 The sum of 3 consecutive odd numbers is 1149 , find them

[Represent the numbers by $2x-1$, $2x+1$ and $2x+3$]

20 The difference between the squares of two consecutive numbers is 513 , find the numbers

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Ex 5 *Divide 100 into two parts, such that $\frac{5}{7}$ of the greater may exceed $\frac{3}{4}$ of the less by 7*

Let $x=$ the greater part,

$100-x=$ the smaller part.

Hence by the condition of the problem

$$\frac{5}{7}x-\frac{3}{4}(100-x)=$$

Multiply by 28, thus

$$20x-21(100-x)=196,$$

whence $x=56$, the greater part,

and $100-56=44$, the smaller part.

Ex. 6 *Divide Rs 1260 between A and B, so that as often as A receives Rs 4, B shall receive Rs 3*

Let x = number of times A receives the money, then also

x = ... B ... ... ,

∴ $4x$ rupees = money A receives,

and $3x$ rupees = ... B ... .

∴ by the question, $4x+3x=1260$, whence $x=180$

Hence A receives $4x$ rupees = Rs 720,

and B ... $3x$ ... = Rs 540

Examples LIII (*Continued*)

21. Divide 36 into 2 parts, such that the sum of 10 times the less and 15 may be equal to the excess of 8 times the greater over 3

22. A horse and saddle together cost Rs 1000, and $\frac{4}{5}$ of the cost of the horse and $\frac{2}{3}$ of the cost of the saddle amount to Rs 800, find the price of each

23. Divide Rs 207 between 2 persons, so that 5 times the share of one together with 6 times the share of the other may be equal to 11 times the excess of the first share over the second

24. A and B together counted Rs 1265, and A counted Rs 125 more than B, what number did each count?

25. At a municipal election 521 votes were given, and of the two candidates, the unsuccessful one had a minority of 31, how many voted for each?

26. Two persons have between them Rs 72, and one of them has Rs 15 more than twice the amount which the other has, how much has each?

27. Divide 349 among three persons, so that the first may have 15 more than the second and 13 less than the third

28. Divide Rs 209 between 2 persons, so that for every three rupees that one receives the other receives two and a half

29. Divide 51s between A and B, so that as often as A receives a crown, B receives 3 shillings and a half

30 Two persons together earned 5 guineas, for every shilling that one earned, the other earned a half-crown, how much did each earn?

Ex 7 *A's age is $\frac{2}{3}$ of B's, 5 years ago, his age was $\frac{3}{5}$ of B's, what are their present ages?*

Let x years $= B$'s present age,

then $\frac{2}{3}x$ years $= A$'s present age

Now 5 years ago, A's age was $(\frac{2}{3}x-5)$ years, and B's age was $(x-5)$ years

by the condition of the problem,

$$\frac{2}{3}x-5=\frac{3}{5}(x-5),$$

whence $x=30$ years $= B$'s age

$\frac{2}{3}$ of 30 yrs $=20$ years $= A$'s age

Examples LIII *(Continued)*

31 Eighteen years hence, A's age will be 3 times his present age, what is his present age?

32 A father's age is 35 years and his son's age is 10 years, when will the son's age be half that of his father?

33 A father's age is 45 years and his son's age is 17 years, when was the father's age 3 times that of his son?

34 A is 3 times as old as B, in 15 years he will be twice as old Find A's present age

35 A is 5 years older than B, 10 years ago $\frac{5}{6}$ of A's age exceeded $\frac{2}{5}$ of B's age by 15 years What are the ages of A and B?

36 The difference of the ages of a father, and his son is 25 years, in 12 years, the father will be twice as old as his son, how old are they?

37 The ages of A and B are together 54 years, $\frac{2}{5}$ of A's age 2 years ago was 7 years more than $\frac{1}{4}$ of B's present age Find their ages

38 The ages of A, B and C are together 4 times what B's age was 7 years ago, if A be 5 years older, and C 3 years younger than B, what is the age of each?

39 A's age is equal to the sum of the ages of B and C, 5 years hence 12 times A's age will be 11 times the sum of the ages of B and C Find the present age of A

40 A is twice and B 5 times as old as C, 2 years ago, B was twice as old as A and C together Find the age of each

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Ex 8 *A and B play with equal sums of money A gains Rs 17 and has then twice as much as B What sum did they begin with ?*

Let x = required sum of money, in rupees,

then $x+17$ = sum of money A has, when the game is over,

and $x-17$ = sum of money B has, when the game is over

By the condition of the problem, A's money is double of B's ;

$$x+17=2(x-17),$$

whence $x=51$

Examples LIII *(Continued)*

41 A and B began to play with equal sums of money, A lost Rs 12, then 15 times A's money was equal to 9 times B's, what sum had they at first ?

42 A has 5 more marbles than B, if he were to give B $\frac{2}{5}$ of his marbles, then B would have 11 more than A How many marbles has each ?

43 A has twice as much money as B, after giving B 12 rupees, he has only $\frac{2}{3}$ of what B then has, what had each at first ?

44. A has 250 rupees and B has 120 rupees, after A has given B a certain sum, B finds that he has 10 rupees more than $\frac{4}{5}$ of the money which A now has, how much did A give B ?

45 A and B have 860 and 1040 rupees respectively when they begin to play After the game is over A, who is the winner, finds that 3 times his money together with 5 times B's money, amount to 7 times the money which B had at first How much did A win ?

46 A has 50 rupees more than B, he pays B a third part of his money, and B pays back two-ninths of the money he then has, it is then found that B has 10 rupees more than A. What had each at first ?

Ex 9 *A post is $\frac{1}{4}$ in the earth, $\frac{2}{3}$ in the water, and 5 cubits above the water what is its length ?*

Let x = the length of the post, in cubits,

then $\frac{1}{4}x+\frac{2}{3}x+5$ = whole length of post $=x$,

whence $x=60$

Ex 10 *A labourer is engaged for 30 days, on condition that for every day he works, he shall receive 8 annas, and for every day he is idle he must pay a fine of 3 annas He receives Rs 10 3as in all How many days does he work?*

Let x = number of days, he works,

then $30-x=$, he is idle,

therefore he receives, as wages, $8x$ annas, and pays, as fine, $3(30-x)$ annas, hence his net receipt will be $8x-3(30-x)$, which is, by the condition of the problem, equal to Rs 10 3 annas or 163 annas

$$8x-3(30-x)=163,$$

whence $$x=23$$

Ex 11 *A bag contains Rs 365, in rupees and eight-anna bits, if the amount of the latter be less than that of the former by Rs 13, how many of each are there?*

Let x = amount of rupees in the bag,

then also x = *number* of rupees in it,

and $x-13$ = amount of eight anna bits in rupees,

$2(x-13)$ = *number* of eight-anna bits required

The total amount in the bag is Rs 365,

$$x+(x-13)=365,$$

whence $x=189$, the required number of rupees,

$2(x-13)=352$ = the required number of eight-anna bits

Note In solving problems concerning *coins*, be careful to remember that the *number* and the *value* of the coins are two *distinct* quantities Hence in an equation, the quantities involved must be either *all numbers* or *all values* When values are involved, be careful to express all in the *same denomination*

Ex 12 *P and Q set out at the same time from A and B respectively to meet each other, P walking 4 miles and Q 5 miles an hour, if the distance between A and B be 27 miles, when and where will they meet?*

Suppose Aa represents 4 miles and Bb 5 miles, then it is clear

A a O b B

that at the end of one hour P will be at a and Q at b, and the distance they will have jointly walked, will be $Aa+Bb=4+5$, or 9

miles Hence if x=number of hours after which they meet, and O the point of meeting, we have

$$4x = \text{distance } P \text{ walks} = AO,$$
$$5x = \text{distance } Q \text{ walks} = BO$$

But $$AO+BO=AB=27 \text{ miles},$$
$$4x+5x=27,$$

whence $$x=3,$$

i e, they meet after 3 hours

And $$AO=4x=12 \text{ miles},$$
$$BO=5x=15 \text{ miles},$$

i e, they meet at a place 12 miles from A or 15 miles from B

Note All problems, concerning space, velocity and time, depend on the Formula xiv of Art 92, viz.,

$$s=vt$$

For from this formula, *any two* of the three things being given, we can find the *third*

Examples LIII (*Continued*)

47 A post is a third of its length in water, a fifth of its length above water and 14 ft in the ground, how long is the post?

48 Two-thirds of the boys of a class are Hindus, one-seventh are Mahomedans and the rest consisting of 8 boys are Christians What is the total number of boys?

49 About one-half of India is under the English, one-third under the Allied Princes, $\frac{209}{1260}$ under independent kings and Hill Tribes, and the rest, about 1200 square miles, is Foreign possessions, what is its area?

50 A party consists of men, women and boys, the men are 2 more than $\frac{1}{6}$ of the whole party, the women 3 less than one quarter of the whole, and the boys 7 more than one-half of the whole How many are there in all?

51 A workman is engaged for 40 days, on condition that for every day he idles he has to pay a fine of 6*d.* He is idle for 10 days and receives £3 10*s* in all What were his daily wages?

52 A man is engaged to work on condition that for every day he works, he shall get 2 rupees and, for every day he idles, shall forfeit Re 1 2 annas If he works 16 days and gets Rs 25 4 as at the end of the stipulated time, for how many days was he engaged?

53 A bill of £21 10*s* was paid with half-crowns and half-sovereigns, and the number of half-sovereigns exceeded 3 times the number of half-crowns by 4, how many were there of each?

13 A market woman bought a certain number of eggs at 2 a penny and as many at 3 a penny, and sold them at the rate of 5 for 2*d*, thus losing 7*d* Find the number of eggs she bought

14 In a sea fight, the number of ships taken was 7 more, and the number burnt 2 fewer, than the number sunk, 21 escaped, and the fleet consisted of 7 times the number burnt Of how many ships did the fleet consist?

15 *A* and *B* start at noon to cycle to a place 7 miles off, *B*'s rate being $\frac{3}{4}$ of that of *A* *A* returning meets *B* at 12 40 P M Find the distance of the place of meeting from the starting point

[*The following are additional problems for exercise*]

16 Divide 200 into 2 such parts that their difference divided by the greater may be $\frac{2}{11}$

17 Find a number such that whether it be divided into 4 or 5 equal parts, the continued product of the parts shall be the same

18 What number is that to which if you add 1, then multiply the sum by 2, next from the product subtract 3, and lastly divide the remainder by 4, the quotient will be the sum of 1, 2, 3 and 4?

19 A person being asked his age replies—If I should live as many years more, half as many years more, and 10 years besides, I should have lived 100 years What is his age?

20 Divide Rs 62 among *A*, *B*, *C*, *D*, so that *A* shall have twice as much as *B* and 8 as more, *C* Rs 2 less than *A*, and *D* Rs 3 more than *B*

21 *A* and *B* have together Rs 57, *B* and *C* Rs 50, *A* and *C* Rs 53, what has each?

22 Divide Rs 800 among 3 persons *A*, *B*, *C*, so that *B* may have $\frac{3}{4}$ths of what *A* will get + 40 rupees, and *C* $\frac{1}{3}$ of what *A* and *B* will get together

23 Two shepherds owning a flock of sheep agree to divide its value *A* takes 72 sheep, and *B* takes 92 sheep and pays *A* £25 What is the value of a sheep?

24 How much water must a wine merchant mix with 50 gallons of wine at 12*s* a gallon, so that by selling the mixture at 10*s* he may gain £1?

25 Find two consecutive numbers such that the 4th and 11th parts of the less together exceed by 1 the 5th and 9th parts of the greater

26 From a basket of oranges, *A* takes one-third, *B* 20 and *C* the rest, it is then found that *A* has 3 more than *C* How many oranges were there?

27 From each of 16 coins an artist filed the worth of half a crown, and then offered them in payment for their original value, but being detected the pieces were found to be really worth no more than 8 guineas What was the original value of each coin?

28 I bought 2 dozen oranges for Re 1 $7a$, some at $1a$ $3p$ each and the rest at $9p$ each, how many of each sort did I buy?

29 A man gained Rs 40 by selling a horse for Rs. 500 and one-third as much as it cost him What was the cost price of the horse?

30 A man has been saving annually a quarter of his income, his income increased by Rs 50 and now he saves a third of his income and finds that he is saving annually Rs 25 more than before What was his original income?

31 Boys equal in number to $\frac{1}{5}$ of a class are promoted out of it and twice as many promoted into it also 4 new boys join the class. It is then found that the number of boys in the class has been increased by one-third Find the numbers in the old and new classes

32 How much gold at Rs 20 a tolah must be mixed with 14 tolahs of gold at Rs 15 a tolah, so that the compound may be worth Rs 18 a tolah?

33 A flock of ewes of which $\frac{1}{10}$ were barren and $\frac{1}{4}$ brought twins, produces 23 lambs Required the number of ewes, none being supposed to produce more than two

34 Three men, A, B and C entered into partnership, A paid in as much as B and $\frac{1}{3}$ of C, B paid in as much as C and $\frac{1}{3}$ of A, and C paid £10 and $\frac{1}{3}$ of A, what did each contribute to the stock?

35 The fore-wheel of a carriage makes 12 revolutions more than the hind-wheel in 130 yards, if the diameter of the latter be $\frac{3}{10}$ as much again as the diameter of the former, find the circumference of each wheel. [See Art 92, vi]

36 Round the edge of a rectangular court which is half as long again as it is broad, runs a path 3 ft wide If the area of the path is 1584 sq ft, find the dimensions of the court

37 A number is formed of 3 consecutive digits, that in the units place being the least of the three, the excess of the number above one fourth of the number obtained by reversing the digits is 36 times the sum of the digits Find the number

CHAPTER X

SIMULTANEOUS LINEAR EQUATIONS

95 Simultaneous Equations If the equation

$$x+y=5 \qquad (1)$$

be considered as an equation in *one* variable, *viz*, x, its solution is

$$x=5-y \qquad (a),$$

and this is the *only solution* of the equation But if we consider

it as an equation in *two* variables, *viz.*, x and y, then it has an *infinite number of solutions*, for giving to y in (a) any value we please, we can get as many values of x as we like. Thus corresponding to the values 1, 2, 3, 4, ... of y, we get for x the values 4, 3, 2, 1, ...

Hence a *single* equation involving *two* variables is **indeterminate,** *i.e.*, it admits of no definite solution.

The same remark holds good in the case of any other *single* equation in *two* variables, as for instance

$$x-y=1 \qquad (2)$$

But if it is known that the x and the y of (1) are the same as the x and the y of (2), then and then only, the two equations (1) and (2) *hold together* and are called *Simultaneous Equations*.

In this case they have *definite solutions* as we shall presently shew.

What we have said above will be made very clear in Chapter XI. There it will be shewn that all simple equations in two variables can be graphically represented by straight lines and that each of these equations is satisfied by an *infinite number of values*, *viz.*, the co-ordinates of the points through which its graph passes. It is only, at the point of their intersection that these equations are *simultaneous* and are satisfied by *definite values*, *viz.*, the co-ordinates of that point.

Definition. If two or more equations are satisfied by the same values of the variables, they are said to be **Simultaneous Equations**.

It is easy to see that the *general form* of all such equations in two variables, is $ax+by+c=0$ (3)

Note. Since simultaneous equations are satisfied by the same values of the variables, any equation obtained by combining them shall also be satisfied by those values.

96. Solutions of Simultaneous Equations. There are *two* methods of solving Simultaneous Equations. The one that is explained below is called the **Method of Elimination**, and the other is called the **Graphical Method** which will be explained in a subsequent Chapter. [See Art. 124.]

To solve the simultaneous equations (1) and (2) of Art. 95.

From (1), $x=5-y$ ... (a),

from (2), $x=1+y$ ... (b).

Since (1) and (2) are simultaneous equations, the x of (a) is the *same* as the x of (b). Hence

$$5-y=1+y,$$

an equation in *one* variable only Solving this equation, we get

$$y=2 \quad \ldots \quad \ldots \quad . \qquad (c)$$

From (*a*) or (*b*), by substituting the value of y from (*c*), we have

$$x=3$$

Hence it is clear that to solve linear equations in two variables we **eliminate** (*i e*, cause to disappear) *one of the variables from the given equations, thus finding a single equation in one variable* which we solve by the methods of Chapter VIII. We then substitute the value of this variable in **either** of the proposed equations, thereby reducing it to an equation in *one* variable and thus solve it

The different methods of *eliminating the variables* will be shewn in the next four articles

Examples LV

Eliminate x and y by turns from the equations

1. $x+y=10$, $x-y=15$
2. $x+3y=4$, $5x-4y=32$
3. $2x-5y=1$, $4x-3y=23$
4. $y=3x-1$, $3y-2x=11$
5. $y=5x+3$, $2x+y=10$
6. $y-4x=0$, $x-4y=3$
7. If $x=5$, find y from the equation $y=2x-3$
8. If $x=-4$, find y from the equation $3x-8y=4$
9. If $x=\frac{1}{3}$, find y from $6x+5y=16-12x$
10. If $y=3$, find x from $\frac{x-1}{2}+4y=13$
11. If $y=-\frac{1}{2}$, find x from $\frac{4y-1}{3}-\frac{2x+3}{4}=1$
12. If $y=3\frac{1}{3}$, find x from $\frac{2y-3x}{5}+\frac{4x+1}{3}=2$

97 First Method of Elimination—Cross Multiplication The principle is to *make the coefficients of the variable to be eliminated the same in both the equations* To do this, we generally multiply the first equation by the cofficient of the variable in the second equation, and the second equation by the coefficient of the *same* variable in the first equation

Then to eliminate the variable, *add* or *subtract* the resulting equations, according as the variable appears with *different* signs or with the *same* sign

Ex 1 Solve $$5x+2y=16 \qquad (1),$$
$$2x+9y=31 \qquad (2)$$

Multiply (1) by 2, the coefficient of x in (2), thus
$$10x+4y=32 \qquad (3),$$
multiply (2) by 5, the coefficient of x in (1), thus
$$10x+45y=155 \qquad (4),$$
now subtract (3) from (4), and x is eliminated, and we get
$$45y-4y=123, \text{ whence } y=3$$
Substituting this value of y in (1) or (2), say in (1), we get
$$5x+6=16, \text{ whence } x=2$$
∴ the required solution is $x=2$ and $y=3$

Note The same result would of course follow by eliminating y first instead of x

Verification When $x=2$ and $y=3$,

(i) $5x+2y=5\times2+2\times3=16$,

(ii) $2x+9y=2\times2+9\times3=31$

Thus both equations are satisfied

Ex 2 Solve $$5x+3y=78 \qquad (1),$$
$$x-13y=2 \qquad (2)$$

Here the coefficient of x in (2) is made equal to that of x in (1) by simply multiplying (2) by 5, thus
$$5x-65y=10 \qquad (3),$$
subtract (3) from (1), thus
$$3y+65y=68, \text{ whence } y=1$$
Substitute this value of y in (1) or (2), thus
$$x-13\times1=2, \text{ whence } x=15$$
Thus the required solution is $x=15$, $y=1$

Ex 3 Solve $$8x-13y=66 \qquad (1),$$
$$12x+3y=54 \qquad (2)$$

Here the coefficients 8 and 12 have a common factor 4, and $8=2\times4$ and $12=3\times4$ Hence to make the coefficients equal, it is enough to multiply (1) by 3 and (2) by 2, instead of by 12 and 8 as before

Thus $$24x-39y=198 \qquad (3),$$
and $$24x+6y=108 \qquad (4),$$
subtract (3) from (4), thus $39y+6y=-90$, or $y=-2$,

from (1), $8x=66+13y=40$, or $x=5$

Hence the complete solution is $x=5$, $y=-2$

Ex 4 Solve
$$\frac{x}{3}+\frac{y}{5}=7 \qquad (1),$$
$$\frac{x}{4}+\frac{2y}{3}=13 \qquad (2)$$

Multiplying (1) by $\frac{1}{4}$, $\frac{1}{12}x+\frac{1}{20}y=\frac{7}{4}$,

" (2) by $\frac{1}{3}$, $\frac{1}{12}x+\frac{2}{9}y=\frac{13}{3}$,

whence by subtraction, $\frac{2}{9}y-\frac{1}{20}y=\frac{13}{3}-\frac{7}{4}$,

multiplying by 9×20, $40y-9y=780-315$,

whence $y=15$,

from (1), $\frac{x}{3}=7-\frac{15}{5}=4$ or $x=12$

Otherwise thus —

From (1), $5x+3y=105$,

from (2), $3x+8y=156$,

now proceed as in other examples given above

Examples LVI

Solve the equations

1 $x+y=59$, $x-y=29$

2 $x-y=0$, $x+y=15$

3 $x-2y=2$, $x+5y=23$

4 $4x-y=22$, $x+3y=25$

5 $3x+y=38$, $x+3y=42$

6 $5x-2y=26$, $2x-5y=23$

7 $8x+7y=17$, $10x-21y=51$

8 $40x+y=43$, $25x-8y=1$

9 $3x=4y$, $10x-8y=48$

10 $6x+5y=31$, $15x+2y=25$

11 $5x+13y=41$, $12x+2y=40$

12 $24x-7y=51$, $30x-5y=75$

13 $x+2y=24$, $y-2x=2$

14 $5x-3y=1$, $5y-3x=9$

15 $3x=26-2y$, $3y=5+4x$

16 $5x+4y=0$, $3x+10y=19$

17 $4x-5y=2$, $x+10y=68$

18 $\frac{x}{3}-\frac{y}{4}=0$, $\frac{x}{5}-\frac{y}{2}=-7$

19 $\frac{x}{6}+\frac{y}{15}=4$, $\frac{x}{3}-\frac{y}{12}=4\frac{3}{4}$

20 $\frac{x}{3}+2y=10$, $\frac{x}{4}-5y=-12$

21 $\frac{2x}{3}=\frac{3y}{4}$, $\frac{x}{18}+\frac{y}{16}=1$

Ex 2 Solve $$\frac{5}{x}+\frac{3}{y}=2 \quad (1)$$
$$\frac{7}{x}-\frac{4}{y}=\frac{1}{15} \quad (2)$$

Here it will be convenient to solve these equations by **finding the reciprocals** of the variables, *viz*, $\frac{1}{x}$ and $\frac{1}{y}$ instead of finding the variables themselves

Multiply (1) by 4, thus $$\frac{20}{x}+\frac{12}{y}=8 \quad (3)$$

Multiply (2) by 3, thus $$\frac{21}{x}-\frac{12}{y}=\frac{1}{5} \quad (4)$$

Add (3) and (4), thus $$\frac{41}{x}=8+\frac{1}{5}=\frac{41}{5}$$
$$x=5$$

Substitute x in (1), thus $$y=3$$

Note The beginner will do well to represent the reciprocals by other letters thus —Let $\frac{1}{x}=u$ and $\frac{1}{y}=v$, thus the proposed equations are

$$5u+3v=2 \text{ and } 7u-4v=\tfrac{1}{15}$$

Ex 3 Solve $$3x-\frac{8}{y}=17 \quad (1)$$
$$5x+\frac{12}{y}=22 \quad (2)$$

Put $\frac{1}{y}=u$, thus from (1) and (2), we have

$$3x-8u=17 \quad (3)$$
$$5x+12u=22 \quad (4)$$

Solving (3) and (4), we get $x=5$ and $u=-\frac{1}{4}$,
required solution is $x=5$, $y=-4$

Ex 4 Solve $$\frac{56}{x}+71y=440 \quad (1)$$
$$\frac{71}{x}+56y=353\tfrac{3}{4} \quad (2)$$

Here we have to find $\frac{1}{x}$ and y and we notice that the sum of the coefficients of $\frac{1}{x}$ and that of the coefficients of y are equal

Add (1) and (2); thus $\frac{127}{x}+127y=793\frac{3}{4}=\frac{3175}{4}$;

divide by 127; thus $\frac{1}{x}+y=\frac{25}{4}=6\frac{1}{4}$ (3)

Again subtract (2) from (1), thus

$$\frac{-15}{x}+15y=86\frac{1}{4}=\frac{345}{4};$$

divide by 15, thus $-\frac{1}{x}+y=\frac{23}{4}=5\frac{3}{4}$

From (3) and (4), by addition and subtraction, we get

$$2y=12 \text{ and } \frac{2}{x}=\frac{1}{2}$$

Examples LVII

Solve the equations

1. $x-8y=17$, $8x+y=201$
2. $2x-9y=11$, $3x-12y=15$
3. $10x-9y=1$, $11y-12x=1$
4. $x=42-8y$, $y=84-8x$
5. $10x+13y=210$, $13x+10y=204$
6. $\frac{2x}{3}=10-\frac{y}{2}$, $\frac{19y}{4}=5x-7$
7. $12x+\frac{y}{5}=98$, $12y-\frac{x}{4}=118$
8. $\frac{x+1}{y}=\frac{1}{2}$, $\frac{x}{y+3}=\frac{1}{3}$
9. $\frac{x}{3}=5-2y$, $\frac{2x-1}{5}-y+1=0$
10. $\frac{x+10}{3}=4y-3$, $\frac{y-2}{7}=x-5$
11. $2x-\frac{y-3}{5}=4$, $3y+\frac{x-2}{3}=9$.
12. $\frac{2x+3y}{6}+\frac{x}{3}=8$, $\frac{7y-3x}{2}-y=11$

Solve the equations

13 $\dfrac{x+y}{x-y}=9,$ $\dfrac{2x-y}{3x+8}=\dfrac{6}{23}$

14 $\dfrac{x+3}{x-8}=\dfrac{y+1}{y-4},$ $10x-13y=4$

15 $\dfrac{x}{8}+\dfrac{y}{9}=42,$ $\dfrac{x}{9}+\dfrac{y}{8}=43$ [See Ex 4]

16 $\dfrac{x}{4}+\dfrac{y}{3}=4+\dfrac{x+y}{6},$ $\dfrac{x}{3}-\dfrac{y}{4}=3+\dfrac{x-y}{6}$

17 $\dfrac{x}{4}+\dfrac{y}{5}+1=\dfrac{x}{5}+\dfrac{y}{4}=23$ [See Ex 4]

18 $4x-\dfrac{15-x}{2}=\dfrac{30y}{12},$ $15x-8y=35-\dfrac{2x+5y}{5}$

19 $\dfrac{x-2}{5}-\dfrac{10-x}{3}=\dfrac{y-10}{2},$ $\dfrac{2y+4}{3}-\dfrac{2x+y}{8}=\dfrac{x+13}{4}.$

20 $\dfrac{1}{x}+\dfrac{1}{y}=8,$ $\dfrac{1}{x}-\dfrac{1}{y}=6$

21 $\dfrac{3}{x}+\dfrac{1}{y}=5,$ $\dfrac{5}{x}-\dfrac{4}{y}=14$

22 $\dfrac{4}{x}+\dfrac{6}{y}=3,$ $\dfrac{6}{x}+\dfrac{4}{y}=2\frac{5}{6}$ [See Ex 4]

23 $\dfrac{4}{x}+\dfrac{10}{y}=2,$ $\dfrac{3}{x}-\dfrac{2}{y}=\dfrac{11}{20}$

24 $\dfrac{8}{x}-\dfrac{5}{y}=1,$ $\dfrac{7}{x}-\dfrac{3}{y}=5$

25 $\dfrac{14}{x}+\dfrac{9}{y}=8,$ $\dfrac{21}{x}-\dfrac{3}{y}=4\frac{2}{3}$

26 $2x+\dfrac{3}{y}=4,$ $3x+\dfrac{2}{y}=5$ [See Ex 4]

27 $6x-\dfrac{2}{y}=10,$ $5x+\dfrac{3}{y}=13$

28 $69x-\dfrac{49}{y}=182\frac{1}{2},$ $49x-\dfrac{69}{y}=112\frac{1}{2}$ [See Ex 4]

29 $\dfrac{54}{x}-5y=38,$ $\dfrac{12}{x}+7y+24=0.$

30 $\dfrac{1}{5x}+\dfrac{y}{9}=5,$ $\dfrac{1}{3x}+\dfrac{y}{2}=14$

31 $\dfrac{5x}{3}+\dfrac{2}{5y}=7,$ $\dfrac{7x}{6}-\dfrac{1}{10y}=3$

32 $\dfrac{25}{x}+\dfrac{24}{y}=1,$ $20\left(\dfrac{2}{x}+\dfrac{3}{y}\right)=7$

Solve the equations

33 $\frac{1}{x}+\frac{1}{y}=\frac{5}{6}$, $3x+2y=2xy$

34 $\frac{2}{y}+\frac{9}{x}=\frac{11}{xy}$, $\frac{1}{x}+\frac{4}{y}=\frac{5}{xy}$

35 $\frac{4}{5x}+\frac{5}{6y}=5\frac{11}{15}$, $\frac{5}{4x}-\frac{4}{5y}=\frac{11}{20}$

36 $2x+3y=8xy$, $5x-y=3xy$

37 $\frac{x}{2}+\frac{y}{3}=\frac{3}{20}xy$, $\frac{y}{2}-\frac{x}{3}=\frac{2}{45}xy$

38 $\frac{x+y}{5}+\frac{x-y}{4}=6$, $\frac{x+y}{4}+\frac{x-y}{5}=6\frac{3}{20}$

[Put $x+y=u$ and $x-y=v$]

39 $\frac{2}{x-1}+\frac{3}{y+1}=2$, $\frac{3}{x-1}+\frac{2}{y+1}=\frac{13}{16}$

[Put $x-1=u$ and $y+1=v$]

40 $\frac{57}{x+y}+\frac{6}{x-y}=5$, $\frac{38}{x+y}+\frac{21}{x-y}=9$

[Put $x+y=u$, $x-y=v$]

41. $2y+\frac{3}{x}-4=5y+\frac{12}{x}+2=y-\frac{2}{x}+4$

42 $2x+4y=1.2$, $3.4x-.02y=.01$

43 $\frac{2}{x}+\frac{3}{y}=\frac{1}{4}$, $\frac{5}{x}-\frac{1}{y}=\frac{1.2}{3}$.

44 $2.4x+.32y-\frac{.36x-.05}{5}=.8x+\frac{2.6+.005y}{25}$, $\frac{.04y+1}{3}=\frac{.07x-1}{6}$

102 Simultaneous Equations in three variables If there are two equations involving three variables, we can eliminate one of the variables from the two equations, thus getting one equation in two variables We have seen [Art 95] that *one* equation involving *two* variables is indeterminate Hence two equations in three variables, cannot be definitely solved We shall see later on that to find *three* variables, *three* independent and consistent equations are necessary and sufficient In the meantime, the method of solution will be seen from the following example

Ex Solve

$$3x+5y-z=26 \qquad (1),$$
$$x-3y+4z=20 \qquad (2),$$
$$4y+3z-12x=8 \qquad (3)$$

Eliminate *one* of the variables from a pair of the above equations. Here we eliminate z from (1) and (2)

Multiply (1) by 4, and add the product to (2); thus

$$13x+17y=124 \qquad (4)$$

Again eliminate the *same* variable z from a different pair, say, from (1) and (3)

Multiply (1) by 3, and add the product to (3), thus

$$-3x+19y=86 \qquad (5)$$

Now (4) and (5) are two equations in two variables, solving which we get $x=3$ and $y=5$ Substitute these values in *any one* of the given equations, say (1), and we have

$$3\times3+5\times5-z=26\text{, whence } z=8$$

Thus the required solution is $x=3$, $y=5$, $z=8$

Examples LVIII

Solve the equations

1 $x+3y+2z=19$,
$2x+4y+z=22$,
$3x+2y+5z=39$

2 $2x+3y+4z=19$,
$8x+5y-3z=12$,
$5x-6y+3z=13$

3 $x+y-z=15$,
$y+z-x=16$,
$z+x-y=17$

4 $x-y+z=12$,
$x+y+z=16$,
$x+y-z=4$

5 $x-y+6z=8$,
$3x+2y-z=11$,
$2x+3y+6z=29$

6 $2x-y+z=9$,
$3x+y-4z=3$,
$6x+2y+z=15$

7 $3x-8y-7z=58$,
$4x-6y+3z=14$,
$9x+5y-4z=19$

8 $x+\frac{1}{2}(y+z)=27$,
$y+\frac{1}{2}(z+x)=29$,
$z+\frac{1}{2}(x+y)=30$

9 $x-y+z=3$,
$6x+2z=31+3y$,
$5x+3z=2(11+2y)$

10 $3x+2z=5y$,
$5x+y=3z+25$,
$2x+3y+4z=10$

11 $4(y-x)+22=5z$,
$3z+4x-2=6y$,
$3y-z+14=10x$

12 $x+y=2$,
$y+z=5$,
$z+x=13$

13 $x+2y=21$,
$y+2z=32$,
$z+2x=22$

14 $3x+4y=63$,
$8x+5z=60$,
$3z-y=0$

Solve the equations

15. $x+\frac{y}{3}=19,$
$y+\frac{z}{4}=17,$
$z+\frac{x}{5}=23.$

16. $x-\frac{y}{6}=17,$
$y-\frac{z}{2}=-14,$
$z-\frac{x}{7}=-10.$

17. $9x+8y=34,$
$4(1-2y)=5z,$
$5z=9x+2.$

18. $8x-3y=13,$
$4y+z=1,$
$3z+6x=3.$

19. $2x-y=3z,$
$3x-z=2y,$
$4x-7y-8z=19.$

20. $\frac{x}{2}+\frac{y}{3}+\frac{z}{4}=9,$
$\frac{y}{2}+\frac{z}{3}-\frac{x}{4}=5,$
$\frac{x}{2}+\frac{z}{3}=8.$

21. $\frac{1}{2}x+y+\frac{1}{2}z=162,$
$\frac{1}{2}x+\frac{1}{2}y=26,$
$5y=4z.$

22. $x-y+z=46,$
$13x-10y=65,$
$11y-5z=53.$

23. $5x-3y=4,$
$8y-9z=2,$
$15z-16x=10.$

24. $y+\frac{1}{3}z=\frac{1}{2}x+5,$
$\frac{1}{2}(x-1)-\frac{1}{3}(y-2)=\frac{1}{10}(z-3),$
$x-\frac{1}{3}(2y-5)=12-\frac{1}{12}z.$

103. The following equations are easily solved by finding the reciprocals of the variables as in Art. 101, Ex. 2. In some of them, however the reciprocals are not apparent and we must transform the equations to find them, as in the following example.

Ex. Solve $\frac{xy}{x+y}=1,\ \frac{xz}{x+z}=2,\ \frac{yz}{y+z}=3.$

Invert the given equations, thus

$$\frac{x+y}{xy}=1,\ \frac{x+z}{xz}=\frac{1}{2},\ \frac{y+z}{yz}=\frac{1}{3},\ \text{or}\ \frac{1}{x}+\frac{1}{y}=1,\ \frac{1}{x}+\frac{1}{z}=\frac{1}{2},\ \frac{1}{y}+\frac{1}{z}=\frac{1}{3} \qquad (a)$$

Add these equations together, thus

$$2\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)=\frac{11}{6}\ \text{or}\ \frac{1}{x}+\frac{1}{y}+\frac{1}{z}=\frac{11}{12} \qquad (b)$$

From (*b*) subtracting each of the equations (*a*), we get

$$\frac{1}{x}=\frac{7}{12},\ \frac{1}{y}=\frac{5}{12},\ \frac{1}{z}=-\frac{1}{12},$$

whence $$x=1\tfrac{5}{7},\ y=2\tfrac{2}{5},\ z=-12$$

Examples LIX.

Solve the equations

1. $\frac{1}{x}+\frac{1}{y}-\frac{1}{z}=5,\ \frac{1}{y}+\frac{1}{z}-\frac{1}{x}=7,\ \frac{1}{x}-\frac{1}{y}+\frac{1}{z}=9$

2. $\frac{x}{6}+\frac{3}{y}-\frac{4}{z}=9,\ \frac{x}{3}-\frac{2}{y}+\frac{5}{z}=6,\ \frac{x}{2}+\frac{4}{y}-\frac{1}{z}=31$

3. $\frac{1}{x}+\frac{2}{y}+\frac{3}{z}=20,\ \frac{2}{x}+\frac{3}{y}-\frac{5}{z}=-7,\ \frac{4}{x}-\frac{5}{y}+\frac{7}{z}=21.$

4. $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=\frac{1}{x}+\frac{3}{y}-\frac{2}{z}=\frac{3}{x}-\frac{1}{y}=\frac{3}{4}$

5. $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=\frac{1}{2x}+\frac{1}{4y}+\frac{4}{z}=\frac{5}{3x}+\frac{3}{4y}-\frac{1}{2z}=1$

6. $\frac{2}{x}+\frac{3}{y}=\frac{3}{x}+\frac{2}{z}=1,\ \frac{2}{y}+\frac{3}{z}=\frac{17}{24}$

7. $\frac{1}{x}+\frac{3}{y}-\frac{2}{z}=6,\ \frac{3}{x}+\frac{1}{z}=5,\ \frac{2}{y}+\frac{5}{z}=16$

8. $xy=x+y,\ yz=y+z,\ zx=z+x$

9. $\frac{xy}{x+y}=\frac{1}{5},\ \frac{xz}{x+z}=\frac{1}{6},\ \frac{yz}{y+z}=\frac{1}{7}$

10. $\frac{4xy}{2x-3y}+2=0,\ \frac{5yz}{4y+6z}=\frac{15}{13},\ \frac{2zx}{4z-3x}=\frac{2}{3}$

11. $15(x+y)=8xy,\ 40(y+z)=13yz,\ 24(z+x)=11zx$

PROBLEMS

104 The problems leading to Simultaneous Equations do not essentially differ from those of Chapter IX Here, as in that Chapter, we have to express symbolically each of the conditions of a problem in the form of an equation We shall thus obtain **as many equations as there are unknown quantities** we have to find , for a problem to be determinate must contain *conditions equal to the number of the unknown quantities* to be found out

Ex 1 *The sum of two numbers is 119, and the remainder when half the smaller is subtracted from the greater is 32 Find the numbers*

Let x=the greater number ,
and y=the smaller number

Then by the *first* condition of the problem,

$$x+y=119 \qquad \ldots \qquad (i)$$

Also by the *second* condition,

$$x-\tfrac{1}{2}y=32 \qquad \ldots \qquad (ii)$$

Subtract (ii) from (i); thus

$$\tfrac{3}{2}y=87, \text{ whence } y=58$$

Hence from either (i) or (ii), we have $x=61$

Thus the required numbers are 61 and 58

Examples LX.

1 Find two numbers such that their sum is 183 and their difference is 13.

2 The sum of two numbers is 65, and 3 times the smaller number subtracted from the greater leaves a remainder 1 What are the numbers ?

3 Five times the sum of two numbers is 235, and twice their difference is 34 Find the numbers.

4 What two numbers are those whose difference is 42 and the quotient of the greater divided by the less is 3 ?

5 Find two numbers such that the sum of the first and half the second is 18, and the sum of the second and half the first is 21

6 The sum of a certain number and $\frac{1}{6}$ of another is 30, and if $\frac{1}{3}$ of the first be subtracted from $\frac{2}{3}$ of the second the remainder is 21 ; what are the numbers ?

7 There are two numbers such that twice the greater added to half the less gives 113 , also if $\frac{1}{4}$ of the greater be subtracted from the less the difference is 5 Find the numbers

8 A number is divided into two parts such that their difference is 28 ; also $\frac{2}{3}$ of the greater part exceeds half the less by 1 What is the number ?

Ex 2 *What fraction is that which becomes $\frac{1}{3}$ when 1 is added to its numerator and $\frac{1}{4}$ when 1 is added to its denominator ?*

Let x=the numerator of the required fraction,
and y=the denominator ;

$$\frac{x}{y}=\text{required fraction} \quad \ldots \quad (i)$$

By the *first* condition of the problem, we have

$$\frac{x+1}{y}=\frac{1}{3} \qquad \text{(ii)}$$

By the *second* condition, we get

$$\frac{x}{y+1}=\frac{1}{4} \qquad \text{(iii)}$$

Solving (ii) and (iii), we find $x=4$ and $y=15$

Thus the required fraction is $\frac{4}{15}$

Verification $\frac{4+1}{15}=\frac{5}{15}=\frac{1}{3}$, thus the first condition is satisfied Again $\frac{4}{15+1}=\frac{4}{16}=\frac{1}{4}$, thus the second condition is satisfied Hence the solution is correct

Examples LX (*Continued*)

9 What fraction is that which becomes $\frac{1}{3}$ if 3 be added to its numerator, and $\frac{1}{5}$ if 5 be subtracted from its denominator ?

10 A fraction becomes 1 if 1 be added to the numerator, but if 1 be subtracted from the numerator and 2 added to the denominator, it becomes $\frac{1}{3}$ What is the fraction ?

11 A fraction is equal to $\frac{3}{4}$, and if 5 be added to its numerator, it becomes $\frac{1}{2}$ Find the fraction

12 A certain fraction is equal to $\frac{2}{3}$ If 10 be subtracted from the numerator and 6 added to the denominator, the resulting fraction is $\frac{1}{3}$ What is the fraction ?

13 If the numerator of a fraction be increased by 1 and the denominator diminished by 1, the result is 1 If the numerator be increased by the denominator and the denominator diminished by the numerator, the result is 4 Find the fraction.

14 If the denominator of a fraction be multiplied by 2, it reduces to $\frac{2}{5}$, but if $1\frac{1}{2}$ be subtracted from the numerator, it becomes $\frac{1}{2}$, what is the fraction ?

15 If 1 be subtracted from the numerator of a fraction, it becomes $\frac{1}{3}$, but if the numerator be multiplied by 2 and the denominator diminished by 4, it reduces to $\frac{5}{4}$ Determine the fraction

16 The sum of the numerator and denominator of a proper fraction, divided by their difference is $4\frac{1}{5}$, what is that fraction ?

Ex 3 *A farmer bought 30 oxen and 19 sheep for* Rs *597, and 25 oxen and 18 sheep for* Rs *504 What is the price of each ?*

Let x=price of each ox, in rupees,

and y= " sheep,

$$30x+19y=597 \qquad \text{(i)},$$

and $$25x+18y=504 \qquad \text{(ii)}$$

Solving (i) and (ii), we have

$$x=18 \text{ and } y=3$$

Thus the price of each ox is 18 rupees and that of each sheep is 3 rupees

Ex. 4 *Fifteen boxes and 8 trunks will exactly fill a room, but if the room were to be half as large again as it is, then 20 boxes and 14 trunks would be necessary to fill it up How many of each will separately fill the room?*

Let V = content of the room

x = ,, each box,

y = ,, trunk.

Hence by the first condition,

$$15x+8y=V \qquad \text{(i)},$$

and by the second condition

$$20x+14y=\tfrac{3}{2}V \qquad \text{(ii)}$$

Divide (ii) by (i), thus

$$\frac{20x+14y}{15x+8y}=\frac{\frac{3}{2}V}{V}=\frac{3}{2},$$

whence $x=\frac{4}{5}y$ (iii), $y=\frac{5}{4}x$ (iv)

From (i) and (iv), $V=15x+8\times\frac{5}{4}x=25x$, i e 25 boxes will fill the room

From (i) and (iii), $V=15\times\frac{4}{5}y+8y=20y$, i e, 20 trunks will fill the room

Examples LX. *(Continued)*

17 If 21 oranges and 15 apples together cost Re 1 13a, and 12 oranges and 16 apples together cost Re 1 8a, what is the price of each?

18 Eighteen sheep and 12 lambs cost Rs 378, and 15 sheep and 16 lambs cost Rs 364 8a Find the cost of each

19 If 16 ducks and 5 geese cost as much as 11 ducks and 7 geese at the same rate, find how many geese are worth 20 ducks

20 Some smugglers discovered a cave, which would exactly hold the cargo of their boat, viz., 13 bales of cotton and 33 casks of rum, while unloading, a custom-house cutter came in sight, consequently they sailed away with 5 of the bales and 9 of the casks, leaving the cave two-thirds full How many bales or casks would it hold?

21 There are 105 coins, each of which is either a florin or a half crown and their total value is £11 8s How many coins of each kind are there?

EX. 5 *A grocer bought for £8 two kinds of tea, the one at 4s and the other at 3s 4d a lb, after mixing them, he sold them at 4s a lb and thus gained 10 per cent How many lbs of each did he buy?*

Let x = required number of lbs of tea at 4s,

and y = " " at 3s 4d

Then $4x + \frac{10}{3}y = £8 = 160s$,

and $(x+y)4$ = selling price $= \frac{110}{100} \times 160s = 176s$

By solving these we have $x=30$ and $y=24$

Thus the grocer bought 30 lbs at 4s and 24 lbs at 3s 4d

EX. 6 *A fruiterer bought for Rs 4 oranges at 3 an anna and guavas at 5 an anna He sold $\frac{3}{4}$ of his oranges and $\frac{5}{6}$ of his guavas for Rs 3 4a, thus gaining 1 anna How many of each did he buy?*

Let x = number of oranges he bought,

and y = " guavas

Then $\frac{x}{3}$ annas is the cost of the oranges and $\frac{y}{5}$ annas is the cost of the guavas, and by the first condition of the problem, these two costs are equal to 64 annas

$$\frac{x}{3} + \frac{y}{5} = 64 \qquad \text{(i)}$$

Again the cost of $\frac{3}{4}x = \frac{\frac{3}{4}x}{3}$ annas $= \frac{x}{4}$ annas, and the cost of $\frac{5}{6}y = \frac{\frac{5}{6}y}{5}$ annas $= \frac{y}{6}$ annas, and by the second condition of the problem, these two costs are equal to 51 annas

$$\frac{x}{4} + \frac{y}{6} = 51 \qquad \text{(ii)}$$

Solving (i) and (ii), we get $x=84$ and $y=180$

Thus the man bought 84 oranges and 180 guavas

EX. 7. *A rectangular bowling-green having been measured, it was observed that, if it were 5 feet broader and 4 feet longer, it would contain 116 square feet more, but if it were 4 feet broader and 5 feet longer, it would contain 113 square feet more Required its area*

Let x = its length, in feet,

and y = its breadth, " ,

xy = its area, in square feet

By the conditions of the problem, therefore,

$$(x+4)(y+5)=xy+116\ ,$$
$$(x+5)(y+4)=xy+113\ ,$$

solving which we get $x=12$ and $y=9$ Therefore the required area $=(12\times 9)$ sq feet $=108$ sq ft

Examples LX. *(Continued)*

22 Two purses together contain Rs 1200 and one of them contains Rs 134 more than the other How many rupees are there in each purse?

23 Find 2 numbers such that if 5 times the greater be added to 4 times the less the sum is 351, and if their difference be multiplied by 48 the product is 432

24 Divide 300 into 3 parts such that $\frac{1}{3}$ of the first, $\frac{1}{4}$ of the second and $\frac{1}{5}$ of the third shall all be equal to one another

25 A father being asked by his son how old he was, replied—When we were 7 years younger, I was 4 times as old as you, and if we both live till we are 7 years older, I shall be twice as old as you. What was the age of each?

26 A man has two sons whose ages differ by 2 years Ten years ago the combined ages of the sons were $\frac{1}{3}$ of their father's age, and 8 years hence the father will be twice as old as the elder son. Find the present age of each

27 A person had two casks, the larger of which he filled with ale and the smaller with cider Ale being half a crown, and cider 11*s* per gallon, he paid £8 6*s* But if he had filled the larger with cider and the smaller with ale, he would have paid £11 5*s* 6*d* How many gallons did each hold?

28 A man spends 4 annas in oranges and 7 annas in apples and the number he gets of both together is 52 If he spends 6 annas in oranges and 8 annas in apples, he will have 4 more oranges than apples How many of each does he buy for an anna?

29 A person spent 12*s* 1*d* in buying apples at 3 for 2*d* and oranges at 9*d* per dozen Had he bought the same number of apples as he bought of oranges and the same number of oranges as he bought of apples, he would have spent 1*d* less Find the number of apples and oranges he bought

30 A dealer bought for Rs 105 sheep at 4 a sovereign and lambs at 9 a sovereign, and sold $\frac{4}{5}$ of the sheep and $\frac{2}{3}$ of the lambs for Rs 85, thus gaining Rs 5 by the transaction. How many of each did he buy?

31 Says *A* to *B*—If you give me 10 guineas of your money, I shall then have twice as much as you will have left But says *B* to

A—Give me 10 of your guineas, and then I shall have 3 times as many as you How many had each?

32 A bill of £26 5*s* was paid with half-guineas and crowns, and twice the number of half-guineas exceeded three times the number of crowns by 17 How many were there of each?

33 A bag contains shillings and half crowns of the value of £8 A third of the shillings are taken out and replaced by half crowns, when it was found that the coins in the bag amount to £9 10*s* How many shillings were there in the bag at first?

34 The area of a room would be increased by 60 sq ft, if each of the sides were 2 ft longer, but if the breadth were 1 ft longer and the length 4 ft shorter, the area would be diminished by 36 sq ft Find the sides of the room

35 In a mixture of wine and water, the wine was 7 gals more than $\frac{1}{5}$ of the mixture and the water was 10 gals less than $\frac{5}{11}$ of the mixture How many gallons of each were there?

36 When A was as old as B is now, he was twice as old as B was then Their united ages are at present 60 years How old are they?

105 Problems relating to Digits The symbols 0, 1, 2, 3, 4, 5, 6, 7, 8 and 9 are called **digits** Every digit when used alone expresses so many units and the number of units denoted is called its **absolute** value In Arithmetic *any number* is expressed by means of these digits by placing them side by side in a row, with this *convention* that each digit acquires a tenfold value as it moves one place to the left, the place of the units' digit being to the extreme right Thus each digit of a number has two values—an absolute value and a **local** value which is due to its place in the row Thus 6 denotes 6 *units* in 456, 6 *tens* in 465 and 6 *hundreds* in 645

In Algebra however there is no such convention to express a number, and therefore *every digit in Algebra has only an absolute value*

Thus 38 is represented *algebraically* by $30+8$, and therefore by $10x+y$, if $x=3$ and $y=8$ Similarly if $x=5$ and $y=4$, 54 is represented by $10x+y$, and so on

Hence the symbolical representation of a number of two digits is $10a+b$, where a represents the tens' digit and b the units' digit

Again 567 is expressed *algebraically* by $500+60+7$, and therefore by $100a+10b+c$, if $a=5$, $b=6$ and $c=7$

Hence the symbolical representation of a number of three digits is $100x+10y+z$, if x denote the hundreds' digit, y the tens' digit and z the units' digit The sum of the digits of this number is of course $x+y+z$ And so on

We have seen that 38 is represented by $10x+y$, where $x=3$ and $y=8$. If the digits are *reversed* we get 83 which is symbolically expressed by $10y+x$, the sum of the digits of either number being $x+y$.

If the student bear in mind that *in Algebra a digit has only an absolute value* he will have no difficulty in solving problems on digits. For if $x=4$ and $y=5$, 45 is represented in Algebra by $10x+y$ and never by xy (which has quite a different meaning). We now proceed to illustrate the above remarks.

Ex. 1. *A number, consisting of two digits, is equal to 4 times the sum of the digits, and if 18 be added to the number the digits will be inverted. Find the number.*

Let $x=$ the tens' digit,

and $y=$ the units' digit.

Thus $10x+y=$ the required number,

and $10y+x=$ the number formed by the inverted digits.

Also $x+y=$ the sum of the digits.

By the first condition, $10x+y=4(x+y)$;

and by the second condition $10x+y+18=10y+x$.

Hence from these two equations, we find the two variables $x=2$ and $y=4$. Thus the required number is 24.

Ex. 2. *Find a number such that if it is divided by the sum of its digits, the result is 6; but if 4 times the reciprocal of the units' digit be subtracted from 10 times the reciprocal of the tens' digit, the difference is 1.*

Let $x=$ the tens' digit,

and $y=$ the units' digit.

Thus $10x+y=$ required number,

and $x+y=$ the sum of its digits.

By the first condition, $\dfrac{10x+y}{x+y}=6$ (i),

and by the second condition, $\dfrac{10}{x}-\dfrac{4}{y}=1$ (ii).

From (i), $10x+y=6x+6y$, or $y=\dfrac{4x}{5}$ (iii).

from (ii) and (iii), $\dfrac{10}{x}-\dfrac{5}{x}=1$, whence $x=5$.

And from (iii), $y=4$.

Thus the required number is 54.

Examples LXI

1 A number of two digits is such that the difference between the digits is 3, and the sum of the number and the number formed by inverting the digits is 143 Find the number

2 A number consists of 2 digits and is such that if 18 be subtracted from it, the digits will be inverted, if the sum of the digits is 16, find the number

3 A number of two digits is equal to 6 times the sum of the digits, and if 9 be subtracted from the number, the digits will be reversed What is the number?

4 In a number of 2 digits, the difference of the digits is 4, and if 1 be subtracted from the units' digit which is the greater of the two, the new number will be 12 times the difference between its own digits Required the number

5 A certain number consisting of 2 digits becomes 110 when the number obtained by reversing the digits is added to it also the first number exceeds unity by 5 times the excess of the second number over unity What is the number?

6 A number consists of 2 digits, whereof the units' digit is greater than the other if the number be divided by the sum of its digits, the quotient is 4, but if the digits be inverted and the number thus formed divided by a number greater by 2 than the difference of the digits, the quotient is 14 Find the number

7 A number of 2 digits is multiplied by 4, and the product is less by 3 than the number formed by inverting its digits, if it be multiplied by 5, the tens' digit in the product is greater by 1 and the units' digit less by 2, than the units' digit in the original number Find the number

8 A number consists of two digits When the number is divided by the sum of the digits, the quotient is 7, and the sum of the reciprocals of the digits is 9 times the reciprocal of the product of the digits Find the number

9 A number is such that if you add 5 to the tens' digit and subtract 5 from the units' digit, the digits will be reversed, but if you subtract 5 from the units' digit only, the number will be 3 times the sum of the digits What is the number?

10 A certain number consisting of 2 digits is multiplied by 4 and the tens' digit of the product is one less and the units' digit one more than the units' digit in the original number, if it be multiplied by 5, the product is one less than the number formed by reversing the order of the digits of the former product Find the number

11 A number has 3 digits, the sum of which is 10, the first and third together exceed the second by 4, and the first and second together exceed the third by 8 Find the number

CHAPTER XI

GRAPHS

Introduction

106 Axes, Co-ordinates Let XOX' and YOY' be two fixed straight lines, intersecting at right angles in O They are usually drawn horizontally and vertically as shewn in the diagram, and are considered as lines of reference.

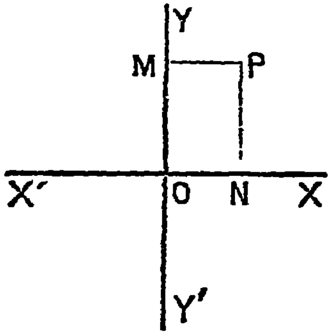

Fig 1

The position of any point in their plane will be determined if its distances from these lines are known For example, if P be a point such that its distance from YOY' is 3 units and that from XOX' is 4 units, we find P by measuring $ON = 3$ units along OX and $OM = 4$ units along OY and drawing the parallels NP, MP, whose point of intersection will be the point P, for PM (distance of P from OY) $= ON = 3$ units and PN (distance of P from OX) $= OM = 4$ units

The lines XOX' and YOY' are called the **axes of co-ordinates** or briefly the **axes**, XOX' being the **axis of x** and YOY' the **axis of y** The point O, where the axes intersect, is called the **origin**

The axes divide the plane of the paper into *four* compartments, called **quadrants**, XOY being the *first* quadrant, YOX' the *second*, $X'OY'$ the *third* and $Y'OX$ the *fourth*

The distance of P from YOY' *measured along the axis of X* (as ON here), is called the **abscissa** of P, and that from XOX' *measured along a perpendicular* to OX (as NP here), is called the **ordinate** of P, and the abscissa and the ordinate together are called the **co-ordinates** of P

If symbols are used, the abscissa of a point is always denoted by x and the ordinate by y Thus the abscissa is often called **the x**, and the ordinate **the y**, of a point, thus here *the x* of P is 3 and *the y* of P is 4

A point whose co-ordinates are a and b is briefly called "the point (a, b)", so also P is the point $(3, 4)$

Note Observe that the x of a point is *always put first* and then its y. Hence the point P cannot be denoted by (4, 3), but always by (3, 4)

107 Signs of co ordinates We have seen above how by means of its co-ordinates the point P has been found But to find P *uniquely* mere *magnitudes* of the co ordinates are not sufficient For by measuring $ON=3$ and $OM=4$ [Fig 2] on *either* side of the origin and drawing parallels, we might obtain the *four* points P, Q, R and S, the co ordinates of each of which are 3 and 4

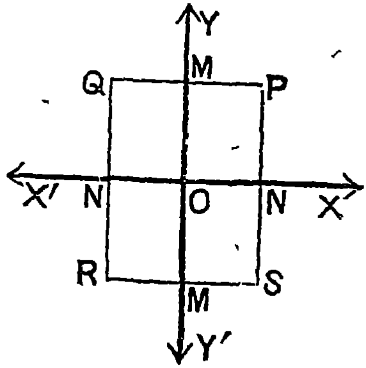

Fig 2

To be unique, each of the points P, Q, R, and S requires to have *separate* co-ordinates Hence the co-ordinates of each are distinguished by the use of the **positive** and **negative** signs

It is agreed that distances measured from O in the direction of OX, *i e*, **to the right**, are *positive* and those measured in the direction of OX' *i e*, **to the left**, are *negative* Also distances measured from O towards OY, *i e*, **upwards**, are *positive*, while those measured towards OY' *i e*, **downwards** are *negative*

Hence P is the point $(+3, +4)$, Q is the point $(-3, +4)$, R is the point $(-3, -4)$, and S is the point $(+3, -4)$ We need not however put the + sign before a co ordinate, thus S is the point $(3, -4)$

The following table gives the signs of the co-ordinates in the several quadrants

I	II	III	IV
+, +	−, +	−, −	+, −

And *conversely*, the signs of the co ordinates indicate the quadrant in which a point is

Note Observe that *all ordinates above* the x axis are *positive* and *all below* are *negative*, and *all* abscissae to the *right* of the y axis are *positive* and *all* to the *left* are *negative.*

PLOTTING OF POINTS

108 Points in the four quadrants We have seen [Art 106] how by drawing parallels we can determine a point whose co-ordinates are given

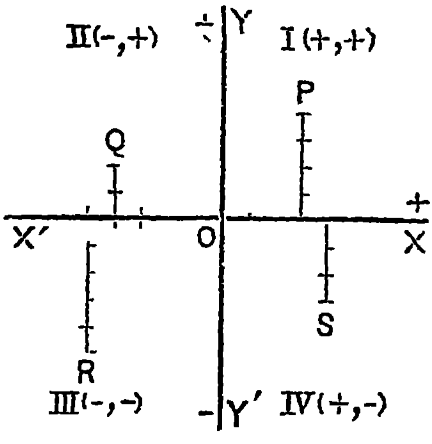

Fig 3

In practice, however, the point P is conveniently found thus —Measure 3 units along OX and then 4 units *upwards* along a perpendicular to OX, we thus come to the point P (3, 4) P is in the *first* quadrant

Similarly other points are found

To find the point $(-4, 2)$

The point is in the *second* quadrant Measure 4 units to the *left* along OX' and then 2 units *upwards* along a perpendicular to OX' Thus Q is the point $(-4, 2)$

To find the point $(-5, -5)$

The point is in the *third* quadrant Take 5 units to the *left* and then 5 units *downwards* along a perpendicular to the OX' Thus R is the point $(-5, -5)$

To find the point $(4, -3)$

The point is in the *fourth* quadrant. Measure 4 units to the *right* and then 3 units *downwards* along a perpendicular to OX Thus S is the point $(4, -3)$

Definition To **plot a point** is to mark its position by means of its co-ordinates

To plot points (and indeed to do all graphical work), **squared paper*** is very convenient, and the student should always use it for this purpose

109 Special Points From Fig 1, it is seen that the point N, being on the x-axis, has *no ordinate* and the point M being on the y-axis has *no abscissa* Thus the co-ordinates of N are 3 and

* "Squared paper" can now be had of all book sellers and stationers In using it, slightly thicken two intersecting lines and take them for the axes A beginner is recommended to use a *tenth inch* squared paper

0, and those of M are 0 and 4 Hence **every point whose y (ordinate) is 0 lies on the x-axis**, and **every point whose x (abscissa) is 0, lies on the y-axis**

The origin, being the *point of intersection* of the axes, *lies on both the axes*, therefore its co-ordinates are $x=0$ and $y=0$, that is, **the origin is the point (0, 0)**

Examples LXII

1 Assuming the side of a square to be the unit, name the point from the diagram [Fig 4] whose co ordinates are

(*a*)	(5, 4)	(*b*)	(8, 10)	(*c*)	(4, −6)	(*d*)	(−7, −3)
(*e*)	(−3, 8)	(*f*)	(7, −7)	(*g*)	(6, 11)	(*h*)	(−8, −13)
(*i*)	(−5, −7)	(*j*)	(6, −14)	(*k*)	(−6, −6)	(*l*)	(−4, 4)
(*m*)	(8, 8)	(*n*)	(−9, 12)	(*p*)	(−10, −14)	(*q*)	(5, −9)
(*r*)	(−1, 13)	(*s*)	(3, 9)	(*t*)	(9, −3)	(*v*)	(−8, 7)

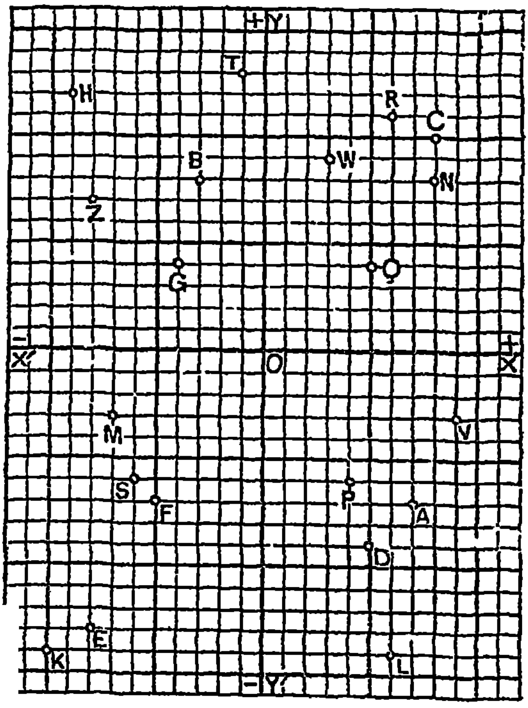

Fig 4

2 Name the co-ordinates of the points A, B, C, D, E, F, G, H, K, L, M, N, O, P, Q, R, S, T, V, W and Z, taking the side of a square as the unit [Fig 5]

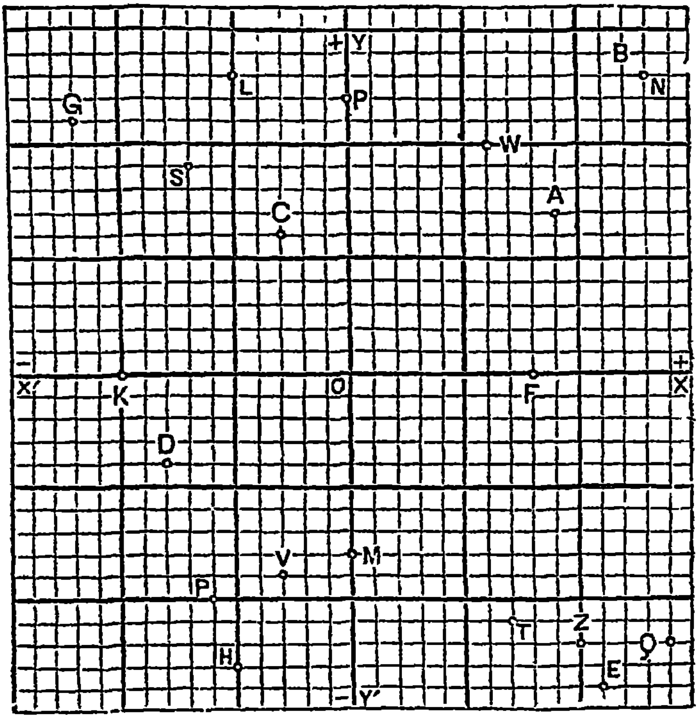

Fig. 5

Ex. 1 *Plot the points* (i) (2, 5), (ii) (−3, 4), (iii) (−6, −3) *and* (iv) (8, −7)

Take the side of a square as the unit of length

As in Art 108, we at once see that the pts are (i) A, (ii) B, (iii) C and (iv) D [Fig 6]

Ex. 2 *Plot the pts* (4, 0), (−3, 0), (0, −5) *and* (0, 6)

Take the side of a square as the unit of length

Thus the pts are E, F, G and H respectively

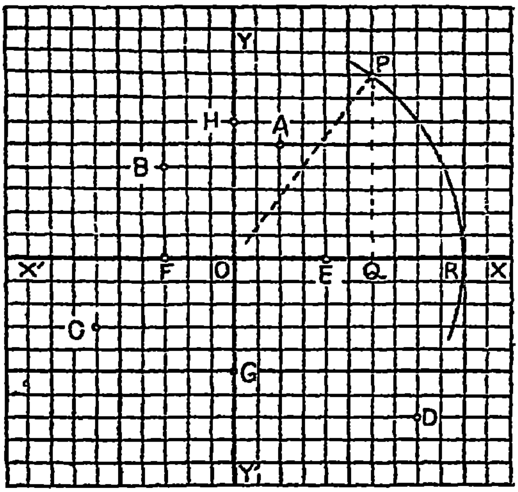

Fig 6

Examples LXII *(Continued)*

3 Plot the following points and name the quadrant in which each is

(*a*) $(4, 5)$ (*b*) $(4, -5)$ (*c*) $(-4, -5)$

(*d*) $(-4, 5)$ (*e*) $(-3, 7)$ (*f*) $(3, -7)$

(*g*) $(3, 7)$ (*h*) $(-3, -7)$ (*i*) $(0, 0)$

(*j*) $(0, 6)$ (*k*) $(5, 0)$ (*l*) $(0, -6)$

(*m*) $(5, 6)$ (*n*) $(0, -4)$ (*p*) $(-5, 0)$

(*q*) $(-4, -8)$, (*r*) $(3, -\frac{2}{3})$ (*s*) $(-\frac{5}{4}, -1)$

(*t*) $(0, \frac{8}{5})$ (*u*) $(5\frac{1}{3}, -2)$ (*v*) $(-2\frac{1}{4}, \frac{3}{4})$

Ex. 3 Plot the points $P(8, 5)$ and $Q(6, 11)$, and read off from the figure the co-ordinates of the point M which is on PQ Measure MP, MQ, they are equal What do you notice about the co-ordinates of M?

M is the mid point of PQ (Fig 7) Its co-ordinates are 7 and 8

The abscissa $7 = \frac{1}{2}(8+6)$

The ordinate $8 = \frac{1}{2}(5+11)$

Note Observe that *the abscissa of the mid point* of a line joining two points *is half the sum of the abscissæ* and *the ordinate is half the sum of the ordinates* of the given points

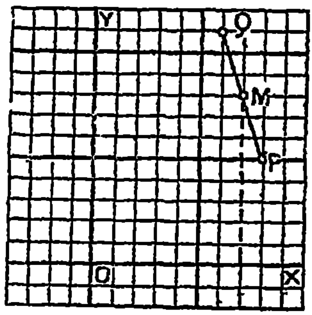

Fig 7

Examples LXII *(Continued)*

4 Plot the following pairs of points and find the co-ordinates of the middle points of their joining lines

(*a*) $(3, 5)$ and $(7, 9)$ (*b*) $(-4, 10)$ and $(-8, 6)$

(*c*) $(-5, -12)$ and $(-9, -8)$ (*d*) $(8, -11)$ and $(10, -7)$

(*e*) $(6, 6)$ and $(-6, -6)$ (*f*) $(3, -4)$ and $(-3, 4)$

(*g*) $(4, 6)$ and $(-4, 2)$ (*h*) $(-5, 8)$ and $(-7, -6)$

(*i*) $(-7, -8)$ and $(5, -4)$ (*j*) $(8, 7)$ and $(4, -11)$

5 Take 10 sides of a square (*i e*, an inch) as the unit and plot the points

(*a*) $(4{\cdot}3, -4{\cdot}2)$ (*b*) $(5{\cdot}5, 0)$

(*c*) $(5{\cdot}0, -0{\cdot}7)$ (*d*) $(-1{\cdot}8, -3{\cdot}5)$

(*e*) $(0{\cdot}2, -6{\cdot}0)$ (*f*) $(-3{\cdot}1, -0{\cdot}4)$

Ex 4. *Plot the point (6, 8), and find its distance from the origin* (i) *by measurement and* (ii) *by calculation*

It is easy to see [Fig 6] that P is the pt (6, 8)

Let the side of a square be taken as the unit

(i) To find OP by **measurement**

With centre O and radius OP, draw an arc of a circle cutting OX in R

Then $OP = OR = 10$ units

(ii) To find OP by **calculation**

Draw PQ perp to OX Thus from the right-angled $\triangle\ OPQ$, we have $OP^2 = PQ^2 + OQ^2 = 8^2 + 6^2 = 100$,

$$OP = 10 \text{ units}$$

Note From this example, it is evident that d, the distance of a pt $P(x, y)$ from the origin, is given by the equation $\mathbf{d^2 = x^2 + y^2}$

Ex 5 *Plot the points* (5, 4), (−3, 10) *and* (−10, −4), *and find the lengths of the sides of the triangle formed by joining them* (i) *by measurement and* (ii) *by calculation*

From Fig 8, we see that A is the point (5, 4), B is (−3, 10) and C is (−10, −4)

To find AB, AC and BC, the unit being the side of a square

(i) By **measurement**

With centre A and radius AB, draw an arc cutting the horizontal through A at D Thus $AB = AD = 10$ units

With centre A and radius AC, draw an arc cutting the horizontal through A at F Thus $AC = AF = 17$ units

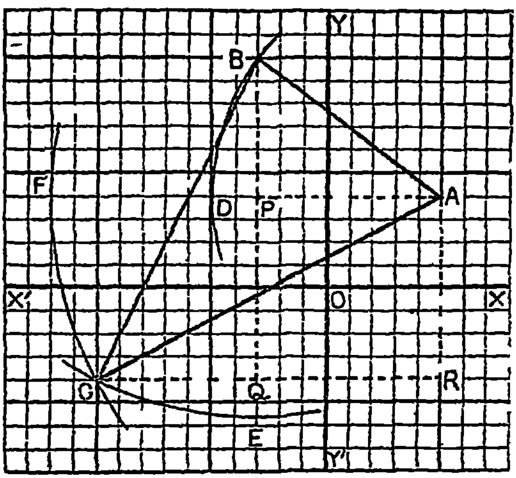

Fig 8

With centre B and radius BC, draw an arc cutting the vertical through B at E Thus $BC = BE = 15.7$ units nearly

(ii) **By calculation**

Through A, draw AP parallel to XX', meeting the ordinate of B in P.

Through C, draw CQR parallel to XX', meeting the ordinate of B in Q and that of A in R

Thus APB, ARC and BQC are right-angled Δs

$AP=8$, $BP=6$, $AR=8$ $CR=15$, $CQ=7$ and $BQ=14$

Hence $AB^2=AP^2+BP^2=8^2+6^2=100$, $\quad AB=10$

$AC^2=AR^2+CR^2=8^2+15^2=289$, $\quad AC=17$.

$BC^2=CQ^2+BQ^2=7^2+14^2=245$, $\quad BC=15.7$ nearly

Examples LXII. *(Continued)*

6 Plot the following pairs of points and find, by measurement or otherwise, the distance between each pair

(*a*) (6, 0) and (0 8), (*b*) (−5, 0) and (0, −12),

(*c*) (8, 0) and (0, −15), (*d*) (8, 10) and (5, 6),

(*e*) (0, 0) and (−8, 6), (*f*) (0, 0) and (−12, −5),

(*g*) (−5, −3) and (−10, −15), (*h*) (2, 5) and (10, −10),

(*i*) (−4, 3) and (−13, −9), (*j*) (7, −5) and (−5, −10)

7 Plot the point (12, −16) and find its distance from the origin

8 Plot the points (15, −20) and (−7, 24), and shew that they are equidistant from the origin Find the distance

9 Plot the following pairs of points (*a*) (5, 0), (0, 6), (*b*) (6, 0), (0, 5), (*c*) (0, 0), (5 6), and shew that the distance between each pair is the same

10 Plot the point (−6, −5), and find by counting squares,* the area of the rectangle formed by the perpendiculars drawn from it to the axes

11 Plot the points (5, 8), (−7, −4), (5, −4) and (−7, 8), and determine, by counting squares, the area of the rectangle formed by the lines that join them

12 Taking the side of a square to be the unit of length, plot the points (−5, 5), (4 5), (−5 −4) and (4, −4) Find the area of the rectangle formed by them by counting the units of area it contains

13 Plot the points (0, 0), (7, 0) and (7, 10), and find, by counting squares, the area of the triangle formed by joining them

Note In counting squares, count as *one* a square which appears to be *more* than half, and as 0 a square which appears to be less than half The square which appears to be half is taken as half

* *i e* squares of "squared paper" that you will use

14 Plot the points $(6, 2)$, $(-4, -2)$ and $(-4, -12)$, and determine, by counting squares, the area of the triangle whose vertices they are.

15 Draw the triangle whose vertices are $(12, 0)$ $(0, 0)$ and $(7, 8)$, and find its area (i) by counting squares and (ii) by calculation Shew that the triangle whose vertices are $(12, 0)$, $(0, 0)$ and $(-3, 8)$ is of equal area Prove this also geometrically

16 Find the perimeter of the triangle formed by joining the points $(0, 6)$, $(8, 0)$ and $(0, -15)$

17 Find the perimeter of the triangle whose vertices are $(5, 15)$, $(10\ 3)$ and $(-5, -5)$

18 Plot the series of points

$(0, 6)$, $(-4, 6)$, $(-7, 6)$, $(3, 6)$ and $(8, 6)$

and shew experimentally that they all lie on a line parallel to the x-axis

19 Plot the series of points

$(-4, -1)$, $(-4, 0)$, $(-4, 3)$, $(-4, -2)$, $(-4, -4)$ and $(-4, 7)$

and shew experimentally that they all lie on a line parallel to the y-axis

20 Plot the points $(5, -10)$ and $(-3, 6)$, and shew experimentally that they lie on a straight line through the origin Name from your diagram the co-ordinates of some other point on the line

21 Plot the points $(1, -3)$ and $(4, 3)$, and shew that the line through them passes through the point $(0\ -5)$

22 Taking the same unit and the same axes, plot the point $(3\ 13)$, $(-5, -11)$, and $(-1, 1)$, $(-3, -5)$, and shew that the lines joining them pass through the point $(0, 4)$

23 Plot the points $(1, 2)$ and $(3, 6)$, join them and produce the straight line Shew that it passes through the origin, and read off from the diagram on squared paper, the ordinate of the point on the line whose abscissa is 4, and the abscissa of the point whose ordinate is -10,

24 Plot the points $(4, 3)$, $(5, 5)$ and $(1, -3)$, and shew experimentally that they all lie on a straight line Name the abscissæ of the points on this line whose ordinates are 7 and -5, and the ordinates of the points whose abscissæ are 2 and 3

25 Plot the points $(2, 10)$, $(-8, -10)$ and $(-3, 0)$, and shew by ruling a line through any two that the third point lies on this line Write down the co-ordinates of one or two other points on this line

26 Find the values of y from the equation $y=3x+5$, when x has the values 0, 4, -2 and -3 Plot the points whose co-ordinates are each pair of values of x and y Mark that they all lie on a

straight line [It will be shewn hereafter that every equation of this form has for its graph a straight line See Art 119]

27 Plot the points $(-8, 15)$, $(0, -17)$, $(8, 15)$, $(-17, 0)$, $(-8, -15)$, $(8, -15)$ and $(0, 17)$, and shew that they all lie on the circumference of a circle whose centre is O Find the radius of this circle

LINEAR GRAPHS

110 Function A quantity which has not always the same value is called a **variable** Thus the population of a town, the rainfall in a district, the height of the barometer, &c are variables

A quantity which always retains the same value is called a **constant** Thus the length of a mile, the number of annas in a rupee, the ratio of the circumference of a circle to its diameter, &c are constants

An expression which involves x and whose value depends on that of x is called a **function of x** Thus x^3 is a function of x, for x^3 will have different values for different values of x Thus if $x=2$, $x^3=8$, if $x=3$, $x^3=27$, &c Similarly $\sqrt{x}$, mx, $2x+3$, x^2+px+q, &c are functions of x

The notation $f(x)$ is used to represent a function of x If necessary other letters may be used instead of f, thus $F(x)$, $\phi(x)$, &c, denote functions of x

If x is a variable, then $f(x)$ is also a variable quantity Hence if y be the value of a function of x, then in the equation

$$y=f(x),$$

the two variables x and y are so related that any change made in the value of x will produce a corresponding change in the value of y For this reason, x is called the **independent variable** and y the **dependent variable**

In the functions mx and x^2+px+q, the quantities m, p and q are constants if they are supposed to retain the same values while x changes from one value to another

111 Graphic Representation of a Function Let $f(x)$ be such that corresponding to the values 1, 2, 3, 4 and 5 of x, it has respectively the values 1, $1\frac{3}{4}$, $2\frac{1}{4}$, 4 and 6

Mark off $ON_1=1$, $ON_2=2$, $ON_3=3$, $ON_4=4$, $ON_5=5$, and draw the perpendiculars N_1P_1, N_2P_2, N_3P_3, N_4P_4, and N_5P_5 Join the points P_1, P_2, P_3, by a free-hand curve This curve therefore

represents $f(x)$, because for any value of x, say $ON_4=4$, the corresponding value of the function $f(x)$, represented by N_4P_4, is 4

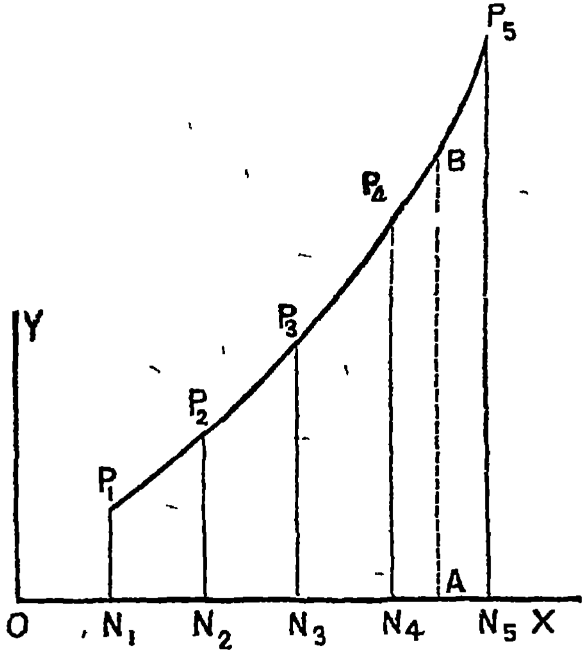

Fig 9

We thus see generally that *to every point N on the axis of x, there corresponds a point P along a perpendicular at N* And all the points P when plotted will be seen to lie on a line, straight or curved This line is called the **graph of the function** $f(x)$ or the **graph of the equation** $y=f(x)$ Thus the graph of any **function** $f(x)$ is the same as the graph of the **equation** $y=f(x)$

Hence to plot the gragh of a function, say $f(x)$, we give to x a series of values represented by abscissæ and find as many values of $f(x)$ represented by corresponding ordinates Each pair of these values will give the co-ordinates of a point on the graph Plot these points, join them by a line drawn free-hand, and the graph required is found Thus to find the graph of $2x$ Denote its value by y, so that $y=2x$

Now when $x=0, 1, 2, 3, 7, \quad , -1, -2, -4, -8,$

we have $\quad y=0, 2, 4, 6, 14, \quad , -2, -4, -8, -16, \quad .$

Thus we get the series of points $(0, 0), (1, 2), (3, 6), \quad (-2, -4)$, which being joined will give the graph required [See Art 118]

Since a graph shews to the eye the nature of the relation between two connected variables, we have the following

Definition A **graph** is a line drawn on a diagram so as to exhibit to the eye the nature of the relation existing between two connected variables

112 The graphical method of representing a function, in which we denote the different values of a variable x by *abscissæ* and the several values of its function by the corresponding *ordinates*, is very important Its advantage is that it enables us to see at a glance the value of a function, represented by the ordinate, corresponding to a particular value of the variable, and *vice versa*

Suppose $OA=4\frac{1}{2}$ and $AB=4\frac{3}{4}$ Let $f(x)=y$ Thus when $x=OA=4\frac{1}{2}$, $f(x)$ or y, represented by AB, is $4\frac{3}{4}$, that is, the ordinate AB represents the value of $f(x)$ or y, corresponding to the value $4\frac{1}{2}$ of x

Examples LXIII

Find the values of y corresponding to the values -4, -3, -2, -1, 0, 1, 2, 3 and 4 of x, and plot the points given by each pair of values What do you notice about the points?

1 $y=5x+3$ 2 $y+x=4$ 3 $y+2x=5$

4 $x-3y=1$ 5 $2x-4y+3=0$ 6 $2y=\dfrac{1-3x}{4}$

Find the integral values, between -15 and $+15$, of x and y that will satisfy the equations

7 $6y=5x$ 8 $2x+5y=3$ 9 $3y=\dfrac{x-1}{4}-3$

113 *The graph of the equation* $x=0$

The equation $x=0$ indicates that the abscissa of every point on the required graph is 0, but every point whose abscissa is 0 lies on the y axis [Art 109], thus the required graph is the axis of y

Hence **the graph of the equation $x=0$ is the axis of y**

Otherwise —Since there is no y in the equation $x=0$, y may have any value whatever Thus the co-ordinates of the pts (0, 0), (0, 1), (0, 12), (0, -5), &c all satisfy the equation $x=0$ When plotted, these pts will be seen to lie on the axis of y

114 *The graph of the equation* $y=0$

The equation $y=0$ indicates that the ordinate of every point on the graph is 0, but every point whose ordinate is 0 lies on the axis of x [Art 109], thus the graph required is the x-axis

Hence **the graph of the equation $y=0$ is the axis of x**

115 *The graph of the equation* $x=a$

For the sake of simplicity we first draw the graph of $x=4$

Take the side of a square to represent the unit

The equation $x=4$ indicates that every pt. on the required graph is at a distance of 4 units from the axis of y

Also as there is no y in the given equation y may have [Fig 10] any value whatever Thus the coordinates of the pts P (4, 6), Q (4,3), N (4, 0), R (4, −3), S (4, −7), all satisfy the equation $x=4$ When plotted all these pts will be seen to lie on the line PS which is parallel to the axis of y and is at a distance of 4 units from it

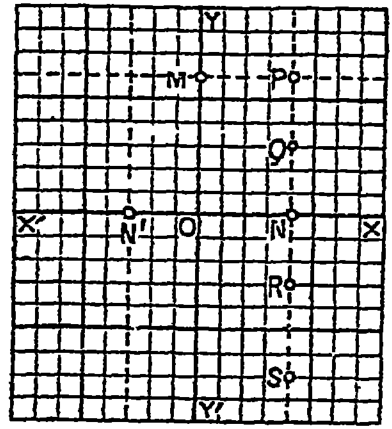

Fig 10

Similarly the graph of $x=-3$ will be the st line through N' parallel to the axis of y and at a distance of −3 units from it.

Hence generally **the graph of the equation $x=a$ is a line parallel to the axis of y** and at a constant distance of a units from it, where a is any number, positive or negative

116 *The graph of the equation* $y=b$.

For the sake of simplicity we first draw the graph of $y=6$

Take the side of a square as the unit

The equation $y=6$ indicates that every point on the required graph is at a distance of 6 units from the x axis

Also as x does not occur in the equation $y=6$, x may have any value whatever Thus [Fig 10] the co-ordinates of the points P (4, 6), M (0, 6), all satisfy the equation $y=6$ When plotted these points will be seen to lie on the line PM which is parallel to the axis of x and is at a distance of 6 units from it

Similarly the graph of $y=-4$ is a straight line parallel to the axis of x and is at a distance of −4 units from it.

Hence generally **the graph of the equation $y=b$ is a straight line parallel to the axis of x** and at a constant distance of b units from it, where b is any number, positive or negative

Note When $a=0$, the graph coincides with the y axis, hence the *graph of $x=0$ is the axis of y*

When $b=0$, the graph coincides with the x-axis, hence *the graph of $y=0$ is the axis of x*

Examples LXIV

1 Plot the graphs of $x=3$, $x=4$, $x=\frac{3}{5}$, $x=-2$, $x=-4$ and $x=-\frac{2}{3}$

Shew that two of the graphs are equidistant from the axis of y

2 Plot the graphs of $y=2$, $y=3$, $y=\frac{3}{4}$, $y=-5$, $y=-6$ and $y=-\frac{3}{4}$

Shew that two of the graphs are equidistant from the axis of x

3 Plot the graphs of $x=5$, $y=8$, $x=-6$ and $y=-12$, and find the area of the rectangle formed by them Also find the area of the rectangle formed by them with the axes in each of the quadrants [Take a square as the unit of area]

117 *To plot the graph of $y=x$*

Take the side of a square to denote the unit.

Since $y=x$, the ordinate of every point on the graph is equal to its abscissa, also $y=0$, when $x=0$ Hence (0, 0), (1, 1), (4, 4) $(-2, -2)$, $(-6, -6)$, are points on the graph Plot them and join them point to point and we get the straight line OP

Further the co-ordinates of *any point P* on the line, which are seen to be 7 and 7, satisfy $y=x$

Also 4 and 6 do not satisfy $y=x$, the point (4, 6) lies outside the line

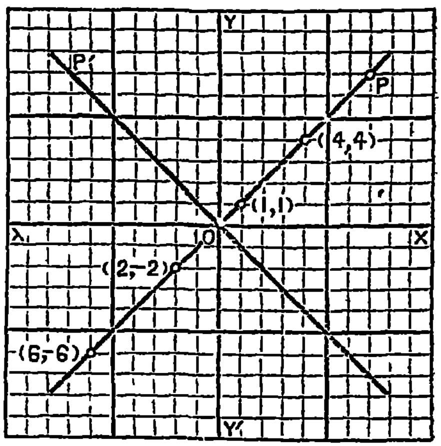

Fig 11

Hence *the graph of the equation* $y=x$ *is the straight line* OP *of unlimited length passing through the origin and lying in the first and third quadrants*

Note Similarly the graph of $y=-x$ will be seen to be the straight line OP' through the origin, and lying in the second and fourth quadrants

REMARK In shewing that a line is a graph of an equation, shew that (i) *every point whose co-ordinates satisfy the given equation lies on the line*, (ii) *the co-ordinates of every point on the line satisfy the given equation*, and (iii) *a point whose co ordinates do not satisfy the equation lies outside the line*

118 *To plot the graph of* $y=2x$

Tabulate the values of x and y

Values of x	0	1	2	3		-1	-2	-4	
Values of y	0	2	4	6		-2	-4	-8	

Take the side of a square for the unit

Thus (0,0), (1,2), (3, 6), $(-2,-4)$, are pts on the required graph Plot them and join them point to point, and we have the line (i) as the required graph

Let P be any point on this graph, its co-ordinates 5 and 10 satisfy the equation $y=2x$

Further 3 and 4 do not satisfy $y=2x$, the pt (3,4) lies outside the graph

Thus *the graph of* $y=2x$ *is the straight line* (i) *which passes through the origin and is of unlimited length*

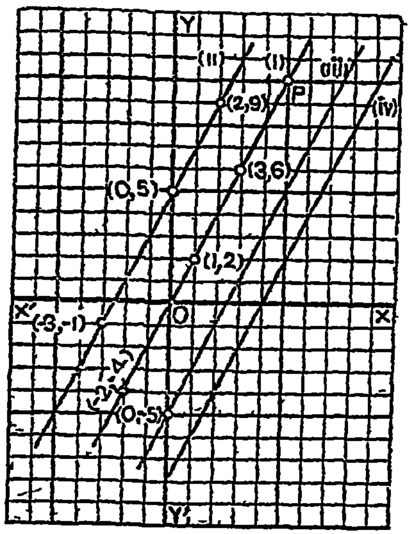

Fig 12

Note 1 Similarly the graph of $y=-2x$ will be seen to be a straight line passing through the origin

Note 2 Hence generally **the graph of the equation** $y=mx$ **is a straight line passing through the origin** where m is any number, positive or negative Also when m is *positive*, x and y have the same sign,

and the line is in the *first* and *third* quadrants, and when m is *negative*, x and y have different signs and the line lies in the *second* and *fourth* quadrants

Definition The number m is called the **slope** or **gradient** of the line whose equation is $y=mx$

119 *To plot the graph of the equation* $y=2x+5$

Tabulate the values of x and y

Values of x	0	2	-3	
Values of y	5	9	-1	

Take the side of a square to represent the unit, and plot the points whose co-ordinates are given by pairs of values in the same column and join them [Fig 12] Thus the graph required is the line (ii), for the co-ordinates of any point on this line will satisfy the equation and the point whose co-ordinates do not satisfy the equation lies outside the line

Hence *the graph of* $y=2x+5$ *is the straight line* (ii), *which passes through the point* (0, 5), thus cutting off an *intercept* of 5 units from the y-axis

Note 1 Thus generally **the graph of the equation** $y=mx+c$ **is a straight line,** *which cuts off an intercept of* c *units from the axis of* y

Note 2 The graph of the equation $y=2x+5$ is often called "the line $y=2x+5$"

Note 3 The ordinate of each point in (ii) is greater than the ordinate of corresponding point in (i) by 5 units, therefore the graph of $y=2x+5$ is parallel to the graph of $y=2x$ For the same reason, the graphs of $y=2x-5$ and $y=2x-8$ are parallel to the graph of $y=2x$ and to one another This is also evident from the diagram [Fig 12] where lines (iii) and (iv) are the graphs of $y=2x-5$ and $y=2x-8$

And generally if m and c are any numbers, the graph of $y=mx-c$ is parallel to that of $y=mx$

Hence **when equations differ only in the constant term, their graphs are parallel to one another**

Thus the graph of $y=mx+c$ is parallel to that of $y=mx+d$, but not to the graph of $y=-mx$ or of $y=-mx+c$, for the coefficients of x, though equal, have different signs

Note 4 The line $y=mx+c$ is fixed in position so long as both m and c are constants If c is constant and m has different values, the line will make different angles with the axis of x, but will still cut the axis of y at the same point (0, c) If m is constant and c has different values, the line will be parallel to $y=mx$, but will cut the axis of y at different points.

Hence the condition of parallelism is otherwise stated thus —**Two lines are parallel, if their equations have the same gradient**

Note 5 Comparing the graphs of the equations of the forms $y=mx$ and $y=mx+c$, we see that

If there is no constant term in an equation its graph passes through the origin *

Examples LXV

Draw the graphs of the following equations, shewing those of each group on one diagram

1 (a) $y=x$,	2 (a) $y=3x$,	3 (a) $2x+y=0$,
(b) $y=x+4$,	(b) $y=3x+4$,	(b) $2x+y=3$,
(c) $y=x-4$	(c) $y=3x-6$	(c) $2x+y+4=0$
4 (a) $y=-4x$,	5 (a) $y=-5x$,	6 (a) $2y=x+8$,
(b) $y=-4x-7$,	(b) $y+5x=6$,	(b) $x-2y=8$,
(c) $y=-4x+5$	(c) $y-5x+3=0$	(c) $6+2y=x$,
		(d) $x+2y+4=0$

120 We have seen that each of the equations $x=0$, $y=0$, $x=a$, $y=b$, $y=mx$ and $y=mx+c$ has for its graph a straight line; and it is easy to see that all these equations are included in the general form $y=mx+c$.

We know that every simple equation in two variables x and y can be reduced to the form $y=mx+c$, where c is the constant term.

Hence the graph of every simple equation in two variables is a straight line

For this reason an equation of the form $y=mx+c$ or $ax+by-c=0$, is called a **linear equation**, and an expression of the form $mx+c$ is called a **linear function** of x.

121 The student should now make himself familiar with the following statements

(i) **The origin is the point (0, 0)**

(ii) **Every point whose $x=0$, lies on the y-axis**, and **every point whose $y=0$, lies on the x-axis**

(iii) **The graph of the equation $x=0$ is the axis of y**

(iv) **The graph of the equation $y=0$ is the axis of x**

(v) **The graph of the equation $x=a$, where a is a constant, is a straight line parallel to the axis of y**

* This is true of equations of *any* degree

(vi) The graph of the equation $y=b$, where b is a constant, is a straight line parallel to the axis of x

(vii) If d is the distance of a point (x, y) from the origin, then $d^2=x^2+y^2$

(viii) The graph of an equation, which has no constant term, passes through the origin

122 Since we can draw a straight line when any two points on it are given, in drawing a linear graph we might *plot only two points* and then rule a line through them

But we must be careful to see that the points plotted be taken fairly far apart, otherwise a slight error in plotting will give the graph a considerable error in direction It is thus advisable to plot a *third* point to test the accuracy of drawing Hence to draw a linear graph *plot at least three points*

Ex 1 *Draw the graph of the expression* $\dfrac{3x-2}{5}$

Let $y=\dfrac{3x-2}{5}$, thus the graph of this equation will be the same as the graph of the given expression

When $x=0$, $y=-\frac{2}{5}$ and when $x=1$, $y=\frac{1}{5}$ Take as unit 5 divisions of squared paper and plot these two points Rule a line through them and the graph required is drawn

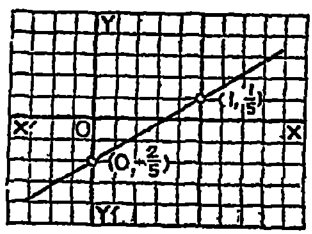

Fig 13

We have not plotted here the *test* point The student should however plot this point to see whether his graph is correctly drawn

Ex 2 *Draw the graph of the equation* $3x+2y=4$

The two points where a linear graph cuts the axes, being easily found from its equation, are sometimes convenient to choose in drawing the graph

Thus putting $y=0$, we get $x=\frac{4}{3}$, and putting $x=0$, we have $y=2$ Now taking 3 divisions of squared paper as unit, we draw the graph required

Note The graph of the given equation is also the graph of the function $\dfrac{4-3x}{2}$

Ex 3 *Plot the graph* of $6x-7y=11$

We have when $y=0$, $x=\frac{11}{6}$ and when $x=0$, $y=-\frac{11}{7}$ Thus the intercepts being fractions these two points are not convenient to

choose We therefore proceed as before to find by trial those co ordinates only which are *integral* It will be seen that when $x=3$, $y=1$, also when $x=-4$, $y=-5$ Thus the graph is found by ruling a line through these two points

Examples LXVI

Draw the graphs of the functions

1	$x+5$	2	$x-1$	3	$2x-3$
4	$3x+4$	5.	$3-4x$	6.	$5-2x$
7	$\frac{1}{3}x+2$	8	$\frac{1}{4}x+\frac{3}{2}$.	9	$\frac{1}{2}x+\frac{1}{3}$
10	$\frac{5-3x}{4}$	11	$\frac{5x-3}{8}$	12	$\frac{4-3x}{6}$

Trace the graphs of the equations

13	$x=2y$	14	$4y=3x$	15	$y=x=4$
16	$y+\frac{x}{2}=3$	17	$x+\frac{y}{3}=2$	18	$2y-x=4$
19	$3x-y=5$	20	$3y=4-5x$	21	$\frac{x}{3}-\frac{y}{5}=1$
22	$\frac{x}{4}=\frac{y}{5}-1$	23	$4x+3y=12$	24	$3x-5y=15$
25	$3x-5y=16$	26	$x=\frac{y-2}{7}+5$	27	$\frac{x-3y}{5}=y+2$

GRAPHICAL SOLUTION OF LINEAR EQUATIONS

123 *Graphical solution of the equation* $2x-8=0$

The graphical solution of such equations is more laborious than the algebraical one But we prefer to illustrate by this simple example the principle explained in Art. 112

Let $y=2x-8$ Now we have to find the value of x for which its function $2x-8$ or y is 0 Evidently $y=0$, *when the graph of* $y=2x-8$ *cuts the axis of* x

When plotted, the graph of $y=2x-8$ is the line (iv) in Fig 12, and we see that it cuts the x-axis at the point whose abscissa is 4 Hence 4 is the value of x for which $y=0$ Thus the solution is $x=4$

124 *Graphical solution of Simultaneous Equations* We know that the graphs of all simple equations in x and y are straight lines and that *two straight lines can intersect at but one point* Thus at the *point of intersection only* (and *nowhere else*) are the x and the y of

one of the equations, **the same** as the x and the y of the other equation, and therefore the equations are then said to hold *simultaneously*. Hence **the co-ordinates of the point of intersection satisfy both the equations and therefore represent their roots**.

Thus simultaneous equations of the first degree in two variables can be solved *by drawing their graphs and reading off the co-ordinates of the point of intersection from the diagram*.

Ex 1 *Solve graphically the equations*

$$4x+5y=7 \text{ and } 2x-3y+13=0$$

From the first equation, we have, when $x=3$, $y=-1$, and when $x=8$, $y=-5$. Plotting these points, and drawing a line through them, we get (i).

From the second equation, we get $y=7$, when $x=4$, and $y=1$, when $x=-5$. Plot the points and we have the line (ii).

They intersect at P, whose co-ordinates are -2 and 3. Thus the roots of the given equations are -2 and 3.

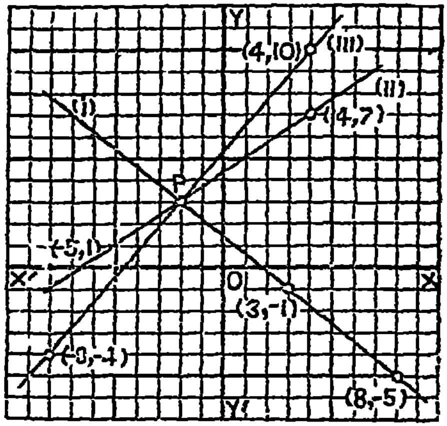

Fig 14

Ex 2 *Shew that the line $6y-7x=32$ passes through the intersection of the lines $4x+5y=7$ and $2x-3y+13=0$*

P is the point of intersection of the graphs of the last two equations [see Ex 1]

From the first equation, we have $y=-4$ when $x=-8$, and $y=10$ when $x=4$. Plotting these points, we get (iii), which is seen to pass through *P* [see Fig 14]

Otherwise —Since $6y-7x=32$ is to pass through $P(-2, 3)$, -2 and 3 will satisfy the equation $6y-7x=32$, and by substituting $x=-2$, $y=3$, we see that the equation is satisfied.

Ex. 3 *Shew that two other lines parallel to the axes also pass through P* [Fig 14] *and find their equations*

Since the co-ordinates of *P* are $x=-2$ and $y=3$, the parallels required are evidently the lines whose equations are $x=-2$ and $y=3$ respectively [Arts 115 and 116]

Ex 4 *Solve graphically* $\frac{7-4x}{5}=\frac{2x+13}{3}$

Draw the graphs of the functions $\frac{7-4x}{5}$ and $\frac{2x+13}{3}$

Let $y=\frac{7-4x}{5}$ and $y'=\frac{2x'+13}{3}$, thus we have

$$4x+5y=7 \qquad (a),$$

and $$2x'-3y'+13=0 \qquad (b),$$

which are identical with the equations in Ex 1

Plotting their graphs, we have the lines (i) and (ii) [Fig 14], which intersect at P Hence at P, the equations (a) and (b) are *simultaneous*, and therefore $y=y'$, $x=x'$ and $\frac{7-4x}{5}=\frac{2x+13}{3}$

Thus the abscissa of P satisfies the given equation

Hence the required solution is $x=-2$

Examples LXVII

Solve graphically the following equations

1 $y=x+5$, $y=2x+7$

2 $x+y=5$, $y=3x+1$

3 $x+y=7$, $2x-y=8$

4 $3x+2y=3$, $x-4y=15$

5 $y=3x$, $y+5x=16$

6 $4y=3x$, $5y=6x+9$

7 $4x+3y=12$, $5x+8y=15$

8 $y=x+5$ $4x+3y=15$

9 $2x+7y+8=0$, $5x-4y+20=0$

10 $y=\frac{2}{7}(11-x)$, $y=\frac{2}{7}(1-3x)$

11 $x=2y+4$, $y+\frac{3}{4}(x+6)=0$

12 $3x=9-2y$, $3y=4x+5$

13 Shew graphically that the line $y=x+1$ passes through the intersection of the lines $y=5x-15$ and $7y=3x+23$, and find the co-ordinates of the point where they intersect

14 Shew by drawing them that the lines $5x+3y=24$ and $3x-5y=28$ cut at right angles, and find the equations of two other lines, parallel to the axes, which pass through their intersection

15 Find graphically the co-ordinates of the point of intersection of the lines $3x-8y=52$ and $8x+3y=17$, and shew that they are perpendicular to each other

16 Shew by drawing them that the lines $2x+3y+18=0$, $3x-4y=7$ and $11y-4x+32=0$ meet at a point and find its co ordinates

17 Shew that the three points $(6, 2)$, $(-6, -6)$ and $(-3, -4)$ lie on a line Where does this line cut the axis ?

18 Find the equation of the line joining the origin to the point of intersection of the lines $3x-4y=13$ and $8x-11y=33$

125 We have hitherto shewn how to plot linear graphs from their equations The converse problem is, however, very useful, *viz*, *to find the equation of a linear graph having given two points on it*

Ex 1 Find the equation of the line passing through the points $(6, -2)$ and $(-3, 4)$, and shew that the point $(2, \frac{2}{3})$ lies on the line

Let $y=mx+c$ be the equation of the required line Since the graph passes through the point $(6, -2)$, its co ordinates will satisfy the equation, thus $\qquad -2=6m+c \qquad$ (i).

Again the graph passes through the point $(-3, 4)$, -3 and 4 will satisfy the equation, thus

$$4=-3m+c \qquad \text{(ii)}$$

Solving (i) and (ii), we get $m=-\frac{2}{3}$ and $c=2$, and thus the required equation is

$$y=-\tfrac{2}{3}x+2,$$

or

$$2x+3y=6$$

Since $x=2$ and $y=\frac{2}{3}$ satisfy $2x+3y=6$, the point $(2, \frac{2}{3})$ lies on the line

Ex 2 Find the equation of the graph that cuts off intercepts of 3 and 4 units on the axes of x and y

Let A and B be the points where the graph AB cuts the axes, so that $OA=3$ units and $OB=4$ units

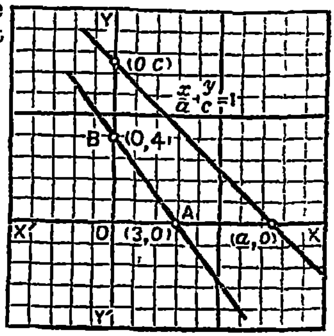

Fig 15

Let $y=mx+c$ be the equation of the graph required

Since the graph passes through A $(3, 0)$, the co-ordinates of A must satisfy the equation, thus

$$0=3m+c \qquad \text{(i)}$$

Similarly, the co ordinates of B $(0, 4)$ also satisfy the equation, thus

$$4=0m+c \qquad \text{(ii)}$$

From (i) and (ii), we get $m=-\frac{4}{3}$ and $c=4$

Substitute the values of m and c in $y=mx+c$, thus $y=-\frac{4}{3}x+4$, whence by dividing by 4 and transposing, the equation required is

$$\frac{x}{3}+\frac{y}{4}=1$$

Note 1. Hence generally if a and b be the intercepts cut off by a graph on the axes of x and y, its equation is $\frac{x}{a}+\frac{y}{b}=1$ [See Fig 15]

Note 2 The signs of a and b will determine the quadrant in which the line lies [see Art 107]

MISCELLANEOUS EXAMPLES LXVIII

1 Plot the points $(0, 0)$, $(-8, 0)$, $(-8, 8)$, and find by counting squares, the area of the triangle formed by joining them

2 Plot the series of points given below

(i) $(-7, 0)$, $(-7, 5)$, $(-7, -1)$, $(-7, 12)$, $(-7, -6)$;

(ii) $(-8, 9)$, $(0, 9)$, $(5, 9)$, $(-3, 9)$, $(6, 9)$,

(iii) $(0, -5)$, $(-9, -5)$, $(4, -5)$, $(8, -5)$, $(-1, -5)$,

and shew that they lie on straight lines parallel to the axes Find the area of the rectangle formed by them and one of the axes

3. Shew that the area of the triangle whose vertices are $(5, 4)$, $(7, 0)$, and $(0, 0)$, is 14 units of area Also shew that the triangle whose vertices are $(0, 0)$, $(7, 0)$, and $(10, 4)$ is of equal area

4. From the equation $y=3x-1$, find the values of y, corresponding to the values $0, 1, 2, -1, -2, \frac{1}{3}$, and $-\frac{2}{3}$, of x

5 Shew by a diagram that the points corresponding to the values of x and y, found in the last example, all lie on a straight line

6 Draw the graphs of the equation $3y=2x$ and $4y=5x$, and find the point of their intersection

7 Plot the points $(-5, -3)$, $(-3, -1)$, $(-2, 0)$, $(0, 2)$, $(2, 4)$, and $(4, 6)$, and shew from the diagram that they lie on a straight line

8 Draw the graphs of $3x+5y=15$, $4x-5y=20$, $2x+y+4=0$, and find the co-ordinates of the vertices of the triangle formed by them

9 Shew that the lines $3x-2y=19$, $4x-y=22$, and $2x+3y=4$, meet in a point, and find the co-ordinates of the point of intersection

10 Shew that two of the lines in the last Example are at right angles to each other

11 What integral values, between $+20$ and -20, of x and y will satisfy the equation $4y-\frac{x+10}{3}=3$? Draw its graph

12 Shew that the points $(-3, 5)$, $(2, -3)$ and $(7, -11)$, lie on a straight line, and find the point where it cuts the axis of y

13 Trace the graphs of the equations $y=3x$ and $y+3x=5$ Are they parallel?

14 *State* by inspection what are the intercepts cut off by the following lines

(i) $\frac{x}{2}+\frac{y}{3}=1$, (ii) $\frac{x}{2}-\frac{y}{3}=1$,

(iii) $-\frac{x}{2}+\frac{y}{3}=1$, (iv) $-\frac{x}{2}-\frac{y}{3}=1$

Name also the quadrant in which each is

15 Trace the graphs of $\frac{x}{4}+\frac{y}{5}=1$ and $\frac{x}{5}-\frac{y}{4}=1$, and shew from your figure that the lines include a right angle

16 Solve graphically the equations $3x+4y=36$ and $5x-y=14$, and find the intercepts that their graphs cut off on the axes

17 Solve graphically the equation $2-\frac{2x}{3}=\frac{3x-47}{5}$

18 Find the equation of the straight line which passes through the points $(0, 0)$ and $(\frac{3}{4}, 1)$

19 Find the equation of the straight line which passes through the points $(\frac{1}{2}, -4)$ and $(8, -\frac{7}{3})$

20 Find the equation of the line joining the point $(1, 1)$, to the intersection of the lines $3x+4y=2$ and $x-2y+5=0$

21 Find the co ordinates of the point at which the line $5x-3y+27=0$ cuts the line joining the points $(3, -5)$ and $(-5, 7)$

126 Measurement on different scales To draw a graph we have hitherto used the same scale for both x and y But sometimes it will happen that for small values of a variable x, the corresponding values of its function (represented by y) will rapidly increase In such cases, if we were to use the *same* scale for both x and y, the graph will assume an inconvenient size It is advisable therefore to use in such cases one scale for x and a far *smaller* scale for y, *i e*, for the variable *that is rapidly increasing* It will then be found that the required graph will be of good size

Example *Draw the graph of* $y=5x+4$.

We shall illustrate the above remark by drawing two graphs of $y=5x+4$, (i) using the *same* unit for both x and y, and (ii) using *different* units

Tabulate the values of x and y

Values of x	0	1	2	3	4
Values of y	4	9	14	19	24

(i) Take a small unit, say 0 1", for both x and y

Plotting the points given by the table we have the graph AB, which is very steep

If a larger unit were used, the diagram would be very large This the student can see for himself

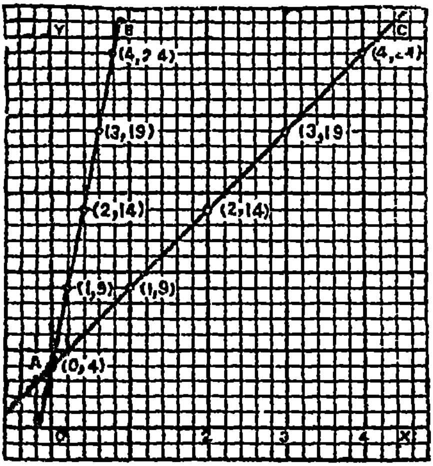

Fig 16

(ii) To remove the difficulty shewn above, we take half an inch as unit for x and one tenth of an inch as unit for y Now plotting the points from the table, we have the graph AC, which is of convenient size

127 We have remarked [Art 112] that the graphic representation of a function of x enables us from the diagram *to read off the value of the function for a particular value of x*, and conversely, *the value of x corresponding to a particular value of the function* We shall now illustrate the remark by an example

Ex Draw the graph of the function $\frac{5x+12}{4}$ and from the diagram read off its values (1) when x is 1 6 and (2) when x is 8

Also read off the values of x, (1) when the value of the function is 4 5 and (2) when it is 1 75

Take 10 divisions of paper or 1 0" to represent x-unit and 4 divisions or 0 4" to represent y-unit.

When $x=0$, $y=3$, we have thus the point (0, 3)

When $x=1$, $y=4\cdot25$, we have thus the point (1, 4·25)

Rule a line through these points and we have the graph $GDEB$.

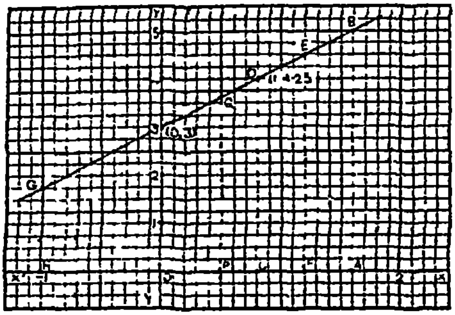

Fig 17

In the diagram

$OA=1\ 6$ and corresponding value of function is $AB=5$

$OC=\ 8$ and corresponding value of function is $CD=4$

Value of function is $4\ 5=EF$ and corresponding value of x is $OF=1\ 2$

Value of function is $1\ 75=GH$, and corresponding value of x is $OH=-1$

Definition **Interpolation** is the process of finding at a glance from the diagram the values of the variables at points other than those plotted Thus we have plotted here the points (0, 3) and (1, 4 25) and found by interpolation the values of x and y at other points, such as B, D,

Again suppose we have to find by interpolation the values of x and y at the point Q The value of x is $OP=\ 5$ and the corresponding values of y is $PQ=3\ 5+\frac{1}{2}$ of $\frac{1}{4}$ nearly $=3\ 62$ nearly

In finding $\frac{1}{2}$ of $\frac{1}{4}$, we judge by the eye what proportion of the unit the graph cuts Here the graph nearly bisects the side of the square which is $\frac{1}{4}$ of the unit

Examples LXIX

1 From the graph of the function $15x-9$, find its value when $x=2\ 2$

[Take 1 0" as unit for x and 0 1" as unit for y]

2 From the graph in Q 1, find the value of x for which (1) the function $=12$, and (2) the function $=0$

3. Draw the graph of the function $\frac{6x+7}{2}$ and read off its value (1) when $x=1.5$ and (2) when $x=0$. [Scale 1.0″ for both x and y.]

4. In Q. 3, find the value of x when the value of the function is 6.5.

5. Draw the graph of the function $\frac{56-5x}{4}$, and find from the graph (1) the value of the function when $x=1.6$, and (2) the value of x for which the function $=6$. [x-unit $=1.0''$ and y-unit $=0.1''$.]

6. In Q. 5, find for what value of x the value of the function is 0.

7. Draw the graph of the function $4x+13$, and from your diagram read off the value of x, when the value of the function is 18.

[Take the same units as in Ex. 1.]

8. Taking the same units for x and y as in Ex. 1, draw the graphs of $10x-8y=1$ and $x+15y=2$. Hence solve the equations graphically.

CHAPTER XII

APPLICATIONS OF GRAPHS

128. Graphical Problems. When two quantities are in proportion, the relation between them can always be represented by means of an equation. Thus if a man walks 4 miles an hour, he will walk 8 miles in 2 hours, 12 miles in 3 hours, and generally $4x$ miles in x hours. Hence if y miles is the distance he walks in x hours, then the relation between x and y is expressed by the equation $y=4x$.

Similarly, the relation between a man's work and his wages, between velocity and time, in fact between two quantities *which are directly proportional* to one another can be expressed by a simple equation. Hence *such relations as the above can always be exhibited by means of a* **linear** *graph*. Thus all problems in which quantities are directly proportional to one another can be solved graphically as the following examples will shew.

Ex 1 A man walks at the rate of 4 miles an hour, construct a graph of his motion

The man walks 4 miles an hour, therefore he walks $4x$ miles in x hours Hence if y represents the distance he walks in x hours, then $y=4x$ is the equation of the graph of his motion

We know that the graph of $y=4x$ is a straight line passing through the origin thus O is a point on the required graph Also when $x=5$, $y=20$, thus the point $(5, 20)$ is also on the graph Thus the required graph is the line through the points O and $(5, 20)$

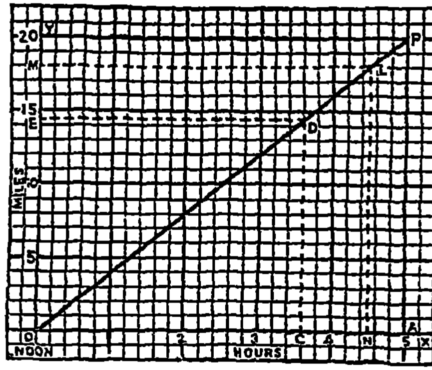

Fig 18

Measure *time* along OX, taking 5 divisions of the paper or half an inch to represent 1 hour, so that each division represents 12 minutes Measure *distance* along OY, taking 1 division or 0 1" to represent 1 mile*

Thus P is the point $(5, 20)$

Join OP, then OP is the graph of the man's motion By this is meant that *any abscissa represents the time and the corresponding ordinate the distance*, performed during that time, and *vice versa*

Otherwise —Since the man walks 4 miles an hour, he walks 20 miles in 5 hours

Take OA along OX equal to 5 hours and draw AP at right angles to OA equal to 20 miles, the scales being the same as before Thus P represents the position of the man at the end of 5 hours

Join OP, then OP is the graph required

[The student will satisfy himself by taking several points on the graph and seeing that the given condition is satisfied]

Remark From the graph we can readily find (i) the distance the man walks in a given time and (ii) the time in which he is at a given distance from the starting point

Question. *If the man starts at noon, read off from the diagram* (i) *the distance he walks in 3 hrs 36 min and* (ii) *the time when he is 18 miles from the starting point*

* In graphical problems, it is often convenient to measure x and y on different scales [See Art 126]

(i) OC represents 3 hours 30 minutes

Draw the ordinate CD which represents the required distance Thus distance required $=CD$ or $OE=14\frac{2}{3}$ miles Here the distance between E and the 15th mile mark is to be estimated by the eye as accurately as possible

(ii) OM represents 18 miles

Let ML be to OX Draw the ordinate LN Then ON represents the required time $=4\frac{1}{2}$ hours

Ex 2 Sugar is selling at the rate of 5 annas a seer, draw a graph to shew the relation between a weight and its corresponding price, and *vice versa*.

A seer of sugar costs 5 annas, therefore x seers cost $5x$ annas Hence if y annas denote the cost of x seers, then $y=5x$ is the equation of the graph required

The graph of this equation passes through the origin, thus O is a point on the graph

Also when $x=10$, $y=50$, thus the point (10, 50) is on the graph Thus the required graph is the line through the points O and (10,50)

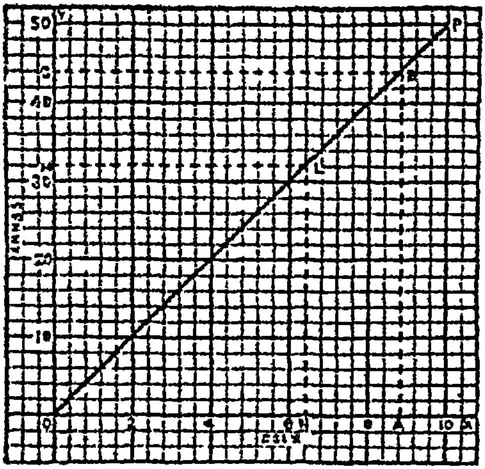

Fig 19

Now measure *seers* along OX, taking 5 divisions or half an inch, to represent 2 seers, and *annas* along OY, taking 1 division, or 0 1" to represent 2 annas

Thus with the above units, P is the point (10, 50)

Join OP Then OP is the required graph, that is, any *abscissa* represents the number of seers whose price is represented by the *corresponding ordinate*, and *vice versa*

[The units chosen are very convenient If the side of a square is taken as unit for both x and y, the graph would be inconveniently large as the student will see by drawing a figure See Art 126]

Remark The graph enables us to see at a glance (i) the price of a given number of seers, and (ii) the number of seers corresponding to a given price

Question *From the diagram read off* (i) *the price of* $8\frac{4}{5}$ *seers and* (ii) *the number of seers that can be had for 2 rupees*

(i) OA represents $8\frac{4}{5}$ seers

Draw the ordinate AB Thus AB or OC represents the required price which is seen to be 44 annas = R2 12a

(ii) OM represents 32 annas or 2 rupees

Draw ML to OX Draw the ordinate LN Then ON represents the quantity required which is seen to be $6\frac{2}{5}$ seers

Ex 3 If 54 English yards = 50 metres, draw a graph to shew the relation between yards and metres

Let x yards = y metres, then $\frac{x}{54}=\frac{y}{50}$

Thus $y=\frac{50}{54}x$ is the equation to the graph required

The graph passes through the origin, thus O is a pt. on the required graph Also $y=50$ when $x=54$, thus the point (54, 50) is on the graph. Hence the graph required is the line passing through the pts O and (54, 50)

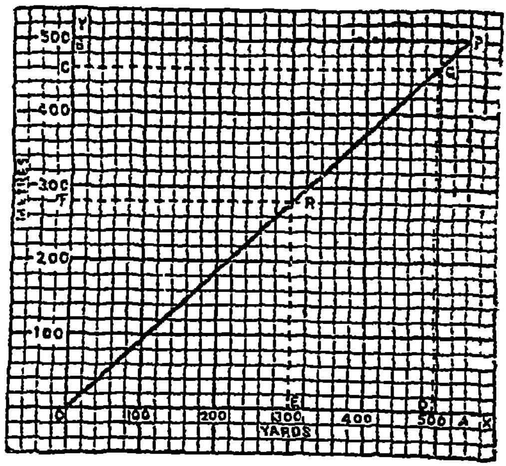

Fig 20

Measure *yards* along OX and *metres* along OY Let OA (i e, 27 sides of a square) represent 54 yds and OB (i e, 25 sides of a square) represent 50 metres, thus P is the pt (54, 50)

Join OP Thus OP is the graph required

This means that *any abscissa represents the number of yards equal to the number of metres represented by the corresponding ordinate*, and vice versa

Remark From the graph, we can readily find (i) the number of yards equal to any given number of metres, and (ii) the number of metres equal to any given number of yards

Question *From the diagram read off* (i) *the number of yards equal to 400 metres, and* (ii) *the number of metres equal to 300 yards*

In the diagram, take half an inch (i e, 5 sides of a square) to represent 100 yards along OX, and the same length to represent 100 metres along OY

(ii) OC represents 460 metres.

Draw CQ parallel to OX. Draw the ordinate QD. Thus OD represents the required number.

That is, 460 metres = 497 yards nearly.

(iii) OE represents 300 yards.

Draw the ordinate ER. Thus ER or OF represents the number required.

That is, 300 yards = 280 metres nearly.

Note. The student will notice that in drawing the graph we have used one set of units, and in solving the above question, a different set as best suited our purpose.

Ex. 4. A starts from O at the rate of 5 miles an hour, and after travelling for 2 hours rests for half an hour and then resumes his journey. Two hours later B starts from O and follows A at the rate of 8 miles an hour. When and where will B overtake A?

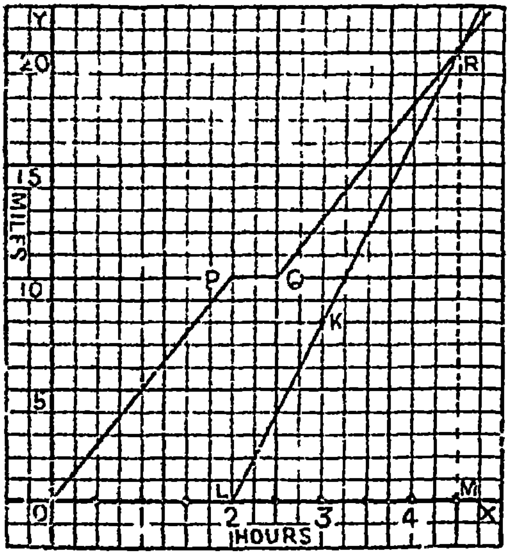

Fig. 21

Measure *time* along OX taking two divisions to represent half an hour.

Measure *distance* along OY, taking each division to represent one mile.

Thus for the first two hours OP is the graph of A's motion. As he rests at P for half an hour, *time advances*, but *his distance from O* (represented on paper by the ordinate of P) *remains the same*, therefore PQ is the graph for the half hour he stops.

His rate remaining the same, the graph of his motion has always the same direction. Hence QR drawn parallel to OP represents the graph of his motion for the remaining part of his journey.

B starts 2 hrs later at 8 miles an hour; KL is the graph of his motion.

The two graphs intersect at R. Hence B overtakes A at R.

From the diagram we see that

Required time $= OM$, i.e., $4\frac{1}{2}$ hrs after A starts, or $= LM$, i.e., $2\frac{1}{2}$ hrs after B starts.

Required distance $= RM = 20$ miles.

Ex. 5. Two men start at the same time, the one from A to B and the other from B to A, AB being 21 miles. If they walk at the rates of 3 and 4 miles an hour respectively, find (i) when they are 10 miles apart for the first time and (ii) when and where they meet.

Since the two men move in *opposite* directions, we take A and B in a *vertical* line.

Let 1·0″ represent 1 hr along AX and 0·1″ one mile along AY.

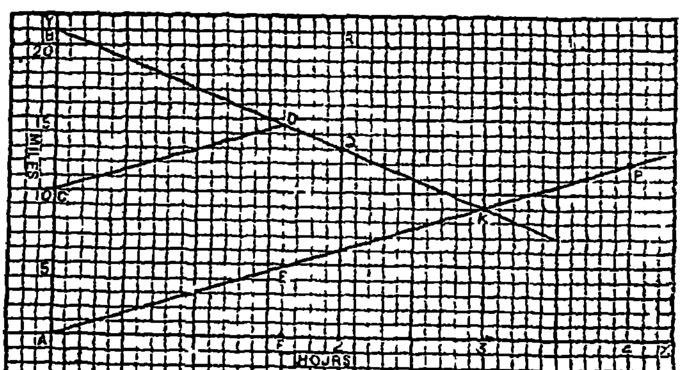

Fig. 22.

Since the first man goes 12 miles in 4 hrs., AP is the graph of his motion, i.e., the ordinate of any pt. in AP denotes the distance the man has walked in the time denoted by the corresponding abscissa.

In drawing the graph of the second man's motion, we measure

hours on the same scale along the horizontal line BM Thus $BR=$ 2 hours Hence the graph of this man's motion is BQ, i e, the distance from BM of any point in BQ denotes the number of miles he has walked in the time denoted by the abscissa of the point

Now draw CD parl to AP meeting BQ in D Thus DE is 10 miles Hence the distance between the men is 10 miles in the time $AF=1\,6$ hours

Again the two graphs cut at K, which indicates that the distance between the men vanishes at K, i e, they meet at K From the diagram we see that they meet in the time $AL=3$ hrs from start, and at the point whose distance from $A=KL=9$ miles and from $B=$the distance of K from $BM=12$ miles

Ex 6 A alone can do a piece of work in 6 hours and B alone in 4 hours In what time could A and B do the work together?

Let $0\,5''$ represent 1 hour along OX, so that $0\,1''=12$ minutes. Also let any distance OT along OY represent the work to be done

Through T, draw TP parallel to OX

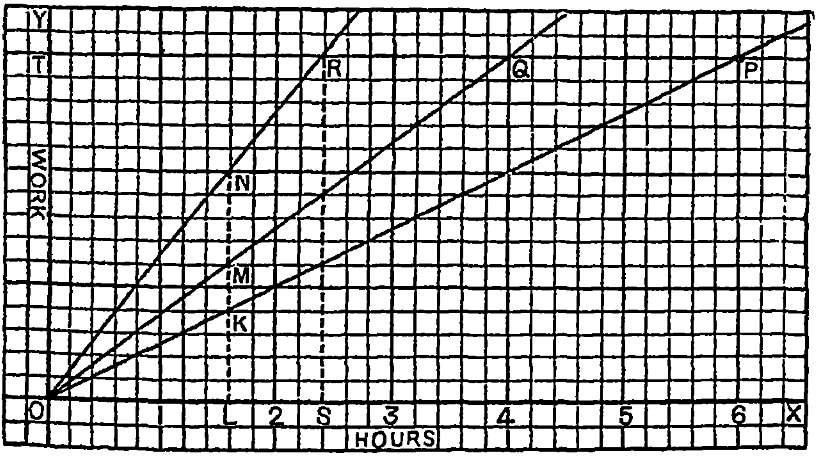

Fig 23

Since A can do the piece of work in 6 hours, OP is the graph of A's work, for $OT=$the ordinate of P at the 6th hour mark Similarly, since B can do the work in 4 hours, OQ is the graph of B's work.

We shall now plot the graph of their *combined* works

Take any point K on OP, and draw the ordinate KL Produce LK to cut OQ in M and make $MN=KL$ Thus ON is the graph of the combined works of A and B

For, by construction $NL = KL + ML$, and KL and ML are the works done by A and B respectively in the time OL

But the work done by two persons together in a given time is equal to the sum of the works done by each in that time

$\therefore$ NL represents the work done by A and B together in the time OL

Hence ON is the graph of the combined works of A and B

Let ON produced cut TP in R Thus $RS (= OT)$ represents the work done by A and B together in the time $OS = 2$ hours 24 minutes

$\therefore$ A and B can do the piece of work in 2 hours 24 minutes

Ex 7 A cistern is filled by a tap A in 30 min and emptied by a tap B in 45 min If the cistern is empty and both taps are opened at the same time, when will the cistern be filled?

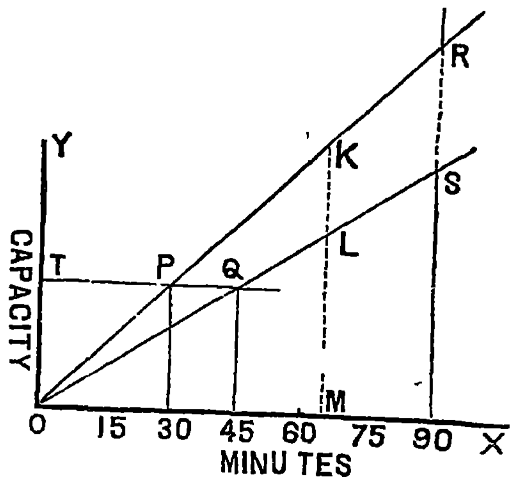

Fig 24

Let each division along OX represent 15 min, and let any convenient length OT along OY represent the capacity of the cistern

Now if TQ be drawn parallel to OX cutting the ordinates at the 30th and 45th min marks at P and Q, then OP and OQ are respectively the graphs of A's capacity for filling the cistern and B's capacity for emptying it, for then OT = ordinate of P = ordinate of Q

Draw any ordinate KLM Then KM represents the proportion of the cistern filled by A and LM represents the proportion emptied by B in the same time Hence KL represents the portion *actually filled* by the joint action of A and B in that time We thus see that *at any instant of time, the* difference *between the ordinates*

indicates the portion actually filled Hence the cistern will be full at the time when that difference $= OT$ (the capacity of the cistern)

Let a parl to OQ through T cut OP at R Draw the ordinate of R, which cuts OQ at S and is seen to pass through the 90th min mark

Now as $RS = OT$, the cistern is filled in 90 min.

Examples LXX.

1. Given that 31 miles are equal to 50 kilometres, construct a graph shewing the relation between a mile and a kilometre From the graph read off the number of miles in 240 kilometres and the number of kilometres in 125 miles

2. Given that £1 = 25 francs, construct a graph from which you can read off to the integer the value of 24*s* in francs, and the value of 24 francs in shillings

3. If 10 centimetres = 3 9 inches, draw a graph to convert centimetres to inches and inches to centimetres Read off from it the value of 2·6 inches in centimetres, and of 8 centimetres in inches

4. If 5 gallons = 22 7 litres, construct a graph to convert gallons to litres and litres to gallons Read off from the graph the value of 18 gallons in litres and the value of 100 litres in gallons

5. If milk sell at 5 seers a rupee, find graphically to the nearest anna the price of $18\frac{1}{2}$ seers Find also the number of seers that can be bought for R5 8*a*.

6. A clerk is employed on *R*25 a month Draw a graph from which you can readily calculate his daily pay taking a month = 30 days. Read off (1) his pay (to the nearest anna) for 22 days, and (2) the number of days for which he can be paid *R*15

7. In an examination, the highest marks awarded were 84 If these are raised to 100, find by means of a graph to the nearest integer the final marks of candidates who obtained 33 and 15 marks respectively.

8. A shopkeeper makes 25 per cent profit by retailing his articles Draw a graph shewing the relation between the cost and retail prices From the diagram, find the cost price of articles which were retailed at 25 annas, *R*3 12*a* and *R*17 8*a* , and the retail price of articles which cost 18*a* , *R*6 and *R*20 12*a*

[Take 4 divisions along OX to denote the cost price and 5 divisions along OY to denote the retail price]

9. If one maund of rice cost *R*6 4*a* , construct a graph to shew the price of any number of seers , and from the graph read off the price of 4 sr , 18 sr and $2\frac{1}{2}$ sr

10. The first 500 copies of a book cost 350 rupees to print , and every subsequent hundred costs only 20 rupees Draw a graph

from which you can read off the cost of 1120 copies and the number of copies you would get for 480 rupees

11 Two trains 180 miles apart approach each other at 35 and 40 miles per hour respectively Draw the graphs of their motion and from the diagram, read off the time when they meet and the distance of the place of meeting When are they 45 miles apart?

12 A man starts to walk 20 miles at a uniform rate of 3 miles an hour If he rests for 15 minutes after walking for 2 hours consecutively, find by means of a graph how long he takes to perform the distance

13 A and B start at noon from two places 23 miles apart to meet each other A walks 3 miles and B 2 miles an hour After walking $1\frac{1}{2}$ hours B reposes for half an hour and then goes on his journey again Find graphically (1) when and where they meet, and (2) when they are 14 miles apart

14 A starts at noon at 5 miles an hour Two hours later B starts from the same place at 15 miles in 2 hours Draw a graph from which you can find (1) when B overtakes A, (2) when A is 6 miles ahead of B, and (3) when B is 5 miles ahead of A

15 A starts at noon from O at 6 miles an hour One hour later B starts from O at 8 miles an hour, and 1 hour after B, C starts from the same place at 10 miles an hour Draw graphs of their motion, and from the diagram, find (1) when B overtakes A, (2) when C overtakes A, (3) when C overtakes B, and (4) the distances between A B and C at 2 30 p m

16 A can do a piece of work in 6 days and B in 8 days In what time could they do it working together?

17 A and B can do together a piece of work in 4 days, and A alone can do it in 6 days When can B alone do it?

18 A train 132 ft long is going at the rate of 30 miles an hour How long will it be in passing completely over a bridge 220 yds long.

19 A cistern has 2 pipes which can fill it separately in 24 and 30 minutes respectively If the cistern is empty, in what time will it be filled by the two pipes running together?

20 The supply pipe can fill a cistern in 20 minutes, while the waste pipe can empty it in 15 minutes In what time will the cistern be empty if when full, the two pipes be opened together?

129 The Method of plotting graphs is general We have seen that the graph of a linear function $f(x)$, i e, of a linear equation $y=f(x)$, is drawn by plotting the points that correspond to pairs of values of x and y obtained from the equation $y=f(x)$, and then drawing a line through these points. The method however is perfectly general, and graphs of functions of higher degrees than the

first, can be drawn exactly in the same way Thus to draw the graph of $y=x^2+1$, we first tabulate the values of x and y

Values of x	0	± 1	± 2	± 3	± 4	. .
Values of y	1	2	5	10	17	.

We thus get the points (0, 1), (1, 2), (2, 5), (−1, 2), (−2, 5), (−3, 10), Plot these points and join them by a free-hand curve, and the graph required is obtained

[The drawing of the curve is left as an exercise to the student]

From exercises below he will see that graphs of other functions of degrees higher than the first can be similarly drawn

Examples LXXI

Draw the graphs of

1 $y=x^2$ 2 $x=2y^2$ 3 $y=x(x-3)$

4 $y-1=(x-1)^2$ 5 $y=x^2-2x-3$

Plot the points given by the tables and draw a graph passing through them

6 $x=-11,\ -9,\ -5,\ 0,\ 7,\ 11,\ 16,$
$y=5,\ 1,\ -2,\ -4,\ -4,\ -3,\ 1$

7 $x=-8,\ -3,\ 0,\ 5,\ 10,\ 15,$
$y=2,\ 3,\ 5,\ 7,\ 6,\ 5$

GRAPHS OF STATISTICS

130 Graphs are often used to exhibit *statistics*, such as the daily height of the barometer, the rise and fall of the thermometer, the population of a country at regular intervals of time, and many other results of observation or experiment In these graphs units of time are generally marked along the axis of x while units of other magnitudes are shewn along the axis of y Thus from the data we plot a series of points which are usually joined by straight lines The broken line thus formed is the graph required

The graphs of statistics are not regular in form like algebraical graphs, and therefore no legitimate conclusions can be drawn as to the values of the variables at points other than those actually plotted If however the data are such as enable us to plot a number of points sufficiently near to one another, then the graph assumes the form of

a curve and then by *interpolation* we can estimate the values of the variables at a point *intermediate* between a pair of plotted points

131 Ex In Calcutta, the temperatures recorded every two hours on a certain day from 8 a m to 6 p m were

At—	8	10	12	2	4	6
Temp —	73°	75 5°	77°	82 6°	82°	79 5°

Draw a graph to shew the variations of temperature

Take 0 5″ along the axis of x to represent 2 hours and 0 1″ along the axis of y to represent 1° of temperature

The data give us the points A, B, C, D, E and F, and the required graph is the broken line $ABCDEF$

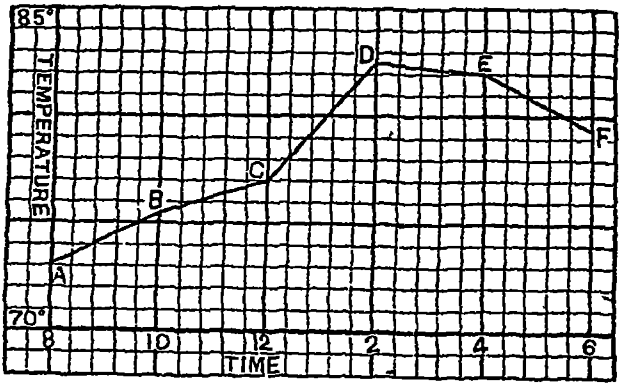

Fig 25

This graph however does not enable us to find *with any accuracy* the temperature at any other times If the temperatures had been taken every half-hour or every quarter of an hour, the plotted points would have been much nearer to one another and the graph would have been obtained by joining them by a curve Thus the graph would give a good idea of what the temperature was between any two readings, but even in this case our estimate would give at best an approximate value

Examples LXXII

1 A boy's position in class is given by the following table

Date—	Jul 1,	Jul 10,	Jul 15,	Jul 18,	Jul 20,	Jul. 23	Jul 30
Place—	4	6	1	1	5	3	7

Draw a graph to illustrate the above

2 The table below records the daily temperature of a patient suffering from remittent fever from the 8th to the 14th day

Day—	8	9	10	11	12	13	14
Temp—	104	103 5	103 8	104 2	102 6	103·4	99 2

Draw a chart to exhibit these variations

3 The average height, in feet and inches, of a boy is furnished by the table below

Age in years—	5	9	11	15	18	20
Height—	3 4	4 1	4 5	5.2	5 7	5.7

Plot a graph to shew the changes Can you guess the probable height in the 7th and 14th years?

4 The population (in millions) of a country at the end of the year noted is given by the following table

Year—	1895	1896	1898	1899	1901	1902
Pop—	30 4	30 7	31 4	31 7	32 5	32 8

Draw a chart from which you can read off (approximately) the population in 1897 and 1900 respectively

5 The following table gives the average weight of a healthy child in infancy during the months noted

Month—	first	third	sixth	tenth	twelfth
Weight—	8 lb	11 6 lb	15 8 lb	21 lb	22 lb.

Draw a graph to illustrate the variations and from it estimate the weight of a child in the 5th month When will a child be 20 lb. in weight?

6 In a Life Insurance Company, the premium to insure R1000 at various ages is given in rupees by the following table

Age—	20	25	30	35	40	45	50
Premium—	22	22 4	27 8	31 3	35 8	41 4	50 6

Draw a graph and estimate, to nearest rupees, the premium for persons whose ages are 23 and 29 years Also find the age of the person whose premium is R 33

7 The times (in hours and minutes) of forenoon high water in the Hughli at Calcutta on successive days are —2 56, 3 44, 4 32, 5 20, 6 8,—, 7 44, 8 32, 9 20,—, 10 56, 11 44

Exhibit these on a graph and deduce the times for the days omitted

8 The average weight in pounds of a boy and of a girl at various ages are shewn in the following table

Age—	0	2	4	6	8	10	12	14	16
Boy—	7	29	36	$42\frac{1}{2}$	$52\frac{1}{2}$	64	75	87	110
Girl—	$6\frac{1}{2}$	27	34	41	50	60	73	$92\frac{1}{2}$	108

Exhibit this graphically on the same diagram Estimate the average weights of a boy and of a girl at the ages of 9, 13 and $15\frac{1}{2}$ years

9 A caterer is asked at short notice to contract for a school picnic He works out the following

Number—	100	150	200	250
Charge per head—2*s*	1*s* 9*d*	1*s* 7*d*	1*s* 6*d*	

Draw a chart which would enable him to quote terms per head for 140, 170, 210 and 240

10 A manufacturer has made boxes of 6 sizes and has fixed the following prices

Length in inches—	8	12	15	18	20	24
Price in *R* and *a* —	1 12	2 8	3 4	4 0	5 0	6 8

He publishes a price list of all sizes from 8 to 24 inches Draw a graph and from it find the approximate prices of 10 inch, and 22 inch boxes

11 The following table gives the population (in millions) of two countries *A* and *B* at the beginning of the various years given in the table

Years—	1821	1831	1841	1851	1861	1871	1881
A—	2 1	2 4	2 6	2 9	3 1	3 4	3 7
B—	6 8	7 8	8 2	6 6	5 8	5 4	5 2

On the same diagram, draw the two graphs (using the same scales) In what year was the population of *B* nearly double that of *A*? When was the population of *B* about $7\frac{1}{2}$ millions ?

12 The area of a circle corresponding to a given diameter is given by the following table

Diameter—	5	6	7	9	10	12
Area—	20	28	38	64	79	113

Construct a graph from which the areas corresponding to the diameters 8 and 11, and the diameter corresponding to the area 133, can be read off

Examples for Revision (C)

1 Simplify $\{d-[b+c-\{a+b-(c+2b+a-d)\}]$

2 If $x=1$, $y=-2$, $z=3$, find the value of

$$\tfrac{1}{2}\{x-\tfrac{1}{3}[y-\tfrac{1}{4}(z-\overline{x-2y})]\}$$

3 Add together

$(a+b)x+(a+c)y$, $(b-c)x+(b-a)y$ and $(c-a)x+(c-b)$

4 Prove that $x^2(x-2y)-y^2(y-2x)=(x-y)(x^2-xy+y^2)$

5 Solve

(i) $\frac{1}{2}(3x-\frac{1}{3})-\frac{1}{5}(2-5x)=0$

(ii) $\frac{x+2}{3}+8y=31, \frac{y+5}{4}+2x=40$

6 Plot the points $(17, -5)$ and $(-13, 11)$, and find (i) by measurement and (ii) by calculation the distance between them

7 By selling an article for a sovereign and one-fourth of what I give for it, I gain R2 4a What was the cost price of the article?

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8 Simplify $a-[3a+c-\{4a-(3b-c)\}+3b]$.

9 Find the value of $a^3+b^3+c^3+3abc$, when $a=02$, $b=08$ and $c=10$

10 Add together $ax-by$, $x+y$ and $(a-1)x-(b+1)y$, and from the sum subtract $(a+1)x-(2b-1)y$

11 Prove that $a(a-2b)^2-b(b-2a)^2=(a-b)(a^2-7ab+b^2)$.

12 Solve

(i) $\frac{4x-5}{7}-\frac{x-30}{5}=\frac{2x+7}{3}$

(ii) $\frac{x}{3}+3y=7, \quad \frac{4x-2}{5}=3y-4$

13 What is that number, which being diminished by 10 and the remainder multiplied by 10, produces the same result, as if it were diminished by 8 and the remainder multiplied by 8?

14 If I buy a things for b shillings, how many do I buy for c florins?

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15 Simplify $2\{4x-[2y+(2x-y)-(x+y)]\}$

16 Express $x^4-2a(a-b)x^2+(a^2+b^2)(a-b)x-a^2b^2$ with numerical coefficients, when $a=4$, $b=8$

17. From $(x+2)(x+3)$ take $(x+1)(x+4)$, and to the result add $2(x-1)(x+1)-x^2$

18 Prove that $a(a-2b)^3-b(b-2a)^3=(a-b)(a+b)^3$

19 Solve

(i) $\frac{3x-11}{16}-\frac{97-7x}{2}=\frac{5(x-1)}{8}-\frac{111-3x}{4}$

(ii) $\frac{2x}{3}+6y=28, \quad 5x+\frac{17y}{2}=27\frac{1}{2}$

20 A circular pond having a diameter of 110 yd has a uniform depth of 10 ft How many cubic feet of earth will be required to fill it up ?

21 How many hens and pigs are there in a farmyard if the total number of heads is 25 and of feet is 70 ?

---

22 Arrange $a^2(x+y-z)-ab(z+x-y)+b^2(z+y-x)$, bracketing the coefficients of x, y and z

23 Simplify $5x-[a-\{x+2a-(3x-7a)\}]$

24 Add together $4x^3+3x^2y-y^3$, $4x^2y-3x^3$ and $7xy^2+9y^3-2x^2y$, and find what must be subtracted from the sum to leave the remainder $2x^3-3x^2y+y^3$

25 Resolve into factors a^2-b^2-2a+1

26 Solve

(i) $(x-1)(x-2)=(x-3)(x-4)$

(ii) $\dfrac{3}{x}+\dfrac{4}{y}=3,\quad \dfrac{6}{x}-\dfrac{2}{y}=1$

27 A ship leaves A for a port B 1200 miles due west After going 300 miles on her course she is driven 450 miles to the south-west and becomes disabled Find graphically to the nearest mile how far she is from A and from B

28 If ducks' eggs cost $4d$ a dozen more than hens' eggs, find the price of each per dozen when 7 ducks' eggs and 19 hens' eggs cost 2s

---

29 Simplify

$$2a-\{c-(a-b+2c)\}-[4b-\{3c+a-(\overline{4a+c}-5b)\}]$$

30 Subtract $2(x-k)(y-k)$ from $(x-k)^2+(y-k)^2$, and shew that whatever value be given to k, this difference is constant

31 Factorize $1+2x(1+2x)+8x^3$, and find its value, when $x=-\frac{1}{2}$

32 Solve (i) $(x+1)(x-2)+(x+3)(x+4)=2(x-4)(x+2)$

(ii) $\frac{1}{2}(x+y)-\frac{1}{3}(x-y)=8$

$\frac{1}{3}(x+y)+\frac{1}{5}(x-y)=18$

33 A body falls from a height of 48 yds How long does it take to reach the ground ?

34 A courier who travels 60 miles a day, had been despatched 5 days, when a second, who travels 75 miles a day, was sent to overtake him When will he overtake the first ?

Solve this problem also graphically

35 Plot the points $(4, -3)$ and $(-8, -12)$, and find the equation of the graph passing through them

---

36 Simplify $a-\{b-(2b+c)\}+\{b-(c-2b)\}$, and find its value when $b=-\frac{1}{2}a$

37 Shew that $\frac{1}{4}(x^2+y^2)+z^2-\frac{1}{2}xy+xz-yz$ and $(y-z)^2$ become identical, when $-x=y=a$

38 From the sum of $a(x-y)+b(y-z)+c(z-x)$ and $ax+by+cz$, subtract $(a-b)x+(b-c)y+(c-a)z$, and in the remainder, bracket the coefficients of x, y and z

39 Prove that $(a+2)^3-4(a+1)^3+6a^3-4(a-1)^3+(a-2)^3=0$

40. Solve (i) $\dfrac{4(x+5)}{25}-\dfrac{10(x-10)-5}{6}=\dfrac{2x-1}{2}$

(ii) $\frac{2}{7}(2x-1)=\frac{1}{3}(y+1)$, $\dfrac{5}{x+1}=\dfrac{4}{y-1}$

41 A person bought $8x$ books at $2a$ annas each and $4y$ books at $4b$ annas each How many rupees did he spend ?

If he sold $8x$ books at $3a$ annas each and $4y$ books at $5b$ annas each, how many rupees did he gain ?

42 Draw the graph of $5x-2y=11$ by finding its intercepts on the axes

---

43 Find the value of $\{a-(b-c)\}^2+\{b-(c-a)\}^2+\{c-(a-b)^2\}$, when $a=1$, $b=3$, $c=5$

44 Factorize $a^2+b^2+2(bc+ca+ab)$

45 Divide the product of (x^2-y^2) and (x^2+xy) by $(x+y)^2$.

46 Solve (i) $\frac{6}{7}(x-1)(x-2)+\frac{2}{7}(x-2)(x-3)$
$=\frac{3}{7}(x-3)(x-4)+\frac{5}{7}(x-4)(x-5)$

(ii) $\dfrac{1}{x}+\dfrac{3}{y}=\dfrac{4}{3x}+\dfrac{2}{y}=1$

47 If 5 men can do a piece of work in 8 days working x hours a day, and 6 men do the same work in 7 days working y hours a day, shew that $20x=21y$

48 Two boys have 210 marbles between them One of the boys arranges his marbles in heaps of 6 each and the other in heaps of 9 each If these heaps be 32 in number, how many marbles has each boy ?

49 Draw the graphs of the functions $\dfrac{x-1}{2}$ and $1+\dfrac{3-x}{3}$ Hence solve the equation $\dfrac{x-1}{2}=1+\dfrac{3-x}{3}$.

---

50 Simplify $5a-3\{a-b-2(a-b)\}$, and find its value, when $a=1$, and $b=\frac{8}{3}$

51 Add together $16a^2-7ab-8b^2+3c$, $6b^2-8c+ab$ and $12ab-8a^2+5c$, and divide the sum by $a+b$

52 If $A=(x+1)^2$, $B=(1-x)^2$ and $C=x^2-1$, find the value of C^2-AB

53 Compare the areas of two circles whose diameters are 10 ft and 15 ft. respectively

54 A man at a party at cards, betted 3s to two upon every deal, after 20 deals he won 5s How many deals did he win?

55 Divide 45 into 3 parts which will be equal to one another if the first be diminished by 2, the second increased by 5 and the third divided by 2

56 A railway company's rates per ton for hauling coal are based on the following

Distances in miles	10	25	35	45	50
Rates	24*d*	27*d*	30*d*	34*d*	37*d*

Draw a graph shewing rates for any distance between 10 and 50 miles, and hence state to the nearest penny the rates for 15, 27, 33 and 40 miles

CHAPTER XIII

LONG MULTIPLICATION AND LONG DIVISION

Multiplication by a Polynomial

132 The method will be seen from the following examples

Ex 1 Multiply $4x^5-8x^4+9$ by x^2+2x+4

Multiply each term of multiplicand by x^2, then by $2x$ and finally by 4 Add the partial products together

$$\begin{array}{lllll}
4x^5-8x^4+9 & & & & \\
x^2+2x+4 & & & & \\
\hline
4x^7-8x^6+9x^2 & & & & \\
\quad +8x^6-16x^5+18x & & & & \\
\qquad\qquad +16x^5-32x^4+36 & & & & \\
\hline
\text{Product } 4x^7 \qquad +9x^2+18x-32x^4+36, & & & &
\end{array}$$

or arranged according to the descending powers of x,

$$\text{Product}=4x^7-32x^4+9x^2+18x+36$$

Verification Assume $x=1$, then

multiplicand $=1-8+9=5$, multiplier $=1+2+4=7$

Thus their product $=5\times 7=35$

Also when $x=1$, product $=1+9+18-32+36=35$

Ex 2 Multiply $x^2-2xy-y^2$ by $y^2+2xy-x^2$

Here the multiplicand is arranged according to the *descending* powers of x, but not the multiplier Arrange the multiplier therefore according to the *descending* powers of x

$$\begin{array}{rrrrr}
x^2-2xy-y^2 & & & & \\
-x^2+2xy+y^2 & & & & \\
\hline
-x^4+2x^3y+x^2y^2 & & & & \\
+2x^3y-4x^2y^2-2xy^3 & & & & \\
+x^2y^2-2xy^3-y^4 & & & & \\
\hline
\text{Product} \quad -x^4+4x^3y-2x^2y^2-4xy^3-y^4 & & & &
\end{array}$$

The arrangement of terms according to powers of some letter is convenient though not necessary For the same result will follow if we multiply out the expressions as they are given, *without arranging them*

$$\begin{array}{l}
x^2-2xy-y^2 \\
y^2+2xy-x^2 \\
\hline
x^2y^2-2xy^3-y^4 \\
\quad +2x^3y-4x^2y^2-2xy^3 \\
\qquad -x^4+2x^3y+x^2y^2 \\
\hline
\text{Product} \quad -2x^2y^2+4x^3y-4xy^3-y^4-x^4
\end{array}$$

Ex 3 Multiply $2a^2-b^2-3ab+bc+4ac$ by $2a+b-3c$

$$\begin{array}{l}
2a^2-3ab+4ac+bc-b^2 \\
2a+b-3c \\
\hline
4a^3-6a^2b+8a^2c+2abc-2ab^2 \\
\quad +2a^2b-3ab^2+4abc+b^2c-b^3 \\
\qquad -6a^2c+9abc-12ac^2-3bc^2+3b^2c \\
\hline
4a^3-4a^2b+2a^2c-5ab^2+15abc-12ac^2-3bc^2+4b^2c-b^3
\end{array}$$

REMARK From Examples 2 and 3, it is obvious that the **product of two homogeneous expressions is homogeneous**, and that the **degree of the product is the sum of the degrees of the factors** *

This principle is very important to remember as it enables the student to test the accuracy of his work Thus in Ex 3, the multiplicand and multiplier being homogeneous and of the *second* and *first* degree respectively, the

* The latter part of the remark is true of *all* expressions [See Ex 1]

product is homogeneous and of the *third* degree. Hence if any term of the product, say the third term, were $3ac$ instead of $3a^2c$, we at once see that it is wrong

Examples LXXIII

Multiply

1 x^2-ax+a^2 by x^2+ax+a^2 2 $a^2+2ab+b^2$ by $a^2-2ab+b^2$

3 $5a^2-7ab-3b^2$ by $2a^2+3ab+9b^2$

4 $x^2+xy-3y^2$ by $x^2-xy-3y^2$ 5 x^2+ab-b^2 by x^2+ab+b^2

6 $x^2-xy+y^2+x+y+1$ by $x+y-1$

7 $a^2+b^2+c^2-ab-bc-ca$ by $a+b+c$

8 $ax^2-2bx+c$ by $px+2q$ 9 x^3-2x+1 by x^3-3x+2

10 $3x^2+xy-y^2$ by $x^2-2xy+3y^2$

11 $2x^3-5x^2y+y^3$ by $y^3+5xy^2+2x^3$

12 $3a^3b-2a^2b^2+ab^3$ by $2a^2-ab-5b^2$

13 $a^2-2ab+b^2+c^2$ by $a^2+2ab+b^2-c^2$

14 $7a-2a^2+8a^3-4a^4$ by $5a+3a^2-7a^3$

15 ax^2-px+q by bx^2+cx-r 16 $bc+ca+ab$ by $a+b+c$

17 $x^4-(p-1)x^3+(q-p+1)x^2-(p-1)x+1$ by $x-1$

18 $1+2x+3x^2+4x^3$ by $4-3x+2x^2-x^3$

19 $1-\frac{x}{2}+\frac{x^2}{3}$ by $1+\frac{x}{3}-\frac{x^2}{3}$

20 $1-\frac{x}{2}+\frac{2x^2}{3}$ by $1+\frac{2x}{3}-\frac{x^2}{2}$

21 x^3-2x^2+4x-8 by $x^3-\frac{x^2}{2}+\frac{x}{4}-\frac{1}{8}$

22 $ax^2+2hxy+by^2$ by $gx+fy$

23 $ax^4+bx^3+cx^2+dx+f$ by $ax^4-bx^3+cx^2-dx+f$

24 $x^2+y^2+2gx+2fy+1$ by $lx+my-1$

133 Definitions An expression in x is said to be **complete** or **natural** when it contains x in all its powers *from the highest to 0* Thus $x^4+ax^3+bx^2+cx+d$ is a complete expression

In a complete expression the last term which does not contain x (*i e*, in which the power of x is 0) is called the **absolute term**, as d in the above expression

An expression in x is said to be **incomplete** when some of the

powers of x are *absent* from it. Thus x^4-3x^2+1 is an incomplete expression.

It is evident that the **number of terms** of a complete expression is *one more than the index of the power that denotes its degree*. Thus x^2+px+q being an expression of the *second* degree contains *three* terms.

The remark applies likewise to incomplete expressions if we suppose the places of absent powers supplied by these powers with zero coefficients. Thus x^3+1 contains *four* terms, for it is equal to x^3+0x^2+0x+1.

Note 1. It is well to remark here that the number of terms of an expression depends on the symbol of reference. Hence *the same expression may contain a different number of terms* according as it is considered with reference to one or other of the involved symbols. Thus the expression

$$(a^2+1)x^3+(a+2)x^2+3x+1$$

consists of *four* terms, viz., $+(a^2+1)x^3$, $+(a+2)x^2$, $+3x$ and $+1$, when the symbol of reference is x, and of *three* terms when it is a, in which case it assumes the form

$$x^3a^2+x^2a+(x^3+2x^2+3x+1),$$

and its terms are $+x^3a^2$, $+x^2a$ and $+(x^3+2x^2+3x+1)$.

Note 2. If an expression of the mth degree be multiplied by another of the nth degree, the **number of terms** of the product will be $m+n+1$, for it will be of $(m+n)$th degree. [See Ex. 1, Art. 134, where the number of terms of the product is $(3+2)+1$ or 6.]

134. The Method of Detached Coefficients. If multiplicand and multiplier contain powers of *one letter* only, multiplication may be conveniently performed by arranging both of them in ascending or descending powers of that letter and then writing down the coefficients only, as illustrated below.

Ex. 1. Multiply $3x^3+2x^2-4x+1$ by x^2+3x+2.

Here the given expressions are *complete*, therefore put down the coefficients as they are given;

$$\begin{array}{r}
3+2-4+1 \\
\underline{1+3+2} \\
3+2-4+1 \\
+9+6-12+3 \\
\underline{+6+4-8+2} \\
3+11+8-7-5+2
\end{array}$$

required product $=3x^5+11x^4+8x^3-7x^2-5x+2$.

For the highest power of x in the product is x^5 which here occurs

in the first term, and the other powers follow in order, the last term evidently not containing any of these powers

When however the two polynomials are *incomplete* expressions, supply with zeros the place of the absent powers

Ex 2 Multiply $2x^3+x-3$ by $5x^2+2$

Here the expressions are *incomplete*, therefore the coefficients are as given below

$$\begin{array}{l} 2+0+1-3 \\ \underline{5+0+2} \\ 10+0+5-15 \\ +0+0+0-0 \\ \underline{+4+0+2-6} \\ 10+0+9-15+2-6 \end{array}$$

required product $=10x^5+0x^4+9x^3-15x^2+2x-6$
$=10x^5+9x^3-15x^2+2x-6$

Remark The second line of multiplication might have been omitted, a all the terms are zero

This method may be used to multiply two polynomials which are *homogeneous in two letters*

Ex 3 Multiply $x^3-4x^2y+2y^3$ by $3x^2+2y^2$

Both the polynomials are written in descending powers of x and ascending powers of y, and the terms containing xy^2 in the multiplicand and xy in the multiplier are absent

$$\begin{array}{l} 1-4+0+2 \\ \underline{3+0+2} \\ 3-12+0+6 \\ \underline{2-8+0+4} \\ 3-12+2-2+0+4 \end{array}$$

required product $=3x^5-12x^4y+2x^3y^2-2x^2y^3+4y^5$

Examples LXXIV

Employ the method of detached coefficients to find the product of

1 $1-x+x^2-x^3$ and $1+x+x^2+x^3$

2 $5x+4x^2+x^3-24$ and $x^2+11-4x$

3 x^3-4x+3 and $3-4x^2+x^3$

4 $x^5-y^5-x^4y+xy^4$ and $x^3+y^3+x^2y+xy^2$

5 $x^5-3x^3+x^2-7$ and $1-3x+x^2$

6 $a^4-2a^2b+b^4$ and $a^3+3ab^2-2b^3$

Also work examples 9, 11, 12, 14, 18 and 19 of LXXIII

135 Multiplication of Several Quantities First multiply any two, then multiply the product by the third, and so on

Ex Multiply $x+a$, $x+b$ and $x+c$ together

$$\begin{array}{l} x+a \\ \underline{x+b} \\ x^2+ax \\ \underline{\quad\;\; +bx+ab} \\ x^2+(a+b)x+ab \\ \underline{x\;+c} \\ x^3+(a+b)x^2+abx \\ \underline{\quad +cx^2+(ac+bc)x+abc} \end{array}$$

Required product $=x^3+(a+b+c)x^2+(ab+ac+bc)x+abc$

Here we first multiply $x+a$ by $x+b$, and then the product $x^2+(a+b)x+ab$ by $x+c$

Examples LXXV

Multiply together

1 $x+1$, $x+2$, $x+3$

2 $x-a$, $x-b$, $x-c$

3 $a+x$, $b+y$, and $c+z$

4 $1-ax$, $1-bx$ and $1-cx$

5 $a+b$, $b+c$ and $c+a$

6 $a-b$, $b-c$ and $c-a$

7 $x+2a$, x^2+4a^2, $x-2a$

8 $1-x$, $1+x$, $1+x^2$, $1+x^4$

9 $x+1$, $x+2$, $3-x$ and $4-x$

10 $x+a$, $x+b$, $x+c$ and $x+d$

11. $x-a$, $x-b$, $x-c$ and $x-d$

12 $1-ax$, $1-by$ and $1-cz$

13 $ax-by$, $ax+cy$ and $ax-dy$

14 $a+b-c$, $b+c-a$ and $c+a-b$

15 $3x-4y$, $x-2y$, $x+2y$ and $3x+4y$

16 $a+x$, $a-x$, a^2-ax+x^2 and a^2+ax+x^2

17 $x+a$, $x-a$, x^2+a^2, x^4+a^4 and x^8+a^8.

18 a^2-ax+x^2, a^2+ax+x^2 and $a^4-a^2x^2+x^4$

19 $a+b+c$, $b+c-a$, $c+a-b$ and $a+b-c$

DIVISION BY A POLYNOMIAL

136. Exact Division. The Rule established in Art. 63 may be applied to cases where one Polynomial is divided by another *exactly*, *i.e.*, without remainder.

Ex. 1. Divide $x^4-10x^2+24x-27$ by x^2-2x+3.

$$
\begin{array}{l}
x^2-2x+3\;)\;x^4-10x^2+24x-27\;(\;x^2+2x-9 \\
\qquad\qquad\quad \underline{x^4-2x^3+3x^2} \\
\qquad\qquad\qquad\quad 2x^3-13x^2+24x \\
\qquad\qquad\qquad\quad \underline{2x^3-4x^2+6x} \\
\qquad\qquad\qquad\qquad\quad -9x^2+18x-27 \\
\qquad\qquad\qquad\qquad\quad \underline{-9x^2+18x-27}
\end{array}
$$

Ex. 2. Divide $x^3+y^3+3xy-1$ by $x+y-1$.

Arrange *both* dividend and divisor according to the *descending* powers of x.

$$
\begin{array}{l}
x-1+y\;)\;x^3+3xy-1+y^3\;(\;x^2+x-xy+1+y+y^2 \\
\qquad\qquad \underline{x^3-x^2+x^2y} \\
\qquad\qquad\quad x^2-x^2y+3xy \\
\qquad\qquad\quad \underline{x^2-x+xy} \\
\qquad\qquad\qquad -x^2y+x+2xy \\
\qquad\qquad\qquad \underline{-x^2y+xy-xy^2} \\
\qquad\qquad\qquad\quad x+xy+xy^2-1 \\
\qquad\qquad\qquad\quad \underline{x-1+y} \\
\qquad\qquad\qquad\qquad xy+xy^2-y \\
\qquad\qquad\qquad\qquad \underline{xy-y+y^2} \\
\qquad\qquad\qquad\qquad\quad xy^2-y^2+y^3 \\
\qquad\qquad\qquad\qquad\quad \underline{xy^2-y^2+y^3}
\end{array}
$$

Ex. 3. Divide $4x^5+5x^3y^2-11x^2y^3-16y^5$ by $2x^2+3xy+4y^2$.

Here both the expressions are *homogeneous* and are arranged according to the *descending* powers of x.

$$
\begin{array}{l}
2x^2+3xy+4y^2\;)\;4x^5+5x^3y^2-11x^2y^3-16y^5\;(\;2x^3-3x^2y+3xy^2-4y^3 \\
\qquad\qquad\qquad \underline{4x^5+6x^4y+8x^3y^2} \\
\qquad\qquad\qquad\quad -6x^4y-3x^3y^2-11x^2y^3 \\
\qquad\qquad\qquad\quad \underline{-6x^4y-9x^3y^2-12x^2y^3} \\
\qquad\qquad\qquad\qquad\quad 6x^3y^2+x^2y^3-16y^5 \\
\qquad\qquad\qquad\qquad\quad \underline{6x^3y^2+9x^2y^3+12xy^4} \\
\qquad\qquad\qquad\qquad\qquad -8x^2y^3-12xy^4-16y^5 \\
\qquad\qquad\qquad\qquad\qquad \underline{-8x^2y^3-12xy^4-16y^5}
\end{array}
$$

REMARK. From Example 3, it is seen that **if dividend and divisor are both homogeneous, the quotient is also homogeneous, and that the degree of the quotient is the difference of the degree of dividend and divisor** *

This principle enables us to test the accuracy of the work. Thus in Ex. 3, each term of the quotient is of $5-2$ or 3 dimensions, so that if the third term were $3xy$ instead of $3xy^2$, it would be wrong.

Hence if dividend is of the mth degree and divisor of the nth degree, quotient is of the $(m-n)$th degree, and the **number of terms** of the quotient is $m-n+1$. [See Ex. 3, where the number of terms of the quotient is $(5-2)+1=4$.]

Examples LXXVI

Divide

1. $x^4+5x^3+11x-6$ by x^2+2x-1
2. $14x^2-17x^3-2x^4-75x-4$ by x^2-3x-4
3. $3x^5+11x^2-8x^3-4x^4-2$ by $1-2x+x^2$
4. $30a^2x^2+8x^4-2a^4-29ax^3-7a^3x$ by $3ax-x^2-2a^2$
5. $1-a^2-b^2+2ab$ by $1+a-b$
6. $x^4-y^4+a^4+2a^2x^2$ by $x^2-y^2+a^2$
7. $x^4+x^2y^2+y^4$ by x^2+xy+y^2 8. a^4+4b^4 by $a^2-2ab+2b^2$
9. $4x^4y^4+1$ by $2x^2y^2-2xy+1$
10. $256x^4+16x^2y^2+y^4$ by $16x^2+4xy+y^2$.
11. $2-x+16x^3-8x^4$ by $2+3x-2x^2$.
12. $a^4-9a^2b^2+18b^4$ by a^2+3b^2-3ab
13. $4x^4-x^2+4x$ by $3x+2+2x^2$
14. $6x^5-19x^4+17x^3-5x$ by $1-3x+3x^2$
15. $a^5-4a^4b^2-8a^3b^3-17ab^4-12b^5$ by $a^2-2ab-3b^2$
16. $14a^4+17a^3b+33a^2b^2+36ab^3+28b^4$ by $7a^2-3ab+14b^2$
17. $9x^4-12x^3y+13x^2y^2-4xy^3+y^4$ by $9x^2-3xy+y^2$
18. $2y^2-5xy+2x^2-ay-ax-a^2$ by $2y-x+a$
19. $x^6+2x^3y^3-y^6$ by $x^2+2xy+y^2$
20. $5x^4+6x^5+1$ by x^2+2x+1
21. $m^6-6mn^5+5n^6$ by $m^2-2mn+n^2$
22. $x^3-3ax^2+3a^2x-a^3+b^3$ by $x-a+b$
23. $a^2+b^2-3ab+1$ by $a+b+1$ 24. $1+x^3-8y^3+6xy$ by $1+x-2y$
25. $8x^3-y^3+z^3+6xyz$ by $y-z-2x$
26. $a^3-b^3-27c^3-9abc$ by $3c-a+b$

* The latter part of the remark is true of *all* expressions. [See Exs. 1 and 2.]

Divide

27 $21x^6-2x^4-70x^3-23x^2+33x+27$ by $3x^3-2x^2-5x-3$

28 $15a^5-17a^4x+20a^3x^2+39a^2x^3-7ax^4-10x^5$ by $5a^2+ax-2x^2$

29 $8a^6+21a^3x^3-x^6-24a^5x$ by $3ax-x^2-a^2$

30 $50x^3y^2-6y^5+25xy^4-45x^2y^3-41x^4y+20x^5$ by $5xy^2-3y^3-4x^2y+5x^3$

31 $x^4+y^4-z^4+2x^2y^2-2z^2-1$ by $x^2+y^2-z^2-1$

32 $a^4+b^4+2a^2b^2-2c^2d^2-c^4-d^4$ by $a^2+b^2-c^2-d^2$

33 $a^4+b^4+c^4-2a^2b^2-2b^2c^2-2c^2a^2$ by $a^2+b^2-c^2+2ab$

34 $x^7-x^4y^3-x^3y^4+y^7$ by $x^4-x^3y-xy^3+y^4$

35 $x^8+x^6y^2+x^4y^4+x^2y^6+y^8$ by $x^4+x^3y+x^2y^2+xy^3+y^4$

[In the following examples, remove the brackets, arrange each example according to powers of some letter, and then proceed as usual]

36 $(a+b)^2-c^2$ by $a+b+c$ 37 $8x^3+y^6$ by $2x+y^2$

38 $x^3-(y-z)^3$ by $x-y+z$ 39 $27a^6-8x^9$ by $3a^2-2x^3$

40 x^3+mx^2-mx-1 by $x-1$ 41 a^5+b^5 by $a+b$

42 $a^5-3a^4x-a^3x^2+4a^2x^3-x^5$ by a^2-x^2

43 $y^5-my^4+ny^3-ny^2+my-1$ by $y-1$

44 $x^3-apx^2+a^2px-a^3$ by $x-a$ 45 $\frac{1}{9}-6a^2+27a^4$ by $1-6a+9a^2$

46 $\frac{1}{3}-\frac{10}{24}x+\frac{5}{8}x^2+\frac{1}{8}x^3-x^4+\frac{3}{4}x^5$ by $\frac{1}{4}-\frac{2}{3}x+\frac{1}{2}x^2$

47 $a^2(b-c)+b^2(c-a)+c^2(a-b)$ by $c^2-(a+b)c+ab$

48 $a^3(b-c)+b^3(c-a)+c^3(a-b)$ by $c^2-(a+b)c+ab$

49 $xy^3+2xyz^2+y^3z+yz^3-2xy^2z-xz^3-y^2z^2$ by $xy-xz+yz$

137 Inexact Division

Ex 1 Divide $2x^3+9x^2+12x+2$ by x^2+3x+1

$$
\begin{array}{r|r|l}
x^2+3x+1 & 2x^3+9x^2+12x+2 & 2x+3 \\
 & \underline{2x^3+6x^2+\ \ 2x} & \\
 & 3x^2+10x+2 & \\
 & \underline{3x^2+\ \ 9x+3} & \\
 & x-1 &
\end{array}
$$

If we continue the process, we get a *fractional* term, *viz*, $\dfrac{1}{x}$ in the quotient and by further continuing the operation, a *series* of fractional terms will be obtained Hence as soon as a *fractional* term enters a quotient, the division ceases to be exact, and the quotient becomes *infinite* In such a case the quotient may be represented

as in Arithmetic, by putting, after the *integral part*, the *last remainder* over the divisor, in the form of a fraction

Thus in the above example, the quotient $=2x+3+\dfrac{x-1}{x^2+3x+1}$

Ex 2 Divide x^2-5x+7 by $x-2$, and find the remainder

$$\begin{array}{r|l|l} x-2 & x^2-5x+7 & x-3 \\ & \underline{x^2-2x} & \\ & -3x+7 & \\ & \underline{-3x+6} & \\ & 1 & \end{array}$$

remainder $=1$

Definition From the above it is easy to see that when dividend and divisor are both arranged according to the *descending* powers of the symbol of reference, *that residue which is of a lower degree than the divisor is called the* **remainder**

Hence if the divisor be of the nth degree, the remainder must be of $n-1$ or of even *lower* dimensions As a particular case, *if the divisor be of the first degree* (say $x+a$), *the remainder cannot contain* x [See Ex 2]

Corollary Since $27=4\times 6+3$ identically, where 4 is the divisor, 6 the quotient and 3 the remainder, we get from Ex. 1,

$$2x^3+9x^2+12x+2=(x^2+3x+1)(2x+3)+(x-1) \text{ identically}$$

Hence generally if D represents the dividend, d the divisor, Q the quotient and R the remainder, we have the identity

$$D=dQ+R$$

Since $D=dQ+R$ is an **identity**, it is satisfied by **any value** whatever of D, d, Q and R

Examples LXXVII

Arrange dividend and divisor in descending powers of x and then find the remainder in the division of

1 $10x^2+17x+8$ by $2x+3$. 2. $8x^2-26xy+14y^2$ by $4x-3y$

3 $82x^2-45x^3+18x^4+37-62x$ by $6x^2+8-7x$

4 $1+x^3-6y^3+6xy$ by $1+x-2y$

5 $x^2-m^2+(m-n)x+1$ by $x-m-n$

6 $6x^5-x^4+7x^2+13x-17$ by $3x^3+4x^2-5$.

7 What number must be added to $x^3+x^2-4(x+3)$ that it may be divisible by $x-6$?

8 If the dividend be x^3+5x^2+6x-4, the divisor x^2+2x-1 and the remainder $x-1$, what is the quotient?

9 The quotient and remainder are respectively $2x^2-x-4$ and $2x-1$, if the dividend is $2x^4-x^3-9$, what is the divisor?

138 The Method of Detached Coefficients As in Art 134, we may use this method to divide one polynomial by another, if they both contain powers of *one* letter only, or if they both be homogeneous in *two* letters only

Ex 1 Divide $x^4-4x^3+5x^2-7x+6$ by $x-2$.

$$
\begin{array}{r|l|l}
1-2 & 1-4+5-7+6 & 1-2+1-5 \\
 & \underline{1-2} & \\
 & -2+5 & \\
 & \underline{-2+4} & \\
 & \quad 1-7 & \\
 & \quad \underline{1-2} & \\
 & \quad\quad -5+6 & \\
 & \quad\quad \underline{-5+10} & \\
 & \quad\quad\quad -4 &
\end{array}
$$

quotient $=x^3-2x^2+x-5$ and rem. $=-4$,

[since the quotient must be of the *third* degree and its first term must contain x^3 as the expressions are arranged in *descending* powers of x]

When the divisor is a binomial, as $x-2$ here, the above work may be more concisely arranged thus —

$$
\begin{array}{rrrr|r}
1 & -4 & 5 & -7 & 6 \\
 & 2 & -4 & 2 & -10 \\
\hline
1 & -2 & 1 & -5 & -4
\end{array}
$$

quotient $=x^3-2x^2+x-5$ and rem. $=-4$

Explanation Arrange the coefficients of dividend in the first line, draw a line and *below* it put the *first* term 1 of the first line Multiply 1 by 2, the second term of divisor, put the product 2 *above* the line and under -4, the difference is -2 Multiply the difference -2 by 2, the second term of divisor, and put the product -4 *above* the line and under 5, the difference is 1 Multiply this 1 by 2, the second term of divisor, and go on as before The quotient is given by the numbers to the *left* of the vertical line and the remainder is the number to the *right* of it

Note The divisor is $x-2$, and we have multiplied by 2 and not by -2. If the divisor were $x+3$, we would multiply by -3 The reason is obvious

Ex 2 Divide x^6-7x+8 by x^2-2x+1

Since the dividend is *incomplete*, the coefficients are to be put down as follows

$$
\begin{array}{l}
1-2+1\,)\;1+0+0+0+0-7+8\;(\,1+2+3+4+5\\
\qquad\quad\;\; \underline{1-2+1}\\
\qquad\qquad\quad 2-1+0\\
\qquad\qquad\quad \underline{2-4+2}\\
\qquad\qquad\qquad 3-2+0\\
\qquad\qquad\qquad \underline{3-6+3}\\
\qquad\qquad\qquad\quad 4-3-7\\
\qquad\qquad\qquad\quad \underline{4-8+4}\\
\qquad\qquad\qquad\qquad 5-11+8\\
\qquad\qquad\qquad\qquad \underline{5-10+5}\\
\qquad\qquad\qquad\qquad\;\; -\;1+3
\end{array}
$$

$\therefore$ quotient$=x^4+2x^3+3x^2+4x+5$, and remainder$=-x+3$

Examples LXXVIII

Divide by the method of detached coefficients

1. $x^3+5x^2-10x+4$ by $x-1$ 2. $3x^3-13x+21$ by $x-3$
3. $8x^3+47x^2-12x-16$ by $x+4$ 4. x^4+10x^2-56 by $x+2$.
5. $x^4-6x^3+9x^2-4$ by x^2-3x+2 6. $32x^4+54xy^3-80y^4$ by $2x+3y$.
7. $3x^5+2x^4+13x-10$ by x^3+2x^2+x-2
8. $2x^5+2x^4y-6xy^4+8y^5$ by $2x^3-6xy^2+5y^3$
9. $2x^6-3x^5+7x^3-16x+15$ by x^4-2x^2+4
10. $x^6+11x-34$ by $x^4+2x^3+x^2-4x-11$.

139 Arrangement in Inexact Division We have seen that in Exact Divisions, the answer is the same whether dividend and divisor are arranged in descending or ascending order But in an Inexact Division an example appears to have *two different* answers according as we arrange it in descending or ascending order

Ex 1 Divide $2x^2+7x+8$ by $2x+1$

Arranged in descending order

$$
\begin{array}{l}
2x+1\,)\;2x^2+7x+8\;(\,x+3\\
\qquad\quad\;\; \underline{2x^2+\;x\quad}\\
\qquad\qquad\quad 6x+8\\
\qquad\qquad\quad \underline{6x+3}\\
\qquad\qquad\qquad\;\; 5
\end{array}
$$

$\therefore$ quot $=x+3$, rem $=5$

Arranged in ascending order

$$
\begin{array}{l}
1+2x\,)\;8+\;7x+2x^2\;(\,8-9x\\
\qquad\quad\;\; \underline{8+16x}\\
\qquad\qquad\; -9x+\;2x^2\\
\qquad\qquad\; \underline{-9x-18x^2}\\
\qquad\qquad\qquad\quad 20x^2
\end{array}
$$

quot $=8-9x$, rem $=20x^2$

This anomaly is explained when we remark that these quotients are only different portions of the real quotient which we cannot find in Inexact Divisions Hence in these divisions, only a limited number of terms can be found

Ex. 2 Divide a by $1-x$, keeping 4 terms of the quotient

$$
\begin{array}{l}
1-x\,)\,a\quad(\,a+ax+ax^2+ax^3+\&c \\
\quad\;\; \underline{a-ax} \\
\qquad\quad ax \\
\qquad\quad \underline{ax-ax^2} \\
\qquad\qquad\; ax^2 \\
\qquad\qquad\; \underline{ax^2-ax^3} \\
\qquad\qquad\qquad ax^3 \\
\qquad\qquad\qquad \underline{ax^3-ax^4} \\
\qquad\qquad\qquad\quad ax^4
\end{array}
$$

required quotient$=a+ax+ax^2+ax^3$

Examples LXXIX

In the following divisions, keep only 4 terms of the quotient

1 $(1+x)\div(1-x)$ 2 $1\div(1-ax)$

3 $(a-x)\div(a+2x)$ 4 $(a-bx)\div(a+cx)$

5 $(1+x^3)\div(1+x)$ 6 $(1-x+x^2)\div(1-x)$

140 Functional Notation In Art 110, we have used the symbols $f(x)$, $F(x)$, $\phi(x)$, &c, to denote functions of x It is to be borne in mind that each of these symbols is only a short way of writing the words "function of x" Moreover it is used as an abbreviation to denote an algebraical function such as x^2-2x+3, just as we might use A for the expression

Any function of x such as $2x^2$, $3\sqrt{x}$, x^2+3x-4, &c may be represented by $f(x)$, but when we once use $f(x)$ to represent a particular function, say x^2+3x-4, we must throughout the same piece of work denote by $f(x)$ this function, and if necessary, other functions in that work must be represented by other symbols, such as $F(x)$, $\phi(x)$, &c

Now in the question "Find $f(a)$, when $f(x)=x^2-5x+6$", what does $f(a)$ mean? Here $f(x)$ being given, $f(a)$ denotes the value which $f(x)$ shall have when x is changed into a For example if $f(x)=x^2-5x+6$, then $f(a)=a^2-5a+6$

Hence if $f(x)=x^2-5x+6$,

then (i) $f(1)=1^2-5\times1+6=2$, (ii) $f(0)=0^2-5\times0+6=6$,

(iii) $f(-1)=(-1)^2-5(-1)+6=12$, and so on

Examples LXXX.

1 If $f(x)=8x-5$, find the value of (i) $f(1)$, (ii) $f(-1)$, (iii) $f(3)$, and (iv) $f(0)$

2 If $F(x)=x^2+1$, find the value of (i) $F(2)$, (ii) $F(-1)$, (iii) $F(0)$ and (iv) $F(5)$

3 If $f(x)=2x^2+3x-1$, find the value of (i) $f(1)$, (ii) $f(-1)$, (iii) $f(2)$ and (iv) $f(0)$

4 If $\phi(x)=(x-1)(x-2)$, find the value of (i) $\phi(0)$, (ii) $\phi(-2)$, (iii) $\phi(2)$ and (iv) $\phi(8)$

5 Given that $f(x)=x^3-x$, determine the value of (i) $f(1)$, (ii) $f(0)$, (iii) $f(-4)$ and (iv) $f(5)$

6 Given that $F(x)=x(x+1)(x+2)$, find the value of (i) $F(1)$, (ii) $F(0)$, (iii) $F(-3)$ and (iv) $F(-6)$

7 Find the value of $\phi(x+1)-\phi(x)$, when $\phi(x)=3x^2-4x+5$

8 Find the value of $f(x+a)+f(x-a)$, if $f(x)=x^2+2x-3$

9 If $f(x)=x^2+px+q$, find the value of $f(x+4)+f(x-4)-2f(x)$.

10 If $\phi(x)=ax^2+bx+c$ and $f(x)=a+bx+cx^2$, shew that

$$\phi(x+1)-f(x+1)=x(a-c)(x+2)$$

141 The Remainder Theorem *When an integral function of x is divided by $x-a$, the remainder is obtained by substituting a for x in that function*

Let Q represent the quotient and R the remainder, when any function $f(x)$ is divided by $x-a$ Thus by Art 137, *Cor*, we have the identity

$$f(x)=Q(x-a)+R \qquad \text{(i)},$$

where the remainder R being of a *lower* degree than $x-a$, cannot contain x [Art 137, *Def*] Now (i) being an identity, is satisfied by *any value* of x, and therefore by the value $x=a$, thus

$$f(a)=Q(a-a)+R=Q\times 0+R=R,$$

that is, $$R=f(a) \qquad \text{(ii)},$$

Thus if $f(x)=x^3+2x^2+3x+1$, then R or $f(a)=a^3+2a^2+3a+1$,

if $f(x)=x^2-px+q$, then R or $f(a)=a^2-pa+q$,

if $f(x)=px^3-qx^2+rx-s$, then R or $f(a)=pa^3-qa^2+ra-s$,

and so on

Again if $x-a$ divides $f(x)$ *exactly*, there cannot be *any remainder*. Thus when $f(a)=0$, then $x-a$ is a factor of $f(x)$

Hence *if an integral function* $f(x)$ *vanish when* $x=a$, *then* $x-a$ *is a factor of* $f(x)$

Ex 1 Find the remainder when $2x^3-7x^2+8x-9$ is divided by $x-4$

By the Theorem

$$R \text{ or } f(4)=2\times4^3-7\times4^2+8\times4-9=39$$

Otherwise [see Art 138, Ex 1] —

$$\begin{array}{rrrr} 2 & -7 & 8 & -9 \\ & 8 & 4 & 48 \\ \hline 2 & 1 & 12 & \vert\ 39 \end{array} \qquad \text{rem} = 39$$

Ex 2 Find the remainder when $27x^3-18x^2+3x-11$ is divided by $3x-2$ Here $3x-2$ vanishes when $x=\frac{2}{3}$

$$R \text{ or } f(\tfrac{2}{3})=27(\tfrac{2}{3})^3-18(\tfrac{2}{3})^2+3\times\tfrac{2}{3}-11=-9$$

Ex 3 Find the remainder when $x^4+5x^3+8x^2+7x+6$ is divided by $x+2$ Since $x+2=x-(-2)$ we have

$$R \text{ or } f(-2)=(-2)^4+5(-2)^3+8(-2)^2+7(-2)+6=0$$

Hence *the given expression is divisible* by $x+2$

Ex 4 Investigate whether $2x^3-3x^2-2x+3$ is divisible by x^2-1

The factors of x^2-1 are $x-1$ and $x+1$ When the divisor is $x-1$

$$f(1)=2-3-2+3=0,$$

$\therefore$ the given expression is divisible by $x-1$ When the divisor is $x+1$,

$$f(-1)=-2-3+2+3=0,$$

$\therefore$ the expression is also divisible by $x+1$ Hence it is divisible by the product of $x-1$ and $x+1$, *i e*, by x^2-1

Ex 5 If one of the factors of x^2+5x+q be $x+8$, determine q

Divide x^2+5x+q by $x+8$, thus

$$f(-8)=(-8)^2-5\times8+q=0 \text{ by supposition},$$

whence $$q=-24$$

Ex 6 For what values of a and b is the expression $3x^3+ax^2+bx+30$ divisible by $x^2+2x-15$?

Let $f(x)=3x^3+ax^2+bx+30$

Divisor $x^2+2x-15=(x-3)(x+5)$

If $f(x)$ is divisible by $x^2+2x-15$, it must be divisible by $x-3$ and $x+5$ separately, and then we must have $f(3)=0$ and $f(-5)=0$

Now $f(3)=3\times 3^3+9a+3b+30=9a+3b+111$,

$f(-5)=3(-5)^3+25a-5b+30=25a-5b-345$

Hence $9a+3b+111=0$, or $3a+b=-37$ (i),

$25a-5b-345=0$, or $5a-b=69$ (ii)

Thus from (i) and (ii), $a=4$, $b=-49$

Examples LXXXI

By the Remainder Theorem find the remainder when

1 x^3+5x^2-16 is divided by $x-2$

2 x^4+2x^3+5x-3 is divided by $x+3$

3 $2x^5+5x^4-9x^3+6x+2$ is divided by $2x+1$

4 $x^{15}+x^8-x+4$ is divided by $x+1$

5 $x^3+10x^2+100x+1000$ is divided by $x+10$

6 x^4-3x^2+1 is divided by $x-c$

7 $x^5+2x^2-8x+10$ is divided by $x+q$

8 $2x^5-3x^4y+5xy^4-4y^5$ is divided by $x-y$

9 Shew that $x-3$, $x+2$ and $x-7$ are the factors of
$$x^3-8x^2+x+42.$$

10 Shew that $2x^3+3x^2-8x+3$ is divisible by $(x-1)(x+3)$

11 Shew that $6x^3-5x^2-8x+3$ is divisible by $(2x-3)(x+1)(3x-1)$

12 Shew that $x^4-2ax^3+(1-a)ax^2+(2a-1)a^2$ is divisible by x^2-a

13 If when x^2-5x+a and $ax^3+13x^2+2x-11$ are divided by $x+3$ the remainders are equal, find a

14 Determine λ so that $3x^4+8x^3-4\lambda x+\lambda$ may be divisible by $x-2$

15 If x^3+x^2+px+q is divisible by $(x-1)(x-2)$, find p and q

16 Determine the values of a and b for which the expression $3x^3+ax^2-74x+b$ is divisible by $x^2+2x-24$

142 Divisibility of $x^n \pm a^n$ by $x \pm a$ where n is a positive integer We have four cases to consider, which we shall do in the following order —

(i) When $x^n - a^n$ is divisible by $x - a$,

(ii) When $x^n + a^n$ is divisible by $x - a$,

(iii) When $x^n - a^n$ is divisible by $x + a$

(iv) When $x^n + a^n$ is divisible by $x + a$

By the Remainder Theorem, we have, when

(i) Dividend $= x^n - a^n$ and Divisor $= x - a$, Rem $= a^n - a^n = 0$ always,

$x^n - a^n$ is **always** divisible by $x - a$,

(ii) Dividend $= x^n + a^n$ and Divisor $= x - a$, Rem $= a^n + a^n = 2a^n$ always,

$x^n + a^n$ is **never** divisible by $x - a$,

(iii) Dividend $= x^n - a^n$ and Divisor $= x + a$ Rem $= (-a)^n - a^n = 0$ or $-2a^n$ according as n is *even* or *odd*,

$x^n - a^n$ is divisible by $x + a$ only when n is **even**,

(iv) Dividend $= x^n + a^n$ and Divisor $= x + a$, Rem $= (-a)^n + a^n = 0$ or $2a^n$ according as n is *odd* or *even*,

$x^n + a^n$ is divisible by $x + a$ only when n is **odd**

Examples LXXXII

Write down the following quotients in

1 $(a^4 - b^4) \div (a \pm b)$ 2 $(a^5 + b^5) \div (a + b)$ 3 $(a^6 - b^6) \div (a \pm b)$

4 $(x^7 + y^7) \div (x + y)$ 5 $(x^8 - y^8) \div (x \pm y)$

6 $(x^9 + y^9) \div (x + y)$ 7 $(a^{10} - x^{10}) \div (a \pm x)$

8 $(a^{11} + x^{11}) \div (a + x)$ 9 $(x^{10} - a^{10}) \div (x \pm a)$

CHAPTER XIV

HARDER FORMULÆ AND FACTORS

43 From Formulæ I and II [Art 64], we have

Formula IX $\begin{cases} (a+b)^2 = (a-b)^2 + 4ab & \text{(i)}, \\ (a-b)^2 = (a+b)^2 - 4ab, & \text{(ii)} \end{cases}$

Formula X
$$a^2 + b^2 = (a+b)^2 - 2ab \qquad \text{(i)},$$
$$= (a-b)^2 + 2ab \qquad \text{(ii)},$$
$$= \tfrac{1}{2}(a+b)^2 + \tfrac{1}{2}(a-b)^2 \qquad \text{(iii)}$$

Formula XI $\begin{cases} 4ab=(a+b)^2-(a-b)^2 & \ldots \quad (i), \\ ab=\left(\frac{a+b}{2}\right)^2-\left(\frac{a-b}{2}\right)^2 & \ldots \quad (ii) \end{cases}$

[The student can easily prove these formulæ]

The form (ii) enables us to express the product of any two factors *as the difference of two squares*

Ex. 1 Find the values of $(x+y)^2$ and $x+y$, when $x-y=2$, $xy=15$, and of $(x-y)^2$ and $x-y$, when $x+y=8$, $xy=12$

$(x+y)^2=(x-y)^2+4xy=2^2+4\times 15=64$, $\quad x+y=\sqrt{64}=8$

$(x-y)^2=(x+y)^2-4xy=8^2-4\times 12=16$, $\quad x-y=\sqrt{16}=4$

Ex. 2 Find the value of $x^2+2xy+y^2$, when $x=625$ and $y=624$

$(x+y)^2=(x-y)^2+4xy=(625-624)^2+4\times 625\times 624$

$=1^2+2500\times 4\times 156=1+10000\times 156=1560001$

Ex 3 Find the value of x^2+y^2, (1) when $x+y=10$ and $xy=24$ (2) when $x-y=3$ and $xy=28$

(1) $x^2+y^2=(x+y)^2-2xy=(10)^2-2\times 24=52$ [from X]

(2) $x^2+y^2=(x-y)^2+2xy=3^2+2\times 28=65$ [from X]

Ex 4 Find the value of x^2+y^2, when $x+y=17$ and $x-y=9$.

$x^2+y^2=\frac{1}{2}(x+y)^2+\frac{1}{2}(x-y)^2=\frac{1}{2}\times(17)^2+\frac{1}{2}\times 9^2=\frac{289}{2}+\frac{81}{2}=185$

Ex. 5 Express $(a+b-2c)^2+2(b-c)(c-a)$ as the sum of two squares

Since $a+b-2c=(b-c)-(c-a)$, we have

Given expn $=\{(b-c)-(c-a)\}^2+2(b-c)(c-a)$

$=(b-c)^2+(c-a)^2$ [by X]

Ex 6 Express $(x-a)(x-3a)$ as the difference of two squares

$(x-a)(x-3a)=\left\{\frac{(x-a)+(x-3a)}{2}\right\}^2-\left\{\frac{(x-a)-(x-3a)}{2}\right\}^2$

[here $a=x-a$, $b=x-3a$]

$=\left(\frac{2x-4a}{2}\right)^2-\left(\frac{2a}{2}\right)^2=(x-2a)^2-a^2$

Ex 7 Show that $8xy(x^2+y^2)=(x+y)^4-(x-y)^4$

$8xy(x^2+y^2)=(4xy)\times 2(x^2+y^2)$

$=\{(x+y)^2-(x-y)^2\}\{(x+y)^2+(x-y)^2\}$

$=(x+y)^4-(x-y)^4$

Ex 8 Shew that

$$(a+b)^2(x-y)^2+4xy(a-b)^2=(a-b)^2(x+y)^2+4ab(x-y)^2$$

Left side $=\{(a-b)^2+4ab\}(x-y)^2+\{(x+y)^2-(x-y)^2\}(a-b)^2$

$$=(a-b)^2(x-y)^2+4ab(x-y)^2+(a-b)^2(x+y)^2-(a-b)^2(x-y)^2$$

$$=(a-b)^2(x+y)^2+4ab(x-y)^2$$

Examples LXXXIII

Find the value of

1 $x^2+2xy+y^2$, when $x-y=4$, $xy=117$

2 $x-y$, when $x+y=5$, $xy=6$

3 $2x+a$, when $2x-a=13$, $ax=7$

4 x^2+y^2, (1) when $x+y=13$, $xy=42$, (2) when $x-y=7$, $xy=24$; (3) when $x+y=18$, $x-y=4$.

5 $m^2+2mn+n^2$, when $m=126$ and $n=125$

6 x^2+y^2, and of xy, when $x=621$ and $y=379$

7 x^2+y^2, when $x=a^2+ab+b^2$, $y=a^2-ab+b^2$

8 x^2+xy+y^2, when $x+y=5$ and $xy=6$

9 x^2-xy+y^2, when $x-y=4$ and $xy=21$

10 If $x=ab+cd$, $y=ab-cd$, find x^2+xy+y^2 and x^2-xy+y^2, in terms of a, b, c and d.

11 Express as the difference of two squares

(1) x^2-10x, (2) $x(x+18)$, (3) $(x+a)(x+5a)$, (4) $(x-a)(x+3a)$, (5) $(x+1)(x+2)(x+3)(x+4)$

12 Shew that $(a^2-b^2)^2=\{(a+b)^2-4ab\}\{(a-b)^2+4ab\}$

13 Shew that $(a+b)^2(x-y)^2+4xy(a-b)^2=(a+b)^2(x+y)^2-16abxy$

144 From Formula IV [Art 66], we have by transposition

Formula XII
$$\begin{cases} a^3+b^3=(a+b)^3-3ab(a+b) & \text{(i)},\\ (a+b)^3-a^3-b^3=3ab(a+b) \quad .. & \text{(ii)} \end{cases}$$

Also from Formula V [Art 66], we get

Formula XIII
$$\begin{cases} a^3-b^3=(a-b)^3+3ab(a-b) & \text{(i)},\\ (a-b)^3-(a^3-b^3)=-3ab(a-b) \quad . & \text{(ii)}. \end{cases}$$

Ex 1 If $x+y=2$ and $xy=3$, find the value of x^3+y^3

$$x^3+y^3=(x+y)^3-3xy(x+y)=2^3-3\times3\times2=-10$$

Ex 2 If $x-y=2$ and $x^3-y^3=20$, find the value of xy

We have $3xy(x-y)=x^3-y^3-(x-y)^3$

$$\therefore 3xy\times 2=20-2^3, \text{ or } 6xy=12, \text{ i.e., } xy=2$$

Examples LXXXIV

1 If $x-y=3$ and $xy=1$, find the value of x^3-y^3

2 If $a+b=2$ and $a^3+b^3=3$, find the value of ab

3 Find the value of x^3+y^3, (1) when $x+y=2$ and $xy=1$, (2) when $x+y=4$ and $xy=2$

4 Find the value of x^3-y^3, when $x-y=2$ and $xy=3$

5 If $2p+3q=4$ and $pq=5$, find the value of $8p^3+27q^3$

6 Find the value of m^3-27n^3 when $m-3n=4$ and $mn=1$

7 Given $x-y=-5$ and $x^3-y^3=10$, find the value of xy

8 Given $2x+y=2$ and $8x^3+y^3=-28$, find the value of xy

145. Formula XIV $(ax+b)(cx+d)=acx^2+(bc+ad)x+bd$.

[The proof will be supplied by the student.]

Ex Multiply $2x+3$ by $5x-4$

Required product $=2\cdot5x^2+(3\cdot5-2\cdot4)x-3\times4$, putting $b=-4$

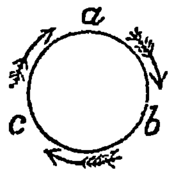
$$=10x^2+7x-12$$

Examples LXXXV

Find the value of

1 $(3x+2)(5x+4)$	2 $(2x+3)(3x+2)$	3 $(4x+5)(8x-3)$
4 $(5x-6)(3x+4)$	5 $(3x-8)(2x-3)$	6 $(9x-1)(2x+5)$
7 $(4a-3)(3a-4)$	8 $(5x+9)(7x-6)$	9 $(2x-15)(8x-7)$

146. Cyclic order Let three letters a, b, c be placed in order round the circumference of a circle as shewn in the annexed diagram. If we start from any letter a and go round the circle in the direction of the arrows we see that a is followed by b, b by c and c by a, that is, the letters follow one another in the order abc; similarly if we start from b, the order of the letters is bca and if from c, the order is cab. When a, b, c follow one another in this way, they are said to be in *cyclic order*.



Hence the products of every two of the letters a, b, c, when written

in cyclic order are ab, bc, ca, if we start from a, but the products ab, ac, bc are not in cyclic order. Similarly if we start from b, the same products, written in cyclic order, are bc, ca, ab.

Again the differences of every two of the letters a, b, c, starting from b, are $b-c$, $c-a$, $a-b$, when written in cyclic order; but not so are $b-c$, $a-b$, $a-c$. And so on.

It is highly desirable to observe cyclic order in arranging the letters of an expression, for then the work will gain in simplicity and elegance. We shall observe cyclic order in the following formulæ.

147. Formula XV. $(x+a)(x+b)(x+c)$

$$=x^3+(a+b+c)x^2+(bc+ca+ab)x+abc$$

If a, b, c are all *negative*, this formula assumes the form

$$(x-a)(x-b)(x-c)=x^3-(a+b+c)x^2+(bc+ca+ab)x-abc$$

Ex. 1. Find the product of $x+3$, $x+4$ and $x+5$.

Required product $=x^3+(3+4+5)x^2+(4\cdot5+5\cdot3+3\cdot4)x+3\cdot4\cdot5$

$$=x^3+12x^2+47x+60$$

Ex. 2. Find the product of $(x-2)(x+5)(x-8)$.

Required product $=x^3+\{(-2)+5+(-8)\}x^2$

$$+\{5(-8)+(-8)(-2)+(-2)5\}x+(-2)5(-8)$$

$$=x^3-5x^2-34x+80$$

Examples LXXXVI.

Find the product of

1. $(m+1)(m+2)(m+3)$	2. $(x-1)(x-2)(x-3)$
3. $(x-1)(x+2)(x+3)$	4. $(x+2)(x+5)(x+8)$
5. $(k+1)(k-3)(k+4)$	6. $(p-1)(p-8)(p+11)$
7. $(z-1)(z+3)(z-4)$	8. $(h-3)(h-4)(h+6)$
9. $(x+a)(x+2a)(x+3a)$	10. $(x+a)(x+2a)(x-3a)$
11. $(x-a)(x-3a)(x-5a)$	12. $(y-3m)(y-2m)(y+m)$
13. $(m-7)(m+1)(m-10)$	14. $(h+2n)(h-3n)(h-4n)$
15. $(3x-1)(3x-4)(3x-10)$	16. $(5x-4)(5x-3)(5x+2)$

148. Formula XVI.

$$\begin{cases} a^2+b^2+c^2+2bc+2ca+2ab=(a+b+c)^2 & \text{(i)}, \\ (bc+ca+ab)^2=b^2c^2+c^2a^2+a^2b^2+2abc(a+b+c) & \text{(ii)} \end{cases}$$

[To prove identity (i). Arrange according to powers of a; thus left side

$$=a^2+2a(b+c)+(b^2+2bc+c^2)$$
$$=a^2+2a(b+c)+(b+c)^2$$
$$=(a+\overline{b+c})^2=(a+b+c)^2$$

To prove identity (ii). Expand left side; thus it

$$=(bc)^2+(ca)^2+(ab)^2+2(ca)(ab)+2(ab)(bc)+2(bc)(ca)$$
$$=b^2c^2+c^2a^2+a^2b^2+2a^2bc+2ab^2c+2abc^2$$
$$=b^2c^2+c^2a^2+a^2b^2+2abc(a+b+c)\ ]$$

Ex. 1. Shew that $4p^2+q^2+16r^2-4pq+16pr-8qr$ is a perfect square and find its value, when $p=4$, $q=12$ and $r=1$.

Arranged according to the powers of p, given expn

$$=4p^2-4p(q-4r)+(q^2+16r^2-8qr)$$
$$=(2p)^2-2(2p)(q-4r)+(q-4r)^2$$
$$=(2p-q+4r)^2$$

Hence required value $=(2\cdot 4-12+4\cdot 1)^2=0$.

Ex. 2. If $x=b+c$, $y=c-a$, $z=a-b$, prove that

$$x^2+y^2+z^2-2xy-2zx+2yz=4b^2$$

Arrange according to powers of some one letter, say x, thus

Left side $=x^2-2x(y+z)+(y^2+z^2+2yz)=x^2-2x(y+z)+(y+z)^2$

$$=\{x-(y+z)\}^2=(x-y-z)^2=(b+c-c+a-a+b)^2=(2b)^2$$

Ex. 3. Prove that $\{(y-z)(z-x)+(z-x)(x-y)+(x-y)(y-z)\}^2$

$$=(y-z)^2(z-x)^2+(z-x)^2(x-y)^2+(x-y)^2(y-z)^2$$

Put $a=y-z$, $b=z-x$, $c=x-y$, thus $a+b+c=y-z+z-x+x-y=0$

Now left side $=(ab+bc+ca)^2=a^2b^2+b^2c^2+c^2a^2+2abc(a+b+c)$

$$=a^2b^2+b^2c^2+c^2a^2, \qquad \because a+b+c=0,$$
$$=(y-z)^2(z-x)^2+(z-x)^2(x-y)^2+(x-y)^2(y-z)^2.$$

Examples LXXXVII

Shew that

1. $4x^2+9y^2+6yz-12xy+z^2-4zx$ is a perfect square.
2. $9m^2-12mn+16p^2-24mp+16np+4n^2$ is a perfect square, and find its value, when $m=2$, $n=3$, $p=-4$.

Find the value of

3. $bc+ca+ab$, when $a+b+c=2$, $a^2+b^2+c^2=1$.
4. $a^2+b^2+c^2$, when $a+b+c=3$, $bc+ca+ab=2$.

Find the value of

5 $a+b+c$, when $a^2+b^2+c^2=9$, $bc+ca+ab=8$

6 $x^2+y^2+z^2+2yz+2zx+2xy$, when $x=b+c-3a$, $y=c+a-3b$, and $z=a+b-3c$

7 $x^2+y^2+4z^2-4yz-4zx+2xy$, when $x=b+c-2a$, $y=c+a-2b$, and $-z=a+b-c$

8 If $x=b-2c$, $y=c-2a$, $z=a-2b$, shew that
$$x^2-12yz+4xy+9z^2-6zx+4y^2=49(a-b)^2$$

9 Shew that $(a-b)^2+(b-c)^2+(c-a)^2+2(b-c)(c-a)+2(c-a)(a-b)$
$$+2(a-b)(b-c)=0$$

149 Formula XVII $\begin{cases}(b-c)+(c-a)+(a-b)=0,\\ a(b-c)+b(c-a)+c(a-b)=0,\\ (b^2-c^2)+(c^2-a^2)+(a^2-b^2)=0\end{cases}$

[These results are very important The student can easily verify them]

Examples LXXXVIII

Simplify

1 $(a+1)(b-c)+(b+1)(c-a)+(c+1)(a-b)$

2 $(x+a)(b-c)+(x+b)(c-a)+(x+c)(a-b)$

3 $a^2(b+c)(b-c)+b^2(c+a)(c-a)+c^2(a+b)(a-b)$

4 $(x+y-z)(x-y)+(y+z-x)(y-z)+(z+x-y)(z-x)$

5 $(x^2-a^2)(b^2-c^2)+(x^2-b^2)(c^2-a^2)+(x^2-c^2)(a^2-b^2)$

6 $(x-b-c)(b-c)+(x-c-a)(c-a)+(x-a-b)(a-b)$

7 $(2a-b-c)(b-c)+(2b-c-a)(c-a)+(2c-a-b)(a-b)$

8 $(la+b+c)(b-c)+(lb+c+a)(c-a)+(lc+a+b)(a-b)$

9 $\{ma-n(b+c)\}(b-c)+\{mb-n(c+a)\}(c-a)$
$$+\{mc-n(a+b)\}(a-b)$$

10 $\{\lambda a+l(b+c)+m\}(b-c)+\{\lambda b+l(c+a)+m\}(c-a)$
$$+\{\lambda c+l(a+b)+m\}(a-b)$$

150 Formula XVIII
$$(a^2+ab+b^2)(a^2-ab+b^2)=a^4+a^2b^2+b^4$$

Conversely $a^4+a^2b^2+b^4=(a^2+ab+b^2)(a^2-ab+b^2)$

[For $a^4+a^2b^2+b^4=(a^4+2a^2b^2+b^4)-a^2b^2$
$$=(a^2+b^2)^2-(ab)^2=(a^2+ab+b^2)(a^2-ab+b^2)$$]

151 Factors of Expressions of the form

$$abx^2+(a^2+b^2)x+ab$$

We have $(ax+b)(bx+a)=abx^2+(a^2+b^2)x+ab$, that is, $abx^2+(a^2+b^2)x+ab$ has two factors $ax+b$ and $bx+a$. Thus if the coefficient of x^2 is *the same as the last term* (irrespective of signs), and the factors of the last term are such that *the algebraic sum of their squares* is the coefficient of x, then the factors of the expression are of the form $(ax\ \ b)$,$(bx\ \ a)$, and can be written down at once, the proper signs being supplied as in Art 74

Ex Factorize $15x^2-16x-15$

The factors of 15 are 1, 15 or 3, 5, we take the latter pair as the algebraic sum of their squares is 16 Hence the required factors are of the form $(5x\ \ 3)$ and $(3x\ \ 5)$ also 3 and 5 must have *different* signs and the sign of 5 must be $-$.

$$\text{given expression}=(5x+3)(3x-5)$$

Examples LXXXIX.

Resolve into component factors

1.	$6x^2+13x+6$	2	$15x^2+34x+15$	3	$12x^2+25x+12$
4	$24x^2-73x+24.$	5.	$12x^2-145x+12$	6	$40x^2-89x+40$
7.	$8x^2+63x-8$	8	$12m^2-7m-12$	9	$15x^2-224x-15$
10	$12x^2-45x-12$	11	$16x^2+255x-16$	12	$11x^2+120x-11$
13	$9x^2+80x-9$	14.	$6x^2-35x+6$	15	$45l^2-56l-45.$
16	$8m^2-65mn+8n^2$	17	$36x^2+21x-36$	18	$6x^2-5xy-6y^2$
19	$14x^2-35xy+14y^2$	20	$7x^2-48xy-7y^2$	21	$10x^2-39xy-40y^2.$

152 In factorizing expressions by suitable rearrangement and grouping of the terms [Art 73], the following two cases should be noticed.

Case I If an expression contains *one of the letters* only in the *first power*, its factors are generally obvious when it is rearranged according to the powers of that letter

Ex 1 Resolve $1+(b-a^2)x^2-abx^3$ into factors

Here b occurs *only* in the *first* power Arrange therefore in powers of b, thus we have

$$\begin{aligned}&(1-a^2x^2)+b(x^2-ax^3)\\&=(1+ax)(1-ax)+bx^2(1-ax)\\&=(1-ax)(1+ax+bx^2)\end{aligned}$$

Ex 2 Resolve $(a^2-b^2)x^2-(2a^2-b^2)bx+a^2b^2$ into factors

Putting $a^2=m$, we see that m occurs only in the first power Arranging according to powers of m, i e, a^2, we have

$$a^2(x^2-2bx+b^2)-(b^2x^2-b^3x)$$
$$=a^2(x-b)^2-b^2x(x-b)$$
$$=(x-b)\{a^2(x-b)-b^2x\}$$
$$=(x-b)(a^2x-b^2x-a^2b)$$

Ex 3 Factorize $a^2-2ab-4ac+2bc+3c^2$

Arrange in powers of b, thus the given expression

$$=(a^2-4ac+3c^2)-2b(a-c)$$
$$=(a-c)(a-3c)-2b(a-c)$$
$$=(a-c)(a-2b-3c)$$

Examples XC

Resolve into factors

1	$a^3-a^2c-ab^2+b^2c$	2	$acx^3+bcx^2+adx+bd$
3	$x^3+(a-b)x^2+(1-ab)x+a$	4	$x^3-2(a+c)x^2+(3a+4c)ax-6a^2c$
5	$x^2-(a+b+c)x+ab+ac$	6	$x^2+2ax-2ab-b^2$
7	$1-2ax-(c-a^2)x^2+acx^3$	8	$15xy+11x+3x^2-5y-4$
9	$x^2-axy-2ax+a^2y+a^2$	10	$x^2+2xz-6xy-2yz+5y^2$
11	$a^2+ab-3ac-2b^2+3bc$	12	$a^2-13ab-20bc+36b^2+5ac$

13 $x^3-(2a+b)x^2+(2ab+a^2)x-a^2b$

14 $a^3-a^2(5b^2+c^2)+ab^2(4b^2+5c^2)-4b^4c^2$

15 $x^4+(p-a)x^3-(ap+q+1)x^2-(p-aq)x+q$

Case II If an expression is of the *second degree in any one of the letters*, it may be rearranged when written according to the powers of that letter

Ex 4 Factorize $a^2-2ab-4ac+2bc+3c^2$ [Ex 3]

Consider this as a quadratic in a, and arrange accordingly, thus the given expression

$$=a^2-2(b+2c)a+c(2b+3c)$$
$$=a^2-\{c+(2b+3c)\}a+c(2b+3c)$$
$$=(a-c)(a-2b-3c) \text{ [Art 74]}$$

Ex 5 Factorize $x^2-2(a+b)x-ab(a-2)(b+2)$

Given expression $=x^2-(2a+2b)x-(ab-2b)(ab+2a)$

$=x^2-\{(ab+2a)-(ab-2b)\}x-(ab+2a)(ab-2b)$

$=\{x-(ab+2a)\}\{x+(ab-2b)\}$ [Art. 68]

$=(x-2a-ab)(x-2b+ab)$

Examples XC. *(Continued)*

Resolve into factors

16.	$x^2+xy-6y^2-x+2y$	17	$x^2-y^2-x+3y-2$
18	$x^2-2(a+1)x-3a(a-2)$.	19	$a^2+3a(2b-1)+2(6b-5)$
20	$a^2+2a(b+c)-3b(b-2c)$	21	$x^2+2(a+1)x-3a(5a+2)$
22	$x^2+(a-b)x-ab(a+1)(b+1)$.	23	$x^2-5ax+6(a+2b)(a-3b)$
24.	$x^2-(a^2+b^2)x-ab(a^2-b^2)$	25	$x^2-4xy+3y^2+4x-10y+3$
26	$a^2+3b^2-4c^2-4ab+4bc$	27	$3x^2+(5b-11a)x+3a(2a-5b)$

[Also work Examples 5–12 inclusive by this method, and Examples 19, 20 and 27 by the method shewn in Case I.]

153 General method of resolving Quadratics The tentative processes of resolving quadratic expressions given in Arts. 74 and 75 are practically very useful We shall however give a general method applicable to all cases, by which we **express a quadratic as the difference of two squares**, and thus resolve it as in Art. 71

A quadratic of the form x^2+px+q or ax^2+bx+c can be expressed as the difference of two squares by *completing the square*, to do which there are two methods—(i) the *common* method, and (ii) *Sridhara's* method.

COMMON METHOD

We know that $x^2+2ax+a^2$ is a perfect square, viz, $(x+a)^2$, where the last term is the square of a which is half the coefficient of x Hence if the first two terms x^2+px of a quadratic are given, x^2+px may be made a complete square $\left(x+\frac{p}{2}\right)^2$ *by adding the square of half the coefficient of* x

Thus x^2+8x is made a complete square $(x+4)^2$ when we add to it $(\frac{8}{2})^2$ or 16, x^2-5x is made a perfect square, if $(-\frac{5}{2})^2$ or $\frac{25}{4}$ be added to it, and so on

SRIDHARA'S METHOD*

The expression $4x^2+4px+p^2$ is a perfect square, viz, $(2x+p)^2$, where the coefficient of x^2 is 4 and the last term is the square of p. Thus if x^2+px is multiplied by 4, *i.e.*, by 4 times the coefficient of x^2 which is 1, and then the square of p, the coefficient of x, is added, then the square is completed. Hence when the first two terms of a quadratic are given, the square may be completed *by multiplying by 4 times the coefficient of x^2 and then adding the square of the coefficient of x*.

Thus x^2+3x is made a complete square, by multiplying it by 4×1, and then adding 3^2 or 9, thus it becomes $4x^2+12x+9$ or $(2x+3)^2$. Similarly x^2-5x is made a perfect square if we multiply it by 4 and then add $(-5)^2$ or 25, we shall then have $4x^2-20x+25$ or $(2x-5)^2$. And so on.

This method is often convenient as it enables us *to avoid fraction in completing the square*.

Ex 1 Express x^2+8x as the difference of two squares.

(i) $x^2+8x=x^2+8x+(\frac{8}{2})^2-(\frac{8}{2})^2=(x^2+8x+16)-(4)^2=(x+4)^2-(4)^2$

Here we add $(\frac{8}{2})^2$, and to preserve the equality, we subtract $(\frac{8}{2})^2$

(ii) $x^2+8x=(x^2+8x)\times 4\div 4=\frac{1}{4}(4x^2+32x)=\frac{1}{4}(4x^2+32x+64-64)$
$=\frac{1}{4}\{(2x+8)^2-(8)^2\}$.

Ex 2 Express $3x^2-16x$ as the difference of two squares.

(i) $3x^2-16x=3\{x^2-\frac{16}{3}x\}$
$=3\{x^2-\frac{16}{3}x+(\frac{8}{3})^2-(\frac{8}{3})^2\}$, $\quad \frac{1}{2}$ of $\frac{16}{3}=\frac{8}{3}$,
$=3\{(x-\frac{8}{3})^2-(\frac{8}{3})^2\}$

(ii) $3x^2-16x=(3x^2-16x)\times 12\div 12=\frac{1}{12}\{36x^2-192x\}$
$=\frac{1}{12}\{36x^2-192x+(16)^2-(16)^2\}$
$=\frac{1}{12}\{(6x-16)^2-(16)^2\}$

Ex 3 Factorize $x^2-13x+40$ by expressing it as the difference of two squares.

(i) $x^2-13x+40=x^2-13x+(\frac{13}{2})^2-(\frac{13}{2})^2+40$
$=\{x^2-13x+(\frac{13}{2})^2\}-\{(\frac{13}{2})^2-40\}$
$=\{x^2-13x+(\frac{13}{2})^2\}-\frac{9}{4}=(x-\frac{13}{2})^2-(\frac{3}{2})^2$
$=(x-\frac{13}{2}+\frac{3}{2})(x-\frac{13}{2}-\frac{3}{2})=(x-5)(x-8)$

* This ingenious method is due to SRIDHARACHARYA, a celebrated Hindu algebraist, and is commonly known as the "HINDU METHOD." Some writer erroneously ascribes it to BHASKAR, who, however, himself lays no claim to it (*see Vijaganita*, § 131)

(ii) $x^2-13x+40=\frac{1}{4}\times 4(x^2-13x+40)$

[Since we multiply by 4 we divide by 4 to keep the value unaltered]

$$=\tfrac{1}{4}(4x^2-52x+160)$$
$$=\tfrac{1}{4}\{4x^2-52x+(13)^2-(13)^2+160\}$$

[adding and subtracting the square of coefficient of x]

$$=\tfrac{1}{4}\{(2x-13)^2-9\}$$
$$=\tfrac{1}{4}(2x-10)(2x-16)$$
$$=\tfrac{1}{4}\,2(x-5)2(x-8)=(x-5)(x-8)$$

Ex 4 Factorize $24x^2-70x-75$ by expressing it as the difference of two squares.

$$24x^2-70x-75=24\{x^2-\tfrac{70}{24}x-\tfrac{75}{24}\}$$
$$=24\{x^2-\tfrac{70}{24}x+(\tfrac{35}{24})^2-(\tfrac{35}{24})^2-\tfrac{75}{24}\}$$
$$=24\{(x-\tfrac{35}{24})^2-(\tfrac{55}{24})^2\}$$
$$=24(x-\tfrac{35}{24}+\tfrac{55}{24})(x-\tfrac{35}{24}-\tfrac{55}{24})$$
$$=6(x+\tfrac{5}{6})\times 4(x-\tfrac{15}{4})$$
$$=(6x+5)(4x-15)$$

Ex. 5 Resolve $2x^2-xy-6y^2+9x+17y-5$ into factors

Given expression $=2x^2-(y-9)x-(6y^2-17y+5)$

$$=\tfrac{1}{8}\{16x^2-8(y-9)x-(48y^2-136y+40)\}$$
$$=\tfrac{1}{8}\{16x^2-8(y-9)x+(y-9)^2-(y-9)^2-(48y^2-136y+40)\}$$
$$=\tfrac{1}{8}\{(4x-y+9)^2-(7y-11)^2\}$$
$$=\tfrac{1}{8}(4x-y+9+7y-11)(4x-y+9-7y+11)$$
$$=\tfrac{1}{8}\times 2(2x+3y-1)\times 4(x-2y+5)$$
$$=(2x+3y-1)(x-2y+5)$$

Examples of this type are, however, *best worked* by *the method of* Art 152 Thus arranged according to powers of x,

Given expression $=2x^2-(y-9)x-(6y^2-17y+5)$

$$=2x^2-(y-9)x-(3y-1)(2y-5)$$
$$=(2x+\overline{3y-1})(x-\overline{2y-5})$$

Examples XCI

Express as the difference of two squares

1 x^2+5x	2 x^2-20x	3 $x(x-18)$
4 $x(x+13)$	5 $2x^2-5x$	6 $5x^2+12x$

Express as the difference of two squares

7 $(x-2m)(x+3m)$ 8 $(x+q)^2-(x+q)$

9 $x(x+1)(x+2)(x+3)$ 10 $x(x-a)(x+2a)(x+a)$

Resolve into factors

11 $x^2+28x+192$ 12 $l^2-21l+90$ 13 $a^2+9a-162$

14 $x^2-18x+80$ 15 $a^2-a-380$ 16 $3x^2-20x-32$

17 $12a^2-23a+10$ 18 $30-31x+5x^2$ 19 $12+17x-7x^2$

20 $10x^2-19xy-15y^2$ 21 $6x^2+23ax-18a^2$

22 $4x^2-51ax+36a^2$ 23 $12-71x-60x^2$

24 $15x^2+32x-775$ 25 $24z^2-406z+1715$

26 $y^2-y-8930$ 27 $8x^2+513x-10935$

[*N B The following examples are more easily worked by the methods of Art* 152]

28 x^2+2x-y^2-4y-3 29 x^2-6x-y^2+2y+8

30 $a^2-6ab+5b^2+4bc-c^2$ 31 $x^2+10xy+9y^2+8yz-z^2$

32 $a^2-2ab-4ac+2bc+3c^2$ 33 $a^2+2ab-2ac-3b^2+2bc$

34 $3x^2-2ax-a^2+4ac-3c^2$ 35 $xy+2y^2+x+5y+3$

36 $2xy-6x+3y^2-5y-12$ 37 $x^2-2y^2-z^2-xy-3yz$

38 $15xy+11x+3x^2-5y-4$ 39 $2y^2-5xy+2x^2-ay-ax-a^2$

40 $x^2+4xy+3y^2+4ax+6ay+3a^2$

154 Formula XIX.

$$a^2(b-c)+b^2(c-a)+c^2(a-b)=-(b-c)(c-a)(a-b)$$

To establish this Formula we have to *resolve* the left-hand expression

Arranged according to the descending powers of a,

$$\begin{aligned}\text{Left side } &=a^2(b-c)-a(b^2-c^2)+bc(b-c)\\ &=(b-c)\{a^2-a(b+c)+bc\}\\ &=(b-c)(a-c)(a-b)\\ &=-(b-c)(c-a)(a-b), \qquad (a-c)=-(c-a)\end{aligned}$$

Otherwise —Since $(b-c)+(c-a)=b-a=-(a-b)$,

$$\begin{aligned}\text{Left side } &=a^2(b-c)+b^2(c-a)-c^2\{(b-c)+(c-a)\}\\ &=a^2(b-c)-c^2(b-c)+b^2(c-a)-c^2(c-a)\\ &=(a^2-c^2)(b-c)+(b^2-c^2)(c-a)\\ &=(b^2-c^2)(c-a)-(b-c)(c^2-a^2)\\ &=(b-c)(c-a)\{(b+c)-(c+a)\}\\ &=(b-c)(c-a)(b-a)=-(b-c)(c-a)(a-b).\end{aligned}$$

It is easy to see that

$$a^2(b-c)+b^2(c-a)+c^2(a-b) \qquad . \qquad . \quad (1)$$

and
$$bc(b-c)+ca(c-a)+ab(a-b) \quad . \qquad \ldots \quad (2)$$

are, when the brackets are removed, only *different forms* of the expression

$$a^2b-ab^2+b^2c-bc^2+c^2a-ca^2 \; . \qquad . \qquad . \quad (3)$$

Thus (1), (2) and (3) are equal to one another Also by removing the brackets it will be seen that

$$a(b^2-c^2)+b(c^2-a^2)+c(a^2-b^2) \qquad . \quad (4)$$
$$=-(a^2b-ab^2+b^2c-bc^2+c^2a-ca^2)$$

Thus (4) is equal to the first three in *absolute value*, but differs from them *only in sign*

Hence Formula XIX may be written in the following forms

$$\left.\begin{array}{l} a^2(b-c)+b^2(c-a)+c^2(a-b) \\ \text{or } bc(b-c)+ca(c-a)+ab(a-b) \\ \text{or } a^2b-ab^2+b^2c-bc^2+c^2a-ca^2 \end{array}\right\} =-(b-c)(c-a)(a-b) \quad \text{(i)}$$

$$\text{Also } a(b^2-c^2)+b(c^2-a^2)+c(a^2-b^2)=(b-c)(c-a)(a-b) \quad \text{(ii)}$$

The student will notice that we have observed *cyclic order* throughout He will see later on the advantage of this arrangement.

Ex 1 Shew that $(a-b)(x-a)(x-b)+(b-c)(x-b)(x-c)$
$$+(c-a)(x-c)(x-a)=(a-b)(b-c)(a-c)$$

Left side$=(a-b)\{x^2-(a+b)x+ab\}+(b-c)\{x^2-(b+c)x+bc\}$
$$+(c-a)\{x^2-(c+a)x+ca\} \text{ [Art 60]}$$
$$=x^2\{(a-b)+(b-c)+(c-a)\}-x\{(a^2-b^2)+(b^2-c^2)+(c^2-a^2)\}$$
$$+\{ab(a-b)+bc(b-c)+ca(c-a)\}$$
$$=x^2\times 0-x\times 0+\{ab(a-b)+bc(b-c)+ca(c-a)\} \text{ [Art 154]}$$
$$=-(b-c)(c-a)(a-b)=\&c$$

Ex 2 Resolve $a^3(b-c)+b^3(c-a)+c^3(a-b)$

Arrange the given expression according to powers of a, which is thus

$$=a^3(b-c)-a(b^3-c^3)+bc(b^2-c^2)$$
$$=(b-c)\{a^3-a(b^2+bc+c^2)+bc(b+c)\}$$

Arrange the second factor according to powers of b, thus

the given expression$=(b-c)\{b^2(c-a)+bc(c-a)-a(c^2-a^2)\}$
$$=(b-c)(c-a)\{b^2+bc-a(c+a)\}$$

Arrange the third factor according to powers of c, thus the given expression $=(b-c)(c-a)\{c(b-a)+(b^2-a^2)\}$

$$=(b-c)(c-a)(b-a)(c+b+a)$$
$$=-(b-c)(c-a)(a-b)(a+b+c)$$

Otherwise.—Given expression

$$=a^3(b-c)+b^3(c-a)-c^3\{(b-c)+(c-a)\}$$

Then proceed as in Formula XIX [Art 154]

Note It is easy to see that

$$a^3b-ab^3+b^3c-bc^3+c^3a-ca^3$$
$$=a^3(b-c)+b^3(c-a)+c^3(a-b)$$
$$=bc(b^2-c^2)+ca(c^2-a^2)+ab(a^2-b^2)$$
$$=-\{a(b^3-c^3)+b(c^3-a^3)+c(a^3-b^3)\},$$

hence each $=-(b-c)(c-a)(a-b)(a+b+c)$

Ex 3 Resolve into factors

$$(b-c)(b+c-2a)^2+(c-a)(c+a-2b)^2+(a-b)(a+b-2c)^2.$$

Assume $x=b+c-2a$, $y=c+a-2b$, $z=a+b-2c$, thus

$$x-y=b+c-2a-(c+a-2b)=3(b-a)=-3(a-b),$$

similarly $\quad y-z=-3(b-c)$ and $z-x=-3(c-a)$

Hence $3(b-c)(b+c-2a)^2=-(y-z)x^2=-x^2(y-z)$,

$3(c-a)(c+a-2c)^2=-(z-x)y^2=-y^2(z-x)$,

$3(a-b)(a+b-2c)^2=-(x-y)z^2=-z^2(x-y)$,

$$3 \text{ times proposed expn} = -x^2(y-z)-y^2(z-x)-z^2(x-y)$$
$$=(y-z)(z-x)(x-y)$$
$$=-3(b-c)\times-3(c-a)\times-3(a-b)$$
$$=-27(b-c)(c-a)(a-b),$$

whence proposed expression $=-9(b-c)(c-a)(a-b)$

Examples XCII

1 Simplify

$(a^2-bc+a)(b-c)+(b^2-ca+b)(c-a)+(c^2-ab+c)(a-b)$

2 $(a^2-bc)(b-c)+(b^2-ca)(c-a)+(c^2-ab)(a-b)$

Resolve into factors

3 $a(b+c)(b-c)+b(c+a)(c-a)+c(a+b)(a-b)$

4 $a(a+1)(b-c)+b(b+1)(c-a)+c(c+1)(a-b)$

5 $(x^2+x+1)(y^2-z^2)+(y^2+y+1)(z^2-x^2)+(z^2+z+1)(x^2-y^2)$

Resolve into factors

6 $(x-a)(x-b)(y-z)+(y-a)(y-b)(z-x)+(z-a)(z-b)(x-y)$

7 $bc(b-c)(x+a)^2+ca(c-a)(x+b)^2+ab(a-b)(x+c)^2$

8 $bc(b-c)+ca(c-a)+ab(a-b)$

9 $(b-c)(b+c)^2+(c-a)(c+a)^2+(a-b)(a+b)^2$

10 $a(b^3-c^3)+b(c^3-a^3)+c(a^3-b^3)$

11 $bc(b^2-c^2)+ca(c^2-a^2)+ab(a^2-b^2)$

12 $(1+ca)(1+ab)(b-c)+(1+ab)(1+bc)(c-a)+(1+bc)(1+ca)(a-b)$

13 $a(b-c)(x-b)(x-c)+b(c-a)(x-c)(x-a)+c(a-b)(x-a)(x-b)$.

14 $(b-c)(b+c-a)^2+(c-a)(c+a-b)^2+(a-b)(a+b-c)^2$

15 If $x=b-c$, $y=c-a$, $z=a-b$, shew that

$$xyz=a(b+c)x+b(c+a)y+c(a+b)z$$

155 Formula XX

$$(a+b+c)(a^2+b^2+c^2-bc-ca-ab)=a^3+b^3+c^3-3abc$$

This formula may be established by ordinary multiplication

Ex 1 Multiply $x^2+4y^2+9z^2-2xy+6yz+3zx$ by $x+2y-3z$

Put $x=a$, $2y=b$ and $-3z=c$, thus required product,

$=\{x^2+(2y)^2+(-3z)^2-x(2y)-2y(-3z)-(-3z)x\}(x+2y-3z)$

$=(a^2+b^2+c^2-ab-bc-ca)(a+b+c)=a^3+b^3+c^3-3abc$

$=x^3+(2y)^3+(-3z)^3-3x(2y)(-3z)=x^3+8y^3-27z^3+18xyz$

Examples XCIII

Write down the value of

1 $(a+b-c)(a^2+b^2+c^2+bc+ca-ab)$.

2 $(2x-y+z)(4x^2+2xy+y^2+z^2+yz-2zx)$

3 $(3x-2y-4z)(9x^2+6xy+4y^2-8yz+16z^2+12zx)$

4 $(1-x+2y)(1+x-2y+x^2+2xy+4y^2)$

5 $(-x-2y+3)(x^2-2xy+4y^2+3x+6y+9)$

6 $(3a-4b-2)(9a^2+16b^2+12ab+6a-8b+4)$

7 $(x+y-1)(x^2-xy+y^2+x+y+1)$

From this Formula, we have conversely

$$a^3+b^3+c^3-3abc=(a+b+c)(a^2+b^2+c^2-bc-ca-ab)$$

Thus we have the **factors of** $a^3+b^3+c^3-3abc$

The factors can, however, be found thus —

$$a^3+b^3+c^3-3abc=(a+b)^3-3ab(a+b)+c^3-3abc \text{ [Art 144]}$$
$$=\{(a+b)^3+c^3\}-\{3ab(a+b)+3abc\}$$
$$=(a+b+c)\{(a+b)^2-(a+b)c+c^2\}-3ab(a+b+c)$$
$$=(a+b+c)\{(a+b)^2-(a+b)c+c^2-3ab\}$$
$$=(a+b+c)(a^2+b^2+c^2-bc-ca-ab)$$

The factors can be put in a different form.

We have $a^2+b^2+c^2-bc-ca-ab$

$$=\tfrac{1}{2}\times 2(a^2+b^2+c^2-bc-ca-ab)$$
$$=\tfrac{1}{2}(2a^2+2b^2+2c^2-2bc-2ca-2ab)$$
$$=\tfrac{1}{2}\{(b-c)^2+(c-a)^2+(a-b)^2\},$$

$a^3+b^3+c^3-3abc$

$$=\tfrac{1}{2}(a+b+c)\{(b-c)^2+(c-a)^2+(a-b)^2\}$$

Hence $a^3+b^3+c^3-3abc$

$$=(a+b+c)(a^2+b^2+c^2-bc-ca-ab) \quad \text{(i)},$$
$$=\tfrac{1}{2}(a+b+c)\{(b-c)^2+(c-a)^2+(a-b)^2\} \quad \text{(ii)}$$

Corollary If $a+b+c=0$, then (i) and (ii) are each equal to 0, thus $a^3+b^3+c^3-3abc=0$ Hence we have the following important theorem

If $a+b+c=0$, *then* $a^3+b^3+c^3-3abc=0$

This corollary may be proved independently, thus —

We have $\quad a+b+c=0,$

ranspose c, thus $\quad a+b=-c, \quad (a)$

ube both sides, thus $\quad a^3+b^3+3ab(a+b)=-c^3,$

from (a), $\quad a^3+b^3+3ab(-c)=-c^3,$

or $\quad a^3+b^3-3abc=-c^3,$

vhence $\quad a^3+b^3+c^3-3abc=0$

Ex 2 Factorize $1+8x^3+18xy-27y^3$

(en expn $=1^3+(2x)^3+(-3y)^3-3\cdot 1\cdot 2x(-3y)$

$$=a^3+b^3+c^3-3abc \text{ [where } a=1,\ b=2x,\ c=-3y]$$
$$=(a+b+c)(a^2+b^2+c^2-bc-ca-ab)$$
$$=(1+2x-3y)(1+4x^2+9y^2+6xy+3y-2x)$$

Or we may proceed directly thus —Given expression

$$=(1+2x)^3-3\cdot 2x(1+2x)-27y^3+18xy$$
$$=\{(1+2x)^3-(3y)^3\}-\{6x(1+2x)-18xy\}$$
$$=(1+2x-3y)\{(1+2x)^2+3y(1+2x)+9y^2\}-6x(1+2x-3y)$$
$$=(1+2x-3y)(1+4x^2+9y^2+6xy+3y-2x)$$

Ex 3 If $x=\cdot 43$ and $y=\cdot 57$, find the value of $x^3+y^3+3xy+1$

Given expression $=(x^3+y^3+3xy-1)+2$

$$=(x+y-1)(x^2-xy+y^2+x+y+1)+2\ ,$$

but $\quad x+y-1=\cdot 43+\cdot 57-1=1-1=0\ ,$

given expression $=0\times(x^2-xy+y^2+x+y+1)+2=2$

Ex 4 Shew that $(y+z)^3+(z+x)^3+(x+y)^3-3(y+z)(z+x)(x+y)$
$$=2(x^3+y^3+z^3-3xyz)$$

Given expression $=\frac{1}{2}(y+z+z+x+x+y)\{(z-y)^2+(x-z)^2+(y-x)^2\}$
$$[\quad (z+x)-(x+y)=z-y,\ \&c\ ]$$
$$=(x+y+z)\times 2(x^2+y^2+z^2-yz-zx-xy)$$
$$=2(x^3+y^3+z^3-3xyz)$$

Ex 5 If $x=a+b+c$, shew that $(x+a)^3+(x+b)^3+(x+c)^3$
$$-3(x+a)(x+b)(x+c)=4(a^3+b^3+c^3-3abc)$$

Given expression $=\frac{1}{2}(3x+a+b+c)\{(b-c)^2+(c-a)^2+(a-b)^2\}$
$$[\quad (x+b)-(x+c)=b-c,\ \&c\ ]$$
$$=2(a+b+c)\times 2(a^2+b^2+c^2-bc-ca-ab)$$
$$[\quad x=a+b+c]$$
$$=4(a^3+b^3+c^3-3abc)$$

Ex. 6 Prove that $(b-c)^3+(c-a)^3+(a-b)^3=3(b-c)(c-a)(a-b)$

Here $(b-c)+(c-a)+(a-b)=0\ ,$

$$(b-c)^3+(c-a)^3+(a-b)^3-3(b-c)(c-a)(a-b)=0\ [Cor\ ],$$

whence $\quad (b-c)^3+(c-a)^3+(a-b)^3=3(b-c)(c-a)(a-b)$

Examples XCIII *(Continued)*

8 Factorize $a^3-3ab+b^3+1$ 9 Factorize $a^3+8b^3+6ab-1$

10. Factorize $x^3+3axy+y^3-a^3$

11 Factorize $(y-z)^3+(z-x)^3+(x-y)^3$

12 Factorize $a^3(b-c)^3+b^3(c-a)^3+c^3(a-b)^3$

13 Factorize $(2x-y)^3-(x+y)^3+(2y-x)^3$.

14 If $x=b+c$, $y=c+a$, $z=a+b$, shew that

$$x^3+y^3+z^3-3xyz=2(a^3+b^3+c^3-3abc)$$

15 If $a=y+z-x$, $b=z+x-y$, $c=x+y-z$, then

$$a^3+b^3+c^3-3abc=4(x^3+y^3+z^3-3xyz)$$

16 Prove that $(x-2y)^3+(2y-1)^3+(1-x)^3=3(x-2y)(2y-1)(1-x)$

17 Prove that

$$(ax-by)^3+(by-cz)^3+(cz-ax)^3=3(ax-by)(by-cz)(cz-ax)$$

18 Prove that $(b+c-2a)^3+(c+a-2b)^3+(a+b-2c)^3$

$$=3(b+c-2a)(c+a-2b)(a+b-2c)$$

19 Find the value of $a^3+b^3+3abc-c^3$, when $a=02$, $b=08$ and $c=10$

20 If $x=\frac{41}{75}$, $y=\frac{34}{75}$, find the value of $3x^3+y^3+3xy-1$

21 If $x=(b-c)(a-d)$, $y=(c-a)(b-d)$, $z=(a-b)(c-d)$, find the value of $x^3+y^3+z^3-3xyz$ [See *App*]

22 If $s=a+b+c$, shew that $(3a-s)^3+(3b-s)^3+(3c-s)^3$

$$=3(3a-s)(3b-s)(3c-s)$$

23 If $2s=a+b+c$, shew that

$$(s-a)^3+(s-b)^3+(s-c)^3-3(s-a)(s-b)(s-c)=\tfrac{1}{2}(a^3+b^3+c^3-3abc)$$

24 Shew that

$$a^3(bz-cy)^3+b^3(cx-az)^3+c^3(ay-bx)^3=3abc(bz-cy)(cx-az)(ay-bx)$$

25 Shew that $(3a-b-c)^3+(3b-c-a)^3+(3c-a-b)^3$

$$-3(3a-b-c)(3b-c-a)(3c-a-b)=16(a^3+b^3+c^3-3abc)$$

26 Shew that the value of $x^2+y^2+z^2-yz-zx-xy$ will not change if x, y, z be *increased* or *decreased by a constant quantity*, e g, *if* $x=x+d$, $y=y+d$, $z=z+d$, or if $x=x-d$, $y=y-d$, $z=z-d$

156 In the following examples the tentative method is to be used We may observe here that if *one* factor of the *first* degree of a cubic, *two* of a biquadratic, &c, be found by trial, the remaining factor in each case must be a *quadratic* which can be resolved by the method of Art 153

Ex 1 Resolve x^3-3x^2-6x+8 into factors

The given expression, when rearranged, may be put in the following forms —

(i) $(x^3+8)-(3x^2+6x)$,

(ii) $x(x^2-3x+2)-8(x-1)$,

(iii) $(x^3-4x^2)+(x^2-4x)-(2x-8)$

Hence

(i) Given expn $=(x+2)(x^2-2x+4)-3x(x+2)$

$=(x+2)(x^2-5x+4)=(x+2)(x-1)(x-4)$

(ii) Given expn $=x(x-1)(x-2)-8(x-1)$

$=(x-1)(x^2-2x-8)=(x-1)(x+2)(x-4)$

(iii) Given expn $=x^2(x-4)+x(x-4)-2(x-4)$

$=(x-4)(x^2+x-2)=(x-4)(x-1)(x+2)$

[Observe that the *three* factors of 8 *must* be 1, 2 and 4, and since it has a + sign, *two* of these factors must have the *same sign* and consequently the third, the sign + Hence if the proposed expression has 3 linear factors, they must be of the form $x\pm1$, $x\pm2$ and $x\pm4$ Now if we put $x=1$, or $x=-2$, or $x=4$ in the given expression, it vanishes, therefore by the Remainder Theorem [Art 141], $x-1$, $x+2$ and $x-4$ are factors of the expression]

Ex 2 Resolve $2x^3-13x^2+27x-18$ into factors

We may rearrange the expression in three ways, thus —

(i) $x^2(2x-3)-(10x^2-27x+18)$,

(ii) $x(2x^2-13x+15)+6(2x-3)$,

(iii) $(2x^3-6x^2)-(7x^2-21x)+(6x-18)$

Taking (i), the given expression

$=x^2(2x-3)-(2x-3)(5x-6)$

$=(2x-3)(x^2-5x+6)=(2x-3)(x-2)(x-3)$

[Find the factors taking (ii) and (iii)]

Ex 3 Resolve $a^3-19ab^2+30b^3$ into factors

Rearranging and grouping the terms, we have

(i) $(a^3-8b^3)-(19ab^2-38b^3)$,

(ii) $a(a^2-9b^2)-10b^2(a-3b)$,

(iii) $(a^3+5a^2b)-(5a^2b+25ab^2)+(6ab^2+30b^3)$

[Here we introduce a 'false' term]

Take (i), thus the given expression

$=(a-2b)(a^2+2ab+4b^2)-19b^2(a-2b)$

$=(a-2b)(a^2+2ab-15b^2)$

$=(a-2b)(a-3b)(a+5b)$

[Taking (ii) and (iii), find the factors]

Ex 4 Resolve x^4-4x+3 into factors

The terms may be grouped in the following ways—

$$\text{(i)}\quad (x^4-1)-4(x-1),$$
$$\text{(ii)}\quad x(x^3-1)-3(x-1)$$

Taking (i), we have $(x-1)(x^3+x^2+x+1)-4(x-1)$

$$=(x-1)(x^3+x^2+x-3)$$
$$=(x-1)\{(x^3-1)+(x^2-1)+(x-1)\}$$
$$=(x-1)(x-1)(x^2+x+1+x+1+1)$$
$$=(x-1)^2(x^2+2x+3)$$

[Take (ii) and find the factors of the given expression]

Ex 5 Resolve $x^4+11x^3+41x^2+61x+30$ into factors

Given expn. $=x^2(x^2+11x+30)+(11x^2+61x+30)$

$$=x^2(x+5)(x+6)+(x+5)(11x+6)$$
$$=(x+5)(x^3+6x^2+11x+6)$$

The second factor $=(x^3+8)+(6x^2+11x-2)$

$$=(x+2)(x^2-2x+4)+(x+2)(6x-1)$$
$$=(x+2)(x^2+4x+3)=(x+2)(x+1)(x+3)$$

Given expn $=(x+5)(x+2)(x+1)(x+3)$

Examples XCIV

Resolve into factors

1	$x^3+7x^2+14x+8$	2	$x^3+10x^2+29x+20$
3	$x^3+9x^2+6x-16$	4	$x^3+x^2-17x+15$
5	$x^3+4x^2+11x+8$	6	$x^3-3x^2-10x+24$
7	$6x^3+17x^2-5x-6$	8	$2x^3-4x^2-8x+16$
9	a^3-3a+2	10	x^3-3x^2+4
11	$x^3-31x-30$	12	$4a^3+ab^2-b^3$
13.	$4x^3+13x^2-9$	14	$8x^3-16x^2+9$
15	$a^3-a^2b-18b^3$	16	$a^3+9a^2b-8b^3$
17	$x^3-28xy^2+48y^3$	18.	$a^3-2a^2b-9b^3$
19	$6a^3-7a^2b+b^3$	20	$25x^3-19x+6$
21	$x^4-13x-42$	22	$x^4+40x-96$
23	$12x^4+x^2-1$	24	x^4-5x^3+54
25	$3x^4-5x^3-8$	26	$3a^4+5a^3b-8b^4$
27	$x^4+2x^3-13x^2-14x+24$	28	$12x^4-49x^3-62x^2+29x+30$

157 Substitution We shall in the present article draw the attention of the student to the fact, which he has perhaps already noticed, that *each letter in a formula may stand for a single letter as well as for an expression* Hence from a result already established, we can deduce *new results*, by substituting for its letters other letters or expressions Thus the *principle of substitution* is a very important element in the Science of Algebra and gives to algebraical results an immense scope for development. The truth of this remark will be seen from the following illustrations

In Formula I, put $2x-y=a$ and $y=b$, thus we get

$$\{(2x-y)+y\}^2=(2x-y)^2+2(2x-y)y+y^2,$$

therefore $(2x-y)^2+2(2x-y)y+y^2=\{(2x-y)+y\}^2=(2x)^2=4x^2$

Again in the same formula, put $x+y=a$ and $x-y=b$, thus we have

$$\{(x+y)+(x-y)\}^2=(x+y)^2+2(x+y)(x-y)+(x-y)^2,$$

$$(x+y)^2+2(x+y)(x-y)+(x-y)^2=\{(x+y)+(x-y)\}^2$$
$$=(2x)^2=4x^2$$

By the same substitutions, we have from Formula II

$$\{(x+y)-(x-y)\}^2=(x+y)^2-2(x+y)(x-y)+(x-y)^2,$$

thus $(x+y)^2-2(x+y)(x-y)+(x-y)^2=\{(x+y)-(x-y)\}^2$
$$=(2y)^2=4y^2$$

Again put $ax+by-cz=A$, $ax-by+cz=B$, thus from Formula III, we get $\{(ax+by-cz)+(ax-by+cz)\}\{(ax+by-cz)-(ax-by+cz)\}$
$$=(ax+by-cz)^2-(ax-by+cz)^2,$$

or $(ax+by-cz)^2-(ax-by+cz)^2=2ax\times 2(by-cz)=4ax(by-cz)$

By the same substitutions again, we have from Formula IV

$$\{(ax+by-cz)+(ax-by+cz)\}^3=(ax+by-cz)^3+(ax-by+cz)^3$$
$$+3(ax+by-cz)(ax-by+cz)\{(ax+by-cz)+(ax-by+cz)\}$$

thus $(ax+by-cz)^3+(ax-by+cz)^3+6ax\{a^2x^2-(by-cz)^2\}$
$$=(2ax)^3=8a^3x^3$$

And so on.

158 The principle explained in the last article may be employed in factorizing expressions whose factors are not apparent

Ex 1 Resolve $x^2-2xy+y^2+x-y$ into factors

Given expression $=(x^2-2xy+y^2)+(x-y)$
$$=(x-y)^2+(x-y)$$
$$=(x-y)(x-y+1)$$

Ex 2 Resolve $4b^2c^2-(b^2+c^2-a^2)^2$ into factors

Given expression $=(2bc)^2-(b^2+c^2-a^2)^2$

$$=(2bc+b^2+c^2-a^2)(2bc-b^2-c^2+a^2)$$
$$=\{(b+c)^2-a^2\}\{a^2-(b-c)^2\}$$
$$=(b+c+a)(b+c-a)(a+b-c)(a-b+c)$$
$$=(a+b+c)(b+c-a)(c+a-b)(a+b-c)$$

Ex 3 Resolve x^4+4y^4 into factors

Given expression $=(x^4+4x^2y^2+4y^4)-4x^2y^2$

$$=(x^2+2y^2)^2-(2xy)^2$$
$$=\{(x^2+2y^2)+2xy\}\{(x^2+2y^2)-2xy)\}$$
$$=(x^2+2xy+2y^2)(x^2-2xy+2y^2)$$

Otherwise thus —Since $a^2+b^2=(a+b)^2-2ab$ [Art. 143], we have $x^4+4y^4=(x^2)^2+(2y^2)^2$

$$=(x^2+2y^2)^2-2(x^2)(2y^2)$$
$$=(x^2+2y^2)^2-(2xy)^2$$
$$=(x^2+2xy+2y^2)(x^2-2xy+2y^2).$$

Ex 4 Resolve x^4-7x^2+9 into factors

We may proceed as usual, or perhaps thus —Since $a^2+b^2=(a-b)^2+2ab$ [Art 143], we have

$$x^4-7x^2+9=\{(x^2)^2+3^2\}-7x^2$$
$$=\{(x^2-3)^2+6x^2\}-7x^2$$
$$=(x^2-3)^2-x^2=(x^2+x-3)(x^2-x-3)$$

Examples XCV

Factorize

1 $1-2x+x^2-a+ax$ 2 $4m^2+4m+4(2m+1)q+4q^2+1$

3 $a^2x^2-2axy-ax+y^2+y$ 4 $3ax^3+6ax^2y+3axy^2-ax^2-axy$

5 $a^2+b^2+2(bc+ca+ab)$ 6 $1-2y+y^2-z^2$

7 $1+2ax+a^2x^2-x^2$ 8 $1-a^2-b^2-2ab$

9 $(a^2+b^2-c^2)^2-4a^2b^2$ 10. $(b^2+c^2)^2-(b^2-c^2)^2-(a^2-b^2-c^2)^2$

11 $x^2+y^2-z^2-u^2+2(xy+zu)$

12 $4(ad+bc)^2-(a^2-b^2-c^2+d^2)^2$

13 x^4+4 14. $4a^4+b^4$ 15 x^4+x^2+1 16 $a^4+a^2b^2+b^4$

17 x^4+2x^2+9 18 $x^4-(p^2+2)x^2y^2+y^4$ [Bom, 1888]

Factorize

19 x^4-26x^2+25

20 $a^8+a^4x^4+x^8$. [Cal, 1887]

21 $9x^4-34a^2x^2+25a^4$

22 $(x+1)^4+4(x-1)^4$

23 $(x^4-2x^3y)-(y^4-2xy^3)$

24 $(a+2b)^2-b^2-3a-9b$

25 $x^2-2xy+y^2-z^2+2x-2y-2z$

26 $x^2-y^2-z^2+2yz+x+y-z$

27 $a^2+a-b^2-b-c^2-c+2bc$

28 $(x^3+12x)^2-(6x^2+8)^2$

29 $(1-3x)^2-(3x^2-x^3)^2$

30 x^6-y^6

31 $x^3-8(y+z)^3$

32 $8x^3-(x^2+1)^3$.

33 $x^6+(x^2-2a^2)^3$

34 $(a^2-bc)^3+8b^3c^3$

35 $x^3-3x^2y+3xy^2-y^3-z^3$

36 $x^3-y^3+3y^2-3y+1$

37 $a^3+3a^2b+3ab^2+2b^3$

38 $(a+b)^3+(a+b)-2$

39. $x^3-7a^3+(x-3a)^3$

40 $a^3+a(b+c)(a+b+c)+(b+c)^3$

Ex 5 Resolve $a^2+2ab+b^2-c^2+a+b-c$ into factors

Given expression $=\{(a^2+2ab+b^2)-c^2\}+(a+b-c)$

$$=\{(a+b)^2-c^2\}+(a+b-c)$$
$$=(a+b-c)(a+b+c)+(a+b-c)$$
$$=(a+b-c)(a+b+c+1)$$

Ex 6 Resolve $(x+y)^3-3(x+y)^2+3(x+y)-1$ into factors

Put $x+y=a$, thus given expression

$$=a^3-3a^2+3a-1=(a-1)^3=(x+y-1)^3$$

Note $(x+y)^3-3(x+y)^2+3(x+y)-1$

$$=\{(x+y)^3-1\}-3(x+y)\{(x+y)-1\}$$

Thus given expression may be resolved as in Art 72

Ex 7 Resolve $(a^2+bc)^3-8b^3c^3$ into factors

Put $x=a^2+bc$, $y=2bc$, thus, because $8b^3c^3=(2bc)^3$, the given expression is seen to be of the form x^3-y^3 Hence

Given expression $=\{(a^2+bc)-2bc\}\{(a^2+bc)^2+(a^2+bc)(2bc)+(2bc)^2\}$

$$=(a^2-bc)(a^4+4a^2bc+7b^2c^2)$$

Ex 8 Resolve $(2a+3b)^3+8a^3+27b^3$ into factors

Given expression $=(2a+3b)^3+(2a)^3+(3b)^3$

$$=(2a+3b)^3+(2a+3b)\{(2a)^2-(2a)(3b)+(3b)^2\}$$
$$=(2a+3b)^3+(2a+3b)(4a^2-6ab+9b^2)$$
$$=(2a+3b)\{(2a+3b)^2+4a^2-6ab+9b^2\}$$
$$=(2a+3b)(8a^2+6ab+18b^2)$$
$$=2(2a+3b)(4a^2+3ab+9b^2)$$

Ex 9 Resolve $8x^4+2x^2-45$ into factors.

Expressions of this type contain *only two powers* of some one letter or expression, *one* of which is the *square* of the other

Put $x^2=X$, thus $x^4=(x^2)^2=X^2$, therefore

given expression $=8X^2+2X-45=(2X+5)(4X-9)$

$=(2x^2+5)(4x^2-9)=(2x^2+5)(2x+3)(2x-3)$.

Ex 10 Factorize $2(a^2+3a+3)^2+3(a^2+3a+3)-5$

Put $a^2+3a+3=X$, thus

given expression $=2X^2+3X-5=(2X+5)(X-1)$

$=\{2(a^2+3a+3)+5\}\{(a^2+3a+3)-1\}$

$=(2a^2+6a+11)(a^2+3a+2)$

$=(2a^2+6a+11)(a+1)(a+2)$

Ex. 11 Factorize $(x^2+7x+4)(x^2+7x+6)-48$

Let $x^2+7x=X$, therefore

given expression $=(X+4)(X+6)-48$

$=X^2+10X+24-48$

$=X^2+10X-24=(X+12)(X-2)$

$=(x^2+7x+12)(x^2+7x-2)$

$=(x+3)(x+4)(x^2+7x-2)$

Ex 12 Factorize $x(x+1)(x+2)(x+3)-15$

Given expression $=x(x+3)\times(x+1)(x+2)-15$

$=(x^2+3x)(x^2+3x+2)-15$

$=(x^2+3x)^2+2(x^2+3x)-15$

$=X^2+2X-15$, where $X=x^2+3x$,

$=(X-3)(X+5)$

$=(x^2+3x-3)(x^2+3x+5)$

Examples XCV *(Continued)*

Factorize

41 x^4+11x^2-12

42 $6x^4+5x^2-4$

43 $2x^4+x^2y^2-3y^4$

44 $8x^6+7x^3-1$

45 $3x^6-x^3y^3-2y^6$

46 $4a^8-63a^4-16$

47. $a^8+3a^4x^4-4x^8$

48 $(x^2+5x)^2+10(x^2+5x)+24$

49 $(x^2+3x)^2-(x^2+3x)-6$

50 $(x^2+4x)^2-2(x^2+4x)-15$

Factorize

51 $(x^2-10x)^2+13(x^2-10x)-264$ 52 $(x^4-2x^3)^2+4(x^2-2)x^2+3$

53 $3(x^2-7x)^2+28(x^2-7x)-96$

54 $(3x^2-2x-10)^2+6(3x^2-2x-10)+8$

55 $(2x+3y)^2+3(2x+3y)(3x+2y)+2(3x+2y)^2$

56 $(x^2+y^2)^2-8(x^4-y^4)-48(x^2-y^2)^2$

57 $(4a+x)^2-5(4a+x)(a-3x)+6(a-3x)^2$

58 $(a^2+ab)^2-3(ab-b^2)(a^2+ab)-4(ab-b^2)^2$

59 $6(2x^2+xy-y^2)^2-(2x^2+xy-y^2)(x^2-xy-2y^2)-12(x^2-xy-2y^2)^2$

60 $(a^2+6a+2)(a^2+6a-4)-27$ 61 $x(x+3)(x+6)(x+9)+56$

62 $(x+1)(x+4)(x+7)(x+10)-40$

63 $(x-1)(x-2)(x-3)(x-4)-120$

64 $(x-a)(x-2a)(2x-a)(2x-3a)-30a^4$

159 Identities In Art 83, we have defined an identity When one side of an identity is shewn by easy transformations to take the form of the other side, the identity is said to be *proved* In proving an identity we take any one of the sides and shew that it is equal to the other side

Ex Prove the identity

$$2(a^2+ab)^2+2(ab+b^2)^2=(a+b)^4+(a^2-b^2)^2$$

Left side$=2\{a(a+b)\}^2+2\{b(a+b)\}^2$

$=2a^2(a+b)^2+2b^2(a+b)^2$

$=(a+b)^2(2a^2+2b^2)=(a+b)^2\{(a+b)^2+(a-b)^2\}$

$=(a+b)^4+(a+b)^2(a-b)^2=(a+b)^4+(a^2-b^2)^2$

Examples XCVI.

Prove the following identities

1 $(a-b)^2-2(a-b)c+c^2=a^2-2a(b+c)+(b+c)^2$

2 $a^3-3a^2(b-c)+3a(b-c)^2-(b-c)^3$
$=(a-b)^3+3(a-b)^2c+3(a-b)c^2+c^3$

3 $(x+y)^2-2(x+y)y+2y^2=x^2+y^2$

4 $x^2-2(x-y)y=(x-y)^2+y^2$

5 $(x+y-z)^2+2(x+y)z=(x+y)^2+z^2$

6 $(x+y)^2+z^2=(x+y+z)^2-2(yz+zx)$

7 $(1+a^2)(1+b^2)-(1-ab)^2=(a+b)^2$

Prove the following identities

8 $(x+2y)^2-4(x+y)y=x^2$

9 $(a-b)^2-2(b-c)(c-a)=(b-c)^2+(c-a)^2$

10 $(a-2b+c)^2+4(a-b)(b-c)=(a-c)^2$

11 $(a+b+2c)^3-(b+c)^3-(c+a)^3=3(b+c)(c+a)(a+b+2c)$

12 $(a-b)^3-(b-c)^3-(a-2b+c)^3=3(a-b)(b-c)(a-2b+c)$

13 $(a+b-c-d)^2=(a+b+c+d)^2-4(a+b)(c+d)$

14 $(a+b)^2+(c+d)^2=(a+b+c+d)^2-2(a+b)(c+d)$

15 $(a+3b)^2+3(a-b)^2=(a-3b)^2+3(a+b)^2$

16 $(a+2)(b+2)+2(a-1)(b-1)=(a-2)(b-2)+2(a+1)(b+1)$

17 $(a+1)^2-6(a^2-1)+9(a-1)^2=4(a-2)^2$

18 $2(a^2-ab)^2+2(ab-b^2)^2=(a^2-b^2)^2+(a-b)^4$

19 $(2x+1)^2+(x+2)^2=(x-2)^2+4x(x+3)+1$

20 $(x-6)^2+(x+1)^2+2x(x-1)=(2x-3)^2+28$

21 $3x^2+(x+1)^2+2x-y^2=(2x+y+1)(2x-y+1)$

22 $(2a-3b)^2+5(a+b)^2-10(a+b)b-4b^2=3a(3a-4b)$

23 $(a+b)^2+(b+c)^2+(c+a)^2=2(a^2+b^2+c^2+bc+ca+ab)$

24 $(a-b)^2+(b-c)^2+(c-a)^2=2(a^2+b^2+c^2-bc-ca-ab)$

25 $(a-b)^2-(b-c)(c-a)=(b-c)^2-(c-a)(a-b)$
$$=(c-a)^2-(a-b)(b-c)$$

26 $(b-c)(c-a)+(c-a)(a-b)+(a-b)(b-c)$
$$=bc+ca+ab-a^2-b^2-c^2$$

27 $(x+y)^2+(y+z)^2+(z+x)^2+(x-y)^2+(y-z)^2+(z-x)^2$
$$=4(x^2+y^2+z^2).$$

28 $(x+y)^2+(y+z)^2+(z+x)^2-(x-y)^2-(y-z)^2-(z-x)^2$
$$=4(yz+zx+xy)$$

29 $(x+y)(x+z)-x^2=(y+z)(y+x)-y^2=(z+x)(z+y)-z^2$

30 $(y+z)^2+(z+x)^2+(x+y)^2-x^2-y^2-z^2=(x+y+z)^2$

31 $(b+c)(b+c-a)+(c+a)(c+a-b)+(a+b)(a+b-c)=2(a^2+b^2+c^2)$

32 $(b-c)(b+c-a)+(c-a)(c+a-b)+(a-b)(a+b-c)=0$

33 $(a+b)^2(a^2+2ab-b^2)+(b+c)^2(b^2+2bc-c^2)+(c+a)^2(c^2+2ca-a^2)$
$$=4a^2b(a+b)+4b^2c(b+c)+4c^2a(c+a)$$

34 $(b-c)(b+c)^3+(c-a)(c+a)^3+(a-b)(a+b)^3$
$$=2bc(b^2-c^2)+2ca(c^2-a^2)+2ab(a^2-b^2)$$

35 $(b-c)(a^2+b^2+c^2+bc)+(c-a)(b^2+c^2+a^2+ca)$
$$+(a-b)(c^2+a^2+b^2+ab)=-(b-c)(c-a)(a-b).$$

Prove the following identities

36 $(1-ax+a^2)(1-ay+a^2)-(1-a^2)^2=a^2(xy+4)-a(1+a^2)(x+y)$

37 $(b+c)^2+(c+a)^2+(a+b)^2+2(b+c)(c+a)$
$$+2(c+a)(a+b)+2(a+b)(b+c)=4(a+b+c)^2$$

38 $(2x-y)^2+(2y-z)^2+(2z-x)^2+2(2x-y)(2y-z)+2(2y-z)(2z-x)$
$$+2(2z-x)(2x-y)=(x+y+z)^2$$

39 $8(x+y+z)^3=(x+2y)^3+(x+2z)^3+6(x+2y)(x+2z)(x+y+z)$

40 $8(x+y+z)^3=(x+y)^3+(x+y+2z)^3+6(x+y)(x+y+z)(x+y+2z)$

41. $(b-c)^2(b+c-2a)+(c-a)^2(c+a-2b)+(a-b)^2(a+b-2c)$
$$=(2a-b-c)(2b-c-a)(2c-a-b)$$

42 $a(b-c)(1-ab)(1+ac)+b(c-a)(1-bc)(1+ba)$
$$+c(a-b)(1-ca)(1+cb)=-abc(b-c)(c-a)(a-b)$$

43 Shew that $x^2(y-z)^2+y^2(z-x)^2+z^2(x-y)^2$
$$=2xy(z-x)(z-y)+2yz(x-y)(x-z)+2zx(y-z)(y-x)$$

44 Prove that $\{(y-z)^2+(z-x)^2+(x-y)^2\}^2$
$$=4(y-z)^2(z-x)^2+4(z-x)^2(x-y)^2+4(x-y)^2(y-z)^2$$

45 If $x=2a-b-c$, $y=2b-c-a$, $z=2c-a-b$, shew that
$$(x^2+y^2+z^2)^2=4(y^2z^2+z^2x^2+x^2y^2)$$

160 Important Identities These identities are very useful in Algebraical transformations and should be carefully borne in mind

Ex 1 Prove that $a^2(b+c)+b^2(c+a)+c^2(a+b)+2abc$
$$=(b+c)(c+a)(a+b)$$

[**Note.** It is easy to see that
$$a^2b+ab^2+b^2c+bc^2+c^2a+ca^2$$
$$=a^2(b+c)+b^2(c+a)+c^2(a+b)$$
$$=bc(b+c)+ca(c+a)+ab(a+b)$$
$$=a(b^2+c^2)+b(c^2+a^2)+c(a^2+b^2)$$
We shall denote, for shortness, each of these equal expressions by S]

Arrange according to powers of a thus left side
$$=a^2(b+c)+a(b^2+2bc+c^2)+b^2c+bc^2$$
$$=a^2(b+c)+a(b+c)^2+bc(b+c)$$
$$=(b+c)\{a^2+a(b+c)+bc\}$$
$$=(b+c)(a+c)(a+b)$$
Hence $S+2abc=(b+c)(c+a)(a+b)$

Ex 2 Prove that $a^2(b+c)+b^2(c+a)+c^2(a+b)+3abc$
$$=(a+b+c)(bc+ca+ab).$$

Left side $=a^2b+ab^2+b^2c+bc^2+c^2a+ca^2+3abc$
$$=(b^2c+bc^2+abc)+(c^2a+ca^2+abc)+(a^2b+ab^2+abc)$$
[rearranging and grouping the terms]
$$=bc(b+c+a)+ca(c+a+b)+ab(a+b+c)$$
$$=(a+b+c)(bc+ca+ab)$$

Hence $S+3abc=(a+b+c)(bc+ca+ab)$

Another form of this identity is
$$(a+b+c)(bc+ca+ab)-abc=(b+c)(c+a)(a+b)$$

For left side $=S+3abc-abc=S+2abc=(b+c)(c+a)(a+b)$ [Ex 1]

Ex 3 Prove that $(a+b+c)^3=a^3+b^3+c^3+3(b+c)(c+a)(a+b)$

$(a+b+c)^3=a^3+b^3+c^3+3a^2b+3ab^2+3b^2c+3bc^2+3c^2a+3ca^2+6abc$
$$=a^3+b^3+c^3+3S+6abc=a^3+b^3+c^3+3(S+2abc)$$
$$=a^3+b^3+c^3+3(b+c)(c+a)(a+b) \text{ [Ex. 1]}$$

Another form of this identity evidently is
$$(a+b+c)^3-a^3-b^3-c^3=3(b+c)(c+a)(a+b)$$

Example 1 Prove that $8(a+b+c)^3-(b+c)^3-(c+a)^3-(a+b)^3$
$$=3(2a+b+c)(a+2b+c)(a+b+2c)$$

Since $8(a+b+c)^3=(2a+2b+2c)^3=\{(b+c)+(c+a)+(a+b)\}^3$,

we have [by putting $x=b+c$, $y=c+a$, $z=a+b$] left side
$$=(x+y+z)^3-x^3-y^3-z^3$$
$$=3(y+z)(z+x)(x+y)$$
$$=3(2a+b+c)(a+2b+c)(a+b+2c)$$

Example 2 Resolve into factors
$$(a+b+c)^3-(b+c-a)^3-(c+a-b)^3-(a+b-c)^3$$

Put $x=b+c-a$, $y=c+a-b$, $z=a+b-c$, thus
$$x+y+z=(b+c-a)+(c+a-b)+(a+b-c)=a+b+c.$$

Hence the given expn $=(x+y+z)^3-x^3-y^3-z^3$
$$=3(y+z)(z+x)(x+y)$$
$$=3(2a)(2b)(2c)=24abc$$

Ex 4 Prove that $a^2(b+c)+b^2(c+a)+c^2(a+b)+a^3+b^3+c^3$
$$=(a+b+c)(a^2+b^2+c^2)$$

Rearranging the terms, the left side

$$=a^3+a^2(b+c)+b^3+b^2(c+a)+c^3+c^2(a+b)$$
$$=a^2(a+b+c)+b^2(b+c+a)+c^2(c+a+b)$$
$$=(a+b+c)(a^2+b^2+c^2)$$

Hence $\qquad S+a^3+b^3+c^3=(a+b+c)(a^2+b^2+c^2)$

Ex 5 Prove that $a^2(b+c)+b^2(c+a)+c^2(a+b)$
$$-a^3-b^3-c^3-2abc=(b+c-a)(c+a-b)(a+b-c)$$

Rearranging the terms, we have

Left side $=a^2(b+c)+a(b^2-2bc+c^2)+b^2c+bc^2-a^3-b^3-c^3$
$$=a^2(b+c)-a^3+a(b^2-2bc+c^2)+bc(b+c)-(b^3+c^3)$$
$$=a^2(b+c-a)+a(b-c)^2+bc(b+c)-(b+c)(b^2-bc+c^2)$$
$$=a^2(b+c-a)+a(b-c)^2-(b+c)(b^2-2bc+c^2)$$
$$=a^2(b+c-a)+a(b-c)^2-(b+c)(b-c)^2$$
$$=a^2(b+c-a)-(b-c)^2(b+c-a)$$
$$=(b+c-a)\{a^2-(b-c)^2\}$$
$$=(b+c-a)(a-b+c)(a+b-c)$$

Hence $\quad S-a^3-b^3-c^3-2abc=(b+c-a)(c+a-b)(a+b-c)$

Ex 6 Prove that $2b^2c^2+2c^2a^2+2a^2b^2-a^4-b^4-c^4$
$$=(a+b+c)(b+c-a)(c+a-b)(a+b-c)$$

Arrange according to powers of a, thus

Left side $=-\{a^4-2(b^2+c^2)a^2+b^4+c^4-2b^2c^2\}$
$$=-\{a^4-2(b^2+c^2)a^2+(b^2+c^2)^2-(b^2+c^2)^2+(b^2-c^2)^2\}$$
$$=-\{(a^2-\overline{b^2+c^2})^2-4b^2c^2\}=(2bc)^2-(a^2-b^2-c^2)^2$$
$$=\{2bc-(a^2-b^2-c^2)\}\{2bc+(a^2-b^2-c^2)\}$$
$$=\{(b+c)^2-a^2\}\{a^2-(b-c)^2\}$$
$$=(b+c+a)(b+c-a)(a+b-c)(a-b+c)$$
$$=(a+b+c)(b+c-a)(c+a-b)(a+b-c)$$

Examples XCVII

Resolve into factors

1 $\quad a^3(b^2-c^2)+b^3(c^2-a^2)+c^3(a^2-b^2)$

2 $\quad a^2(b^3-c^3)+b^2(c^3-a^3)+c^2(a^3-b^3)$

Resolve into factors

3 $a^4(b^2-c^2)+b^4(c^2-a^2)+c^4(a^2-b^2)$

4 $b^2c^2(b^2-c^2)+c^2a^2(c^2-a^2)+a^2b^2(a^2-b^2)$

5 $(a+b+c)^3-a^3-b^3-c^3$ 6 $(b+c)(c+a)(a+b)+abc$

7 $a(b-c)^2+b(c-a)^2+c(a-b)^2+8abc$

8 $a(b+c)^2+b(c+a)^2+c(a+b)^2-3abc$

9 $a(b-c)^3+b(c-a)^3+c(a-b)^3$

10 $a^4(b-c)+b^4(c-a)+c^4(a-b)$

11 $bc(b^2-c^2)+ca(c^2-a^2)+ab(a^2-b^2)$

12 $a(b^4-c^4)+b(c^4-a^4)+c(a^4-b^4)$

13 $a(b-c)(bc-a^2)+b(c-a)(ca-b^2)+c(a-b)(ab-c^2)$

14 $bc(b-c)(1-a^2)+ca(c-a)(1-b^2)+ab(a-b)(1-c^2)$

15 $(b-c)(b+c)^3+(c-a)(c+a)^3+(a-b)(a+b)^3$

16 $(b+c)(b-c)^3+(c+a)(c-a)^3+(a+b)(a-b)^3$

161 Multiplication by rearrangement of the terms Much of the labour of multiplication is often saved, and the work neatly performed, by arranging the multiplicand and multiplier according to the powers of some *one* letter (called the *symbol of reference*)

Ex 1 Multiply $bc+ca+ab$ by $a+b+c$

Arrange multiplicand and multiplier in powers of a

$$
\begin{array}{l}
a(b+c)+bc \\
\underline{a+(b+c)\qquad} \\
a^2(b+c)+abc \\
\underline{\qquad\qquad +a(b+c)^2+bc(b+c)} \\
a^2(b+c)+a(b^2+c^2+3bc)+bc(b+c) \\
=a^2b+ab^2+b^2c+bc^2+c^2a+ca^2+3abc
\end{array}
$$

Ex 2 Multiply $x^2-xy+y^2+x+y+1$ by $x+y-1$

Here consider x as the symbol of reference

$$
\begin{array}{l}
x^2-(y-1)x+(y^2+y+1) \\
\underline{x\ +(y-1)\qquad\qquad\qquad} \\
x^3-(y-1)x^2+(y^2+y+1)x \\
\underline{\qquad +(y-1)x^2-(y^2-2y+1)x+(y^3-1)} \\
x^3 \qquad\qquad\qquad +3yx \qquad +(y^3-1)
\end{array}
$$

$$=x^3+y^3+3xy-1$$

Examples XCVIII

Find the product of

1 $x^2+ax+bx+ab$ and $x-c$

2 $cx^2+ax+bx+a$ and $x-1$

3 $a^2-bc+ca+ab$ and $a-b+c$

4 $x^2-ax-bx+ab$ and $x+a-b$

5 $(a^2-a+1)x^2+(a-1)x+1$ and $(a+1)x-1$

6 $(x-1)a^2-(x-1)a+3$ and $(x^2+x+1)a-(x+1)$

7 $a^2+b^2+c^2+bc-ca+ab$ and $a-b+c$

8 $x^2-xy+y^2-2x+y+1$ and $x+y-1$

9 $a^2+bc-ca-ab$ and $b^2-c^2-ca+ab$

10 $a^3-(m-1)a^2+(m+1)a-2$ and $(m+1)a^2+a-m$

11 $(a^2+ab+b^2)x^2-(a+b)x-ab$ and $(a-b)x^2+2x-1$

12 $2x^3-(a+b)x^2+abx-a+b$ and $(a-b)x^2+abx+a+b$

13 $bc(b-c)+ca(c-a)+ab(a-b)$ and $bc+ca+ab$

162 Division by rearrangement of the terms The labour of division is considerably shortened, and the operation much neatly performed, by arranging the dividend and divisor according to the powers of some *one* letter (called the *symbol of reference*)

Ex 1 Divide $a^3-ab^2-b^2c+bc^2+c^2a+2ca^2+abc$ by $a^2+bc+ca+ab$

Arrange dividend and divisor in powers of a

$$\begin{array}{r|l|l}
a^2+a(b+c)+bc & a^3+2a^2c-a(b^2-bc-c^2)-bc(b-c) & a-(b-c) \\
 & \underline{a^3+a^2(b+c)+abc} & \\
 & \quad -a^2(b-c)-a(b^2-c^2)-bc(b-c) & \\
 & \quad \underline{-a^2(b-c)-a(b^2-c^2)-bc(b-c)} &
\end{array}$$

Ex 2 Divide $x^3+y^3+3xy-1$ by $x+y-1$

Arrange dividend and divisor according to the descending powers of x

$$\begin{array}{r|l|l}
x+(y-1) & x^3+3xy+(y^3-1) & x^2-(y-1)x+(y^2+y+1) \\
 & \underline{x^3+(y-1)x^2} & \\
 & \quad -(y-1)x^2+3xy & \\
 & \quad \underline{-(y-1)x^2-(y-1)^2x} & \\
 & \qquad (y^2+y+1)x+(y^3-1) & \\
 & \qquad \underline{(y^2+y+1)x+(y^3-1)} &
\end{array}$$

Ex 3 Divide $(x^3-1)a^3-(x^3+x^2-2)a^2+(4x^2+3x+2)a-3(x+1)$ by $(x-1)a^2-(x-1)a+3$

Here take a as the *symbol of reference*

$$
\begin{array}{l}
(x-1)a^2-(x-1)a+3)\ (x^3-1)a^3-(x^3+x^2-2)a^2+(4x^2+3x+2)a-3(x+1)\ ((x^2+x+1)a-(x+1) \\
\underline{(x^3-1)a^3-(x^3-1)a^2+(3x^2+3x+3)a} \\
\qquad\qquad -(x^2-1)a^2+(x^2-1)a-3(x+1) \\
\qquad\qquad \underline{-(x^2-1)a^2+(x^2-1)a-3(x+1)}
\end{array}
$$

Examples XCIX.

Divide

1 $a^3+b^3-3a^2+3a-1$ by $a+b-1$

2 $a^3+b^2(2a+b)+ab(2a+c)-c^3$ by $a+b-c$

3 $x^3-a^2(x-b)+(a-b)bx-ab^2$ by $(x+a)(x-b)$

4 $a^3(b+c)+b^3(c+a)+c^3(a+b)+abc(a+b+c)$ by $bc+ca+ab$

5 $x^3-(a+b+c)x^2+(ab+ac+bc)x-abc$ by $(x-a)(x-b)$

6 $a^3(b-c)+b^3(c-a)+c^3(a-b)$ by $a^2(b-c)+b^2(c-a)+c^2(a-b)$

7 $x^3+(1-a-b)x^2-2bx-a^2+b^2$ by $x-a-b$

8 $b^2c^2(b-c)+c^2a^2(c-a)+a^2b^2(a-b)$ by $bc+ca+ab$

9 $a^4-(x^2-y-z)a^2-(y-z)ax+yz$ by a^2+ax+y

10 $x^3-2ax^2+(a^2-ab-b^2)x+a^2b+ab^2$ by $(x-a)(x+b)$

11. $(a-b)x^3-(x-b)a^3+(x-a)b^3$ by $x^2-(a+b)x+ab$

12 $x^3+(4ab-b^2)x-(a-2b)(a^2+3b^2)$ by $x-a+2b$

13 $(a^3-1)x^3+(2a^2+a)x^2+2ax+1$ by $(a-1)x+1$

14 $a^3+b^3+c^3-3abc$ by $a+b+c$

15 $1+x^3-8y^3+6xy$ by $1+x-2y$

16 $(a+b)(x+y)(ax+by)-bx-ay-1$ by $ax+by-1$

17 $x^4-a^4+2ax^3+2na^3x-(n^2-1)a^2x^2$ by $x^2+a^2-(n-1)ax$

18 $x(x-1)a^3+(x^3+2x-2)a^2+(3x^2-x^3)a-x^4$ by $a^2x+2a-x^2$

19 $x^3(a+1)-xy(x-y)(a+b)-y^3(b-1)$ by $x(a+1)-y(b-1)$

20 $(2x-y)^2a^4-(x+y)^2a^2x^2+2(x+y)ax^4-x^6$

by $(2x-y)a^2-(x+y)ax+x^3$.

163 Division by Resolution into Factors The following examples will sufficiently illustrate the method

Ex. 1. Divide a^3-3a+2 by $a+2$

Dividend $=a^3+8-3a-6=(a^3+8)-(3a+6)$

$=(a+2)(a^2-2a+4)-3(a+2)$ [Art. 67]

$=(a+2)(a^2-2a+4-3)=(a+2)(a-1)^2$,

$\therefore$ required quotient $=(a-1)^2$ or a^2-2a+1

Ex. 2. Divide $x^3-3ax^2+3a^2x-a^3+b^3$ by $x-a+b$

Dividend $=(x-a)^3+b^3$ [Art. 67]

$=(x-a+b)\{(x-a)^2-(x-a)b+b^2\}$

$=(x-a+b)\{x^2-(2a+b)x+a^2+ab+b^2\}$,

$\therefore$ required quotient $=x^2-(2a+b)x+a^2+ab+b^2$

Ex. 3. Divide $x^5-1-5(x-1)$ by $(x-1)^2$ [*Punj.*, 1893]

Dividend $=(x-1)(x^4+x^3+x^2+x+1)-5(x-1)$

$=(x-1)(x^4+x^3+x^2+x-4)$

$=(x-1)\{(x^4-1)+(x^3-1)+(x^2-1)+(x-1)\}$

$=(x-1)^2\{(x^3+x^2+x+1)+(x^2+x+1)+(x+1)+1\}$

$=(x-1)^2(x^3+2x^2+3x+4)$,

$\therefore$ required quotient $=x^3+2x^2+3x+4$

Examples C

Divide

1. $a^2(a-2b)-b^2(b-2a)$ by $a-b$
2. $(x-a)(x-b)-(y-a)(y-b)$ by $(x-a)+(y-b)$
3. $(a-b)(b-c)+(a-d)(c-d)$ by $b-d$.
4. $(1+x)^2(1+y^2)-(1+x^2)(1+y)^2$ by $1-xy$
5. $x(x-1)(x-2)+y(y-1)(y-2)-6xy$ by $x+y$
6. $x^4+y^4-z^4+2x^2y^2-2z^2-1$ by $x^2+y^2-z^2-1$
7. $a(b^2+c^2-a^2)+b(c^2+a^2-b^2)$ by $(b+c-a)(c+a-b)$
8. $a^3+b^3-3a^2+3a-1$ by $a+b-1$
9. $b(x^3+a^3)+ax(x^2-a^2)+a^3(x+a)$ by $(a+b)(x+a)$
10. $(1+a)^3+(1+a)^2x+(1+a)x^2+x^3$ by $1+a+x$
11. $(a^2+ab+b^2)^4-(a^2-ab+b^2)^4$ by $(a^2+b^2)(a^4+3a^2b^2+b^4)$
12. $(x+y)^3+(1-x-y)3xy-1$ by $x+y-1$
13. $(x^2-1)^4-3(x^2-1)^2+1$ by x^4-3x^2+1
14. $(a+b)^3+(c-a)^3-(b+c)^3$ by $a^2+a(b-c)-bc$

Divide

15 $x^3+8y^3+6xy-1$ by $x+2y-1$

16 $(a-b)^2c^2+(a-b)c^3-(c^2-a^2)b^2+(c-a)b^3$ by $(a-b)c^2-(c-a)b^2$ [Cal, 1883]

17 $x^8+a^4x^4+a^8$ by x^2-ax+a^2

18 $x(x+1)(x+2)(x+3)-3$ by x^2+3x+3

19 Shew that $(x^2+xy+y^2)^3+(x^2-xy+y^2)^3$ is divisible by $2x^2+2y^2$.

20 Shew that

$(bx+cy+az)^3-(cx+ay+bz)^3$ is divisible by $(b-c)x+(c-a)y+(a-b)z$

21 Shew that

$(x^2-1)(y^2-1)(z^2-1)+(x+yz)(y+zx)(z+xy)$ is divisible by $xyz+1$

22 Divide $x(1+y^2)(1+z^2)+y(1+z^2)(1+x^2)+z(1+x^2)(1+y^2)+4xyz$ by $1+yz+zx+xy$ [Cal, 1878]

[*The following examples will further illustrate the foregoing formulæ*]

23 Simplify $(a+b-1)(3a-b)-2(a+b-1)(a-b+2)$

24 Simplify $(a+b+c)^2+(b+c-a)^2+(c+a-b)^2+(a+b-c)^2$

25 Simplify $(a+b+c+d)^2-(a+b-c-d)^2+(a-b+c-d)^2$
$-(a-b-c+d)^2$.

26 Multiply together $x-a$, $x-b$, $x-c$ and $x-d$, and from the product deduce the value of $(x+2)^4$

27 What number must be added to $3x^3-13x^2+6x-5$, that the result may be divisible by $3x-4$?

28 If $c^2=a^2+b^2$, find the value of
$(a+b+c)(b+c-a)(c+a-b)(a+b-c)$

29 Find the product of $x-a$, $x-b$ and $x-c$, and factorize the result when $-x^2=bc+ca+ab$

30 Find the value of
$(x-a)^3+(x-b)^3+(x-c)^3-3(x-a)(x-b)(x-c)$,
when $3x=a+b+c$

31 Find the value of $\dfrac{384\times384\times384-383\times383\times383}{384\times384+384\times383+383\times383}$

32 Find the value of $x^3+y^3+3xy-1$ when $x=15$, $y=85$

33 Find the coefficient of x^3 in the product of
$5x^3-4x^2+3x-2$ and $6x^2+8x+3$

[The terms that involve x^3 may be obtained, first, by multiplying the terms of the *third* degree by those that do not involve x, and

next, the terms of the *second* degree by those of the *first* degree Therefore the terms involving x^3 in the product, are $+15x^3$, $-32x^3$ and $+18x^3$ The *algebraic sum* of these terms

$$=15x^3-32x^3+18x^3=x^3, \qquad \text{coefficient required}=1]$$

34 Find the coefficient of x^2 in the above example

35 Find the coefficient of x^4 in the product of

$$x^3-\frac{3x^2}{4}+\frac{2x}{3}+\frac{7}{12} \text{ and } 2x^3+\frac{2x^2}{3}-\frac{x}{2}+\frac{5}{6}$$

36 Find the coefficient of x^3 in the above example

37 Find the coefficient of x^4 in the product of

$$x^4-ax^3+bx^2-cx+d \text{ and } x^2+px+q$$

38 Find the coefficient of x^4 in the product of

$$1-\frac{x}{2}+\frac{x^2}{4}-\frac{x^3}{8}+\frac{x^4}{16} \text{ and } 1+\frac{x}{2}+\frac{x^2}{4}+\frac{x^3}{8}+\frac{x^4}{16}$$

39 If $a+b+c=0$, show that

(i) $a^2(b+c)+b^2(c+a)+c^2(a+b)+3abc=0$

(ii) $a(b+c)^2+b(c+a)^2+c(a+b)^2-3abc=0$

(iii) $a(b-c)^2+b(c-a)^2+c(a-b)^2+9abc=0$

Decompose into elementary factors

40 $(a^2-1)(b^2-1)+4ab$

41 $(a^2-bc)^3-(b^2-ca)^3$

42 $4(x-a)^3-27a^2x$

43 $a^3+2b^3-3ab^2$

44 $x^4+x^2-2ax+1-a^2$

45. $(a^2-b^2)(x^2+y^2)+2(a^2+b^2)xy$.

46 $x^4-(a+b)x^3+(a^2b+ab^2)x-a^2b^2$

CHAPTER XV

HIGHEST COMMON FACTOR

164 *Definitions* **A common factor** of two or more expressions is one which divides each of the latter without remainder Thus a is a common factor of $2a^2$, $3a^3b$ and a^2b^2

The Highest Common Factor of two or more expressions is the expression of the *highest dimensions* which divides each of the given expressions without remainder Thus a^2 is the Highest Common Factor of $2a^2$, $3a^3b$ and a^2b^2

The Highest Common Factor is termed by some writers Greatest Common Measure The corresponding abbreviations are H C F and G C M

Note If A and B be two expressions whose H. C. F. is F, then *every factor of F is a common factor of A and B*, and conversely, *every common factor of A and B is a factor of F*. This is self evident in the case of monomial factors.

REMARK. The Arithmetical term Greatest Common Measure is, not appropriate in Algebra. Here we have to see whether the expression, found as the G. C. M. is of *highest possible degree*, without any reference to its *numerical value*. Thus $x+a$, which is the Algebraical G. C. M. of x^2-a^2, and $(x+a)^2$ [as the student will presently see], is not their Arithmetical G. C. M. when $x=12$, $a=4$. In this case x^2-a^2, or its value 128, is the required G. C. M.

165. H. C. F. of Monomials. Since the required H. C. F. must be that factor of *highest dimensions which is common* to all the proposed expressions we have the following.

RULE.—*Take all the factors common to the given expressions and raise each to the lowest power in which it occurs; the product of these powers will be the required* H. C. F.

If there be *numerical* coefficients, find their G. C. M. as in Arithmetic, and put it before the H. C. F.

Ex. 1. Find the H. C. F. of $60a^4b^3x$, $72a^2by^3$ and $84a^3b^2z$.

The common factors here are a and b; the lowest powers of these occurring in the proposed expressions are a^2 and b respectively; also 12 is the G. C. M. of the numerical coefficients 60, 72, 84;

$$\therefore \text{H. C. F.} = 12\times a^2\times b = 12a^2b$$

Examples CI

Find the Highest Common Factor of

1. a^2b and ab^3. 2. $2a^2x$ and abx^3. 3. $2a^3b$ and $3a^2c$.
4. a^5b^2 and a^2b^6. 5. $a^2b^3c^4$ and ab^2c^3. 6. $75ac^2$ and $30a^2x$.
7. $105ab^2x$ and $84ax$. 8. ab^3, a^2bx and a^3b^2.
9. $8xy$, $12xz$ and $20yz$. 10. $16m^2n^3$, $48m^3n$ and $80m^2p$.
11. $35a^3b^3x^3y^4$ and $49a^2b^4x^4y^3$.
12. $12x^2yz$, $18xy^3z^2$, $27x^2y^2z^3$ and $33x^2yz^4$.
13. $5a^4b^3c^3y^2$, $15a^3c^3d^3x$, $10a^5b^4c^5y$ and $18a^2b^2c^4x^3$.
14. $12m^3p^3qx^3$, $16a^3m^4x^3y$, $40a^3m^2q^2x^5$, $32b^3m^5x^4y^3$ and $20m^4px^3y^2$.

In a similar way, we can find the H. C. F. of a Monomial and a Binomial or Polynomial.

Ex 2 Find the H C F of $2a^2x$ and ax^2-axy

Now $2a^2x=2a\cdot ax$ and $ax^2-axy=ax(x-y)$, and the common factors are a and x

$$\text{H C F required}=a\times x=ax$$

Examples CI. *(Continued)*

Find the Highest Common Factor of

15 $6m^2n^2p^2$ and $8mn^2p^2-12m^2p^3$.

16 $2ab^2x$ and $3a^2x+3ab^2x$

17 $72m^2n^2x^3$ and $54m^4n^5x^3-36m^2n^4y^2$

18 $108a^3m^2x^3$ and $72a^2m^2x-54a^2mx^2+90a^3m^2x$

166 H C F. of expressions readily resolved into Factors The method being precisely the same as that of the last article, we here follow the same rule

Ex 1 Find the H C F of $6a^2-12ax$ and $8ax-16x^2$

Here $6a^2-12ax=6a(a-2x)$ and $8ax-16x^2=8x(a-2x)$

Thus the highest common factor is $(a-2x)$, also the G C M of 6 and 8 is 2,

$$\text{H C F required}=2(a-2x)$$

Examples CII

Find the Highest Common Factor of

1 a^2+ax and $ax+x^2$ 2 a^3-ax and a^3-ax^2

3 $2a^2x-2abx$ and $2a^2x-4abx$. 4 x^2+2cx and $3ax+6ac$

5 $ax+x^2$ and $3ax+2xy$ 6 a^2-1 and $ab-b$

7 $5x^2-10xy$ and $2xz-4yz$ 8 x^2y+xy^2-xyz and y^2z-yz^2+xyz

---

Ex 2 Find the H C F of $12a^3(x+1)^3(x-2)^3$

and $15a^2b(x+1)^2(x-2)$

The common factors, each raised to the lowest power in which it occurs in the given expressions, are a^2, $(x+1)^2$ and $(x-2)$, and the G C M of the numerical coefficients is 3,

$$\text{H C F required}=3\times a^2\times(x+1)^2\times(x-2)$$
$$=3a^2(x+1)^2(x-2)$$

Examples CII *(Continued)*

Find the Highest Common Factor of

9 $(a+b)^3x^3$ and $(a+b)^2x^4$ 10 $4a^2(a-b)^2$ and $5a(a-b)^4$

11 $a^3(p+q)^4x$ and $2ab(p+q)x^4$

12 $4(a-1)^3(x-a)^2$ and $8(a-1)^2(x-a)^3$

13 $18a^3b^2(x-a)^4(x^2-y^2)$, $24b^2c^3(x-a)^3(x^2-y^2)^2$, and $42a^2bc(x-a)^2(x^2-y^2)^3$

14 $16(a+x)^2$ and $40(a^2-x^2)$ 15 $9(a^3+b^3)$ and $6(a^2-b^2)$

16 $16(x^4-x^3+x^2)$ and $56(x^4+x^2+1)$

17 $15ab(a^2-b^2)^3$ and $27a^2(a-b)(a+b)^2$

18 $10(x+1)^2(x^2-4)$ and $15(x^2-1)(x+2)^2$

19 $8a^2(a-x)^3$, $12ax(a^2-x^2)$ and $16a^3x^2(a^3-x^3)$

20 $6(a^3+b^3)(a-b)^3$, $9(a^4-b^4)(a-b)^2$ and $12(a^2-b^2)^3$.

21 $8(x+1)(x^3+8)$ and $4(x+1)(x^2-5x-14)$

Ex 3 Find the H C F of $2a^4x-5a^3x^2-3a^2x^3$ and $4a^3x^2+14a^2x^3+6ax^4$

First expn $=a^2x(2a^2-5ax-3x^2)=a^2x(2a+x)(a-3x)$,

Second expn $=2ax^2(2a^2+7ax+3x^2)=2ax^2(2a+x)(a+3x)$,

H C F required $=ax(2a+x)$

Examples CII *(Continued)*

Find the Highest Common Factor of

22 x^2+5x+6 and x^2-2x-8 23 $2x^2+x-1$ and $6x^2+x-2$

24 x^2+3x+2 and x^3+2x^2+3x+6

25. $2+2x$, $1-x^2$ and $1+x+y+xy$

26 x^2-9, $(x+3)^2$ and x^2+x-6

27 a^2-c^2, $a^2-2ac+c^2$ and $a^2+ab-ac-bc$

28 $1-x^2$, x^3+1 and $1-x-2x^2$

29 x^2-x-2, x^2+x-6 and x^2-3x+2

30 $2x^2-5x+2$, $3x^2-2x-8$ and $4x^2-5x-6$.

31 $px^2-(p+1)x+1$ and $qx^2-(q-1)x-1$.

32 $20x^4+x^2-1$ and $25x^4+5x^3-x-1$

33 $x^4-(m+1)x^3+(m+1)x-1$ and $x^4-(n+1)x^3+(n+1)x-1$

34 $ay(x^2+b^2)+bx(by^2+a^2x)$ and $ax(y^2+b^2)+by(bx^2+a^2y)$

Find the Highest Common Factor of

35 $1-abx^3+(b-a^2)x^2$ and $1+acx^3-(c-a^2)x^2-2ax$

36 $ab+2a^2-3b^2-4bc-ac-c^2$ and $9ac+2a^2-5ab+4c^2+8bc-12b^2$

The investigation of the *general rule* for finding the H C F of Polynomials depends on two *Lemmas* which we establish in the next two articles

This rule is necessary when the expressions are not readily resolved into factors

167 Theorem *If two expressions have a common factor, it will divide the sum or difference of any integral multiples whatever of them*

Let D be a common factor of A and B, then by hypothesis $A=mD$, $B=nD$; therefore $pA=mpD$, $qB=nqD$, therefore

$pA\pm qB=(mp\pm nq)D$ Thus D divides $pA\pm qB$

Cor Evidently D divides $A+B$ or $A-B$

168 Theorem *If an expression B divide an expression A leaving a remainder R, then the* H C F *of B and R will be the* H C F *of A and B*

Let Q be the quotient when A is divided by B, thus we have

$$A=BQ+R \qquad (\text{i})$$

Now every common factor of B and R, will divide $BQ+R$ or A, that is, every common factor of B and R will divide A and B

Again from (i), we have by transposition

$$R=A-BQ \qquad (\text{ii})$$

Hence every common factor of A and B will divide $A-BQ$ or R, that is, every common factor of A and B will divide B and R

Thus B and R have exactly the same factors as A and B, that is, the H. C F. of B and R is the H. C F of A and B.

169 H C F. of two Polynomials Let A and B be two expressions, say, in x, arranged according to the *descending* powers of x, and let A be not of lower dimensions than B

Divide A by B, and let p be the quotient and C the remainder divide B by C, and let q be the quotient and D the remainder, and so on

$$\begin{array}{l} B)\,A\,(p \\ \underline{pB} \\ C)\,B\,(q \\ \underline{qC} \\ D)\,C\,(r \\ \underline{rD} \\ 0 \end{array}$$

Now since A and B are arranged in decending powers of x, it is clear that each successive remainder shall be of a lower degree than the corresponding divisor and consequently we must at last arrive at a stage of division where there shall be either no *remainder*, or if there be any, it shall be a

constant (*i e*, a quantity not involving x) In the latter case, of course, A and B *have no* H C F

Suppose, then, for the sake of simplicity, there is no remainder at the next stage of the process

Divide C by D, and let r be the quotient

Thus from Art 168, the H C F of B and C will be the H C F of A and B, and the H C F of C and D will be the H C F of B and C, therefore the H C F of C and D will be the H C F of A and B, but the H C F of C and D is D itself, for no expression *higher* than D can divide D, thus D is the H C F required

Hence we have the following rule for finding the H C F of two polynomials—

RULE —*Arrange the polynomials according to the* DESCENDING *powers of some common letter, divide the one of a higher degree by the other, take the remainder, if any, after this division for a new divisor and the preceding divisor for dividend, and so on, until there is no remainder, the last divisor will be the* H C F *required*

If there be no remainder after the first division, then one of the proposed expressions, *viz*, that of a lower degree, will be the H C F required

Note 1 If the chain of division for finding the H C F *terminates with a zero remainder, the last divisor will be the* H C F *required*, but if it *terminates with a constant as remainder, the polynomials have no* H C F

Note 2 In the chain of division for finding the H C F *the* H C F *of any divisor and the corresponding dividend, will always be the* H C F *required*

Ex 1 Find the H C F of x^2-3x+2 and x^3-4x^2+6x-4

$$
\begin{array}{r|l}
x^2-3x+2)\,x^3-4x^2+6x-4 & (x-1 \\
x^3-3x^2+2x & \\
\hline
-x^2+4x-4 & \\
-x^2+3x-2 & \\
\hline
x-2 &
\end{array}
$$

Take this remainder for divisor, thus

$$
\begin{array}{r|l}
x-2)\,x^2-3x+2 & (x-1 \\
x^2-2x & \\
\hline
-x+2 & \\
-x+2 & \\
\hline
\end{array}
$$

H C F required $=x-2$

Examples CIII

Find the Highest Common Factor of

1 x^2-2x-3 and x^3-2x^2-2x-3

2 x^2+3x-4 and x^3+5x^2+3x-9

3 $3x^2-11x-4$ and $6x^3-25x^2+3$

4 $2x^3+x^2-5x-3$ and $8x^3+6x^2-21x-18$

5 $x^3+6x^2+2x-15$ and $x^3+5x^2-2x-10$

6 $6x^3-19x^2+13x-2$ and $12x^3-32x^2+19x-2$

7 $x^4+x^3-x^2-2x-2$ and $x^4+2x^3-x^2-2x-3$

8 $x^4-4x^3-30x^2-28x+17$ and $3x^4-11x^3-86x^2-81x+49$

Ex 2 Find the H C F of $2x^3+7x^2+2x-3$ (i),

and $3x^3+8x^2-2x+3$ (ii)

The given expressions are of the same degree, hence it is immaterial which of these is considered the dividend If we take (ii), it is evident that the first term in the quotient will be a fraction, *viz*, $\frac{3}{2}$ to avoid which we multiply (ii) by 2, which is the coefficient of x^3 in (i) and which is *not a factor* of the divisor (i)

$$\begin{array}{rl}
 & 3x^3+\ 8x^2-2x+3 \\
 & \underline{2\qquad\qquad\qquad\quad} \\
2x^3+7x^2+2x-3) & 6x^3+16x^2-4x+6\ (\ 3 \\
 & \underline{6x^3+21x^2+6x-9} \\
 & -5|\underline{-5x^2-10x+15} \\
 & \qquad x^2+\ 2x-3
\end{array}$$

The remainder has a factor -5, *which is not a factor of* (i) the divisor, and since the required H C F will be the H C F of (i), and this remainder [NOTE 2], the factor -5 cannot affect the required H C F, and may therefore be rejected The required H. C F. will therefore be the H C. F of (i) and x^2+2x-3.

$$\begin{array}{rl}
x^2+2x-3\ ) & 2x^3+7x^2+2x-3\ (\ 2x+3 \\
 & \underline{2x^3+4x^2-6x} \\
 & \qquad 3x^2+8x-3 \\
 & \qquad \underline{3x^2+6x-9} \\
 & \qquad\quad 2|\underline{2x+6} \\
 & \qquad\qquad x+3
\end{array}$$

For the reason given above, *reject the factor* 2

$$
\begin{array}{r}
x+3\;)\;x^2+2x-3\;(\;x-1 \\
\underline{x^2+3x\qquad} \\
-\;x-3 \\
\underline{-\;x-3} \\
\end{array}
$$

required H. C F $=x+3$ [NOTE 1]

REMARK From this example it is seen that we can *introduce or reject* with certain restrictions, a factor at any stage of our operation For as the H C F of two polynomials will always be the H C F of any divisor and the corresponding dividend [NOTE 2], *we can multiply a dividend when necessary, by a quantity which is not a factor of the divisor*, and *vice versa*. Similarly *we can remove from a divisor, a factor which is not a factor of the dividend*, and *vice versa* In the latter case, however, *if a factor be common to both the divisor and the dividend, we may remove and reserve it, to be introduced afterwards into the* H C F *of the remaining factors* [See Ex. 4, below.]

Ex 3 Find the H C F of $2x^4-3x^3-2x^2+6x+3$ (i),

and $2x^4-7x^3-10x^2+x+2$ (ii)

$$
\begin{array}{r}
2x^4-3x^3-2x^2+6x+3\;)\;2x^4-7x^3-10x^2+x+2\;(\;1 \\
\underline{2x^4-3x^3-2x^2+6x+3} \\
-1\,|\,\underline{-4x^3-8x^2-5x-1} \\
4x^3+8x^2+5x+1
\end{array}
$$

$$
\begin{array}{r}
4x^3+8x^2+5x+1\;)\;2x^4-3x^3-2x^2+6x+3 \\
\underline{2\qquad\qquad\qquad\qquad} \\
4x^4-6x^3-4x^2+12x+6\;(\;x \\
\underline{4x^4+8x^3+5x^2+x\qquad} \\
-14x^3-9x^2+11x+6 \\
\underline{2\qquad\qquad\quad} \\
-28x^3-18x^2+22x+12\;(\;-7 \\
\underline{-28x^3-56x^2-35x-7} \\
19\,|\,\underline{38x^2+57x+19} \\
2x^2+\;3x+\;1
\end{array}
$$

$$
\begin{array}{r}
2x^2+3x+1\;)\;4x^3+8x^2+5x+1\;(\;2x+1 \\
\underline{4x^3+6x^2+2x\qquad} \\
2x^2+3x+1 \\
\underline{2x^2+3x+1}
\end{array}
$$

H C F. required $=2x^2+3x+1.$

Examples CIII (*Continued*)

Find the Highest Common Factor of

9. $3x^2+16x-12$ and $x^3+7x^2+4x-12$

10 $x^3-19x^2+119x-245$ and $3x^2-38x+119$.

11 $4x^3+x-1$ and $2x^3+3x^2-1$ 12 $2x^3-19x+3$ and $3x^3-10x^2+9$

13 $2x^3+x^2-3x+6$ and $3x^3+7x^2-3x-10$

14 $3x^3-13x^2+23x-21$ and $6x^3+x^2-44x+21$

15 $2x^3-x^2-x-3$ and $x^5-x^3-4x^2-3x-2$

16 $4x^4-9x^2+6x-1$ and $6x^3-7x^2+1$

17 $3x^3+14x^2+12x+16$ and $2x^4+7x^3-4x^2-x-4$

18 $6x^4-2x^3+7x^2-x+2$ and $6x^4-12x^3+21x^2-6x+9$

19 $2x^4+17x^3+30x^2+8x-5$ and $x^4+4x^3-18x^2-29x-10$

20 $x^5-x^3+4x^2-3x+2$ and $5x^4-3x^2+8x-3$

Ex 4 Find the H C F of $2x^3+3x^2-35x$ and $10x^3-33x^2-7x$

Evidently x is a factor of the given expressions, remove and reserve it, we have thus the two remaining factors $2x^2+3x-35$ and $10x^2-33x-7$ Hence the required H C F will be the H C F of these two factors *multiplied by the reserved factor* x [See Remark, Ex.2]

The student will easily find the H C. F of these factors to be $2x-7$,

$$\text{H C F required} = x(2x-7) = 2x^2-7x$$

Examples CIII (*Continued*)

Find the Highest Common Factor of

21. $18x^3+45x^2+63x+54$ and $24x^3+24x^2+6x+36$

22 $4x^3-8ax^2-20a^2x+24a^3$ and $6x^3+24ax^2+6a^2x-36a^3$

23 $3x^4-8x^3+8x$ and $4x^4-19x^2+6x$

24 $12x^4-14x^3+2x$ and $8x^4+10x^2-6x$

25 $3x^5+5x^4+2x^3+8x^2$ and $x^5-13x^3-14x^2+8x$

26 $8a^4b^2-34a^2b^4+24ab^6$ and $12a^5b-26a^4b^2+18a^2b^4$

Ex 5 Find the H C F of $x^4-9a^2x^2+10a^3x$ (i),

and $ax^3-a^2x^2-4a^4$ [*Cal*, 1873] (ii)

First expression $=x(x^3-9a^2x+10a^3)$

Second expression $=a(x^3-ax^2-4a^3)$.

Thus x is a factor of (i) but not of (ii), and a is a factor of (ii) but not of (i) Hence x and a cannot enter the required H C F and must therefore be rejected [see Remark, Ex 2] The required H C F. will thus be the H C F of $x^3-9a^2x+10a^3$ and $x^3-ax^2-4a^3$, which will be found to be $x-2a$

$$\text{H C F required}=x-2a$$

Examples CIII (*Continued*)

Find the Highest Common Factor of

27 $2x^3-10x^2+20x-16$ and $3x^3-12x^2+21x-18$

28. $8x^3-10x+4$ and $6x^3+21x^2-6$

29 $2x^4+8x^3-7x^2+15x$ and $3x^4+17x^3+12x^2+13x+15$

30 $6x^5-21x^4y+6x^3y^2-6x^2y^3-3xy^4$ and $5y^5-10xy^4+20x^2y^3-15x^3y^2+10x^4y$

31 $a^5-4a^4b+27ab^4$ and $a^4b-4ab^4+3b^5$

32 $x^3-6a^2x-9a^3$ and $2x^3-10ax^2+9a^2x+9a^3$

33 $3x^3-15x^2y+xy^2-5y^3$ and $6x^4-25x^2y^2-9y^4$

34 $7a^3-6a^2b-18ab^2+4b^3$ and $14a^3-19a^2b-32ab^2+28b^3$

35 $3x^3-22x-15$ and $5x^4-17x^3+18x$

36 $x^4-9x^2-30x-25$ and $x^5+x^4-7x^2+5x$

37. $4x^6+8x^5-56x^4-12x^3$ and $6x^3-6x^2-36x$

38 $6x^4y+x^3y^2-xy^4$ and $4x^3-6x^2y-4xy^2+3y^3$

39 $6a^4x^3-10a^3x^4y-9a^3x^2y^2+15ax^3y^3$ and $10a^4xy^2-15a^3y^4+8a^2x^2y^3-12axy^5$

40 $2x^5-4x^4+8x^3-12x^2+6x$ and $3x^5-3x^4-6x^3+9x^2-3x$

41 $x^4+4x^3y-21x^2y^2+10xy^3-y^4$ and $x^4+12x^3y+33x^2y^2-12xy^3+y^4$

42 $x^4-px^3+(q-1)x^2+px-q$ and $x^4-qx^3+(p-1)x^2+qx-p$

43 $x^4+px^3-(a-1)x^2-apx-a$ and $x^4-px^3-(a+1)x^2+apx+a$

44 $x^4-2a(a-b)x^2+(a^2+b^2)(a-b)x-a^2b^2$ and $x^4-(a-b)x^3+(a-b)b^2x-b^4$

45 $3x^5-10x^3+15x+8$ and $x^5-2x^4-6x^3+4x^2+13x+6$

46. $6x^5-4x^4-11x^3-3x^2-3x-1$ and $4x^4+2x^3-18x^2+3x-5$

47 $x^5+x^4-8x^3+12x^2-x-21$ and $x^5-3x^3+9x^2-4x-3$

48 $2x^5-11x^2-9$ and $4x^5+11x^4+81$

49 $a^6-3a^3b^3-8b^6$ and $a^6-5a^3b^3-12b^6$

170 H. C F of several Polynomials From the process of finding the H C F of two expressions A and B [Art. 169], it is seen that if C, D, . be the successive remainders, we have

$$C = A - pB. \quad . \qquad (\text{i}),$$
$$D = B - qC \qquad (\text{ii})$$

From (i), we see that every common factor of A and B is a factor of C, that is, a common factor of B and C; and from (ii), that every common factor of B and C is a factor of D, hence every common factor of A and B is a factor of D

Thus if D be the H C F of A and B, then it follows that *every common factor of two expressions is a factor of their* H C F

Hence if G is the H C F of A and B, every common factor of G and C is a common factor of A, B and C, therefore the H C F of G and C (say H) is the H C F of A, B and C Similarly we see that H C F of H and D is the H C F of A, B, C and D And so on Hence the following

RULE —*First find the* H C F *of any two of the polynomials, then find the* H C F *of this result and the third polynomial, and so on The last* H C F *will be the* H C F *required.*

Ex Find the H C F of $x^3-x^2-10x-8$, $x^3+6x^2+11x+6$ and $x^3+4x^2-11x-30$

The H C F. of the first two expressions will be found to be x^2+3x+2 The required H C F will therefore be the H C F of this and the third expression, which will be seen to be $x+2$

required H C F $= x+2$

Examples CIV

Find the Highest Common Factor of

1 $x^2-3x-70$, $x^3-39x+70$ and $x^3-48x+7$

2 $a^3-9a^2+26a-24$, $a^3-10a^2+31a-30$ and $a^3-11a^2+38a-40$

3 $a^3+4a^2b-5b^3$, $a^3-3ab^2+2b^3$ and $a^3+4a^2b-8ab^2+3b^3$

4 x^2-2a^2-ax, x^2-4a^2, x^2-6a^2+ax and x^2-8a^2+2ax

5 $1-16x^4$, $1-5x^2+4x^4$, $1+2x-3x^3-6x^4$ and $1+2x-4x^3-8x^4$

6 $6x^3-23x^2+29x-12$, $10x^3-19x^2+9$ and $15x^3-26x^2-x+12$

7 $4x^3-28x^2+39x+27$, $6x^3-47x^2+96x-27$

and $12x^3-52x^2-11x+9$

171 Theorem *If* A *and* B *be any two expressions in* x, *then the* H C F *of* lA+mB *and* pA+qB *will be the same as the* H C F *of* A *and* B, *where the quantities* l, m, p *and* q *do not involve* x, *and are such that* lq−mp *is not*=0

Let the H C F of A and B be G and that of $lA+mB$ and $pA+qB$ be H

Every common factor of A and B which involves x, is a factor of $lA+mB$ and of $pA+qB$ [Art 167], therefore their *highest common* factor G is a common factor of $lA+mB$ and $pA+qB$ Thus either $G=H$, or G is a factor of H [Art 170] and consequently of a *lower* degree than H

Again *every common* factor of $lA+mB$ and $pA+qB$ which involves x, is a factor of

$$q(lA+mB)-m(pA+qB)$$

and of $$l(pA+qB)-p(lA+mB) \text{ [Art 167]},$$

and consequently their *highest common* factor H is a common factor of these expressions

now $$q(lA+mB)-m(pA+qB)=(lq-mp)A,$$

and $$l(pA+qB)-p(lA+mB)=(lq-mp)B$$

therefore H is a common factor of $(lq-mp)A$ and $(lq-mp)B$, and since by supposition $lq-mp$ is not $=0$, and does not contain x, H must be a common factor of A and B Hence either $H=G$, or H is a factor of G and therefore G is of a *higher* degree than H

Thus G is at once higher and lower than H, which is absurd

$$\therefore\ H=G$$

This Theorem enables us to find the H C F rather easily In applying it to examples, the principle is to *destroy alternately the highest and lowest terms* by choosing suitable values of the multipliers l, m, p and q

Ex 1 Find the H C F of

$$A=2x^3+7x^2+2x-3,$$

and $$B=3x^3+8x^2-2x+3$$

Now $$A+B=5x^3+15x^2=5x^2(x+3),$$

and $$3A-2B=3(2x^3+7x^2+2x-3)-2(3x^3+8x^2-2x+3)$$
$$=5(x^2+2x-3)=5(x-1)(x+3)$$

Obviously $5x^2$ and 5 are not factors of A and B, reject them

Also by the Remainder Theorem [Art 141], $x-1$ is not a factor of A.

H C F required $=x+3$

Ex 2 Find the H C F of

$$A=4x^3+9x^2+3x-2,$$

and $$B=4x^3-3x^2-5x+2$$

We have $$A+B=8x^3+6x^2-2x=2x(4x^2+3x-1),$$

and $$A-B=12x^2+8x-4=4(3x^2+2x-1)$$

Reject $2x$ and 4 which are not factors of A and B and put

$$A_1=4x^2+3x-1 \text{ and } B_1=3x^2+2x-1.$$

Now $\qquad A_1-B_1=x^2+x=x(x+1),$

and $\qquad 3A_1-4B_1=3(4x^2+3x-1)-4(3x^2+2x-1)=x+1.$

Hence $x+1$ is the required H. C. F., for it is the H. C. F. of A_1-B_1 and $3A_1-4B_1$ and therefore of A_1 and B_1, i. e., of $A+B$ and $A-B$ and consequently of A and B.

Ex. 3. Find the H. C. F. of

$$A=2x^5-11x^2-9,$$

and $\qquad B=4x^5+11x^4+81$ [Ch. III, Ex. 48].

Now $\qquad 2A-B=-11x^4-22x^2-99=-11(x^4+2x^2+9),$

and $\qquad 9A+B=22x^5+11x^4-99x^2=11x^2(2x^3+x^2-9).$

Obviously -11 and $11x^2$ are not factors of A and B; therefore the required H. C. F. is the H. C. F. of $x^4+2x^2+9=A_1$ and $2x^3+x^2-9=B_1$, that is, of B_1 and

$$A_1+B_1=x^4+2x^3+3x^2=x^2(x^2+2x+3).$$

As x^2 cannot form part of the reqd. H. C. F., the reqd. H. C. F. is the H. C. F. of B_1 and $x^2+2x+3=A_2$, or of B_1 and $3A_2+B_1$, that is, of $3A_2+B_1$ and A_2+B_1.

Now $\qquad 3A_2+B_1=2x^3+4x^2+6x=2x(x^2+2x+3).$

Hence the required H. C. F. evidently is x^2+2x+3.

CHAPTER XVI

LOWEST COMMON MULTIPLE

172. Definitions. A **common multiple** of two or more quantities is that quantity which is divisible by each of the latter without remainder. Thus 16 is a common multiple of 2, 4 and 8; $5ab$, is a common multiple of 5, a and b.

The **Lowest Common Multiple** of two or more quantities is the quantity of *lowest dimensions* which is divisible by each of the latter without remainder. Thus $5ab$ is the lowest common multiple of 5, a and b, but not so are $5a^2b$, $10a^2b^4$, &c.

The term Lowest Common Multiple is often shortened into L. C. M.

173. L. C. M. of Monomials. Since the required L. C. M. must be that multiple of *lowest dimensions*, which is common to all the given expressions, we have the following

RULE —*Take the several factors occurring in the given expressions, each raised to the highest power which it has in any one of them the product of these powers will be the* L C M *required*

If there be numerical coefficients, find their L C M. as in Arithmetic, after which put the L C M of the literal factors

Ex Find the L C M of $8x^2y$, $12y^2z$ and $16xz^3$

Here the factors are x, y and z of which the indices are 2, 2 and 3 respectively, and L C M of coefficients is 48,

$$\text{L C M required} = 48x^2y^2z^3$$

Examples CV

Find the Lowest Common Multiple of

1. $5a^2b$ and $10ab^2$ 2. $6xy^2$ and $8xy$ 3. $12a^3$ and $16a^2b$
4. $9ax^3$ and $21a^2x^4$ 5. $24x^3y^4$ and $28x^2y^6$ 6. xy, yz and xz
7. $2a^2x^2$, ax^3 and a^3x 8. ab^3, 20 and $15a^2x^2$.
9. $5m^3$, $10m^2n$ and $25n^3$ 10. $38a^2xy$, $57ax^2y^3$ and $95bxy^3$
11. 12, $3a^2$, $6ax$ and $8x^3$ 12. $2a^3x$, $3a^2x^2$, $4ax^3$ and $5x^4$
13. $32a^2xy^3$, $48x^2yz$, $64ay^2z$ and $80x^2y^3z^2$

174 L C M found by inspection. The method is the same as given in the last article, we therefore follow the same Rule

Ex. Find the L C M of $4a^2x^2$, $6(a^2+ax)$ and $8(ax-x^2)$

Second expression $=6a(a+x)$, third expression $=8x(a-x)$

Thus the factors are a, x, $a+x$ and $a-x$ whose highest indices are 2, 2, 1 and 1, and L C M of coefficients is 24,

$$\text{L C M required} = 24a^2x^2(a+x)(a-x) = 24a^2x^2(a^2-x^2)$$

Examples CVI

Find the Lowest Common Multiple of

1. $3ax^2$ and $2x+x^2$ 2. axy and $a(xy-y^2)$ 3. x^2+xy and y^2+xy
4. $2(ax+ay)$ and $3(ax-ay)$ 5. $4(x^2+x)$ and $6(x^2-1)$
6. $15y(a-x)$ and $20(a^2x-x^3)$ 7. $30(a^2-b^2)$ and $42(a^3-b^3)$
8. $4x^2-1$ and $8x^3+1$ 9. $2(x^2-y^2)$ and $10(x+y)^2$
10. a^3+x^3 and $a^4+a^2x^2+x^4$ 11. $14(x^3-y^3)$ and $21(x^3+x^2y+xy^2)$
12. $2ax$, $a+x$ and $a-x$. 13. $x+y$, x^2-y^2 and x^3+y^3
14. $x+1$, x^2+1 and x^3+1 15. $1+a$, $1-a^2$ and $1-2a+a^2$
16. $2(x^3-x^2y)$, $3(xy^2+y^3)$ and $4(x^2y-xy^3)$

Find the Lowest Common Multiple of

17 $(x-y)(x-z)$ and $(y-x)(y-z)$.

18 ax^2-x^3 and ax^3-a^3

19 x^2+x, $(x+1)(x+2)$ and x^3+2x^2

20 $6(x^2+xy)$, $8(xy-y^2)$ and $10(x^2-y^2)$.

21 x^2-4, x^2+4x+4 and $(x-2)^3$

22 $3(x+y)(x+2y)$, $15(x+2y)(x+3y)$ and $24(x+y)(x+3y)$.

23 $6(x^2y+xy^2)$, $9(x^3-xy^2)$ and $4(y^3+xy^2)$

24 $2x$, $3y$, $1+a$ and $4(1-a^2)$ 25. a^3x^2, a^2x^3, $ax-xy$ and a^2+ay.

26 $x+1$, $x-1$, x^2+x+1 and x^2-x+1

27 x^2-y^2, $4(x+y)^2$, $6(x-y)^2$ and $15(x^3+y^3)$

28 x, $x-1$, x^2-1, x^3-1 and x^4-1

29 $x^2(x-y)^2$, $y^2(x+y)^2$, x^4-xy^3, x^3y+y^4 and $x^3y^2-y^5$

175 L C M of two Polynomials Let A and B be any two expressions whose H. C F is H and whose L C M. is M

Divide A and B respectively by H, and let the quotients be a and b, so that $A=aH$ and $B=bH$

Then since H is the H C F of A and B, a and b cannot have a *common* factor, and therefore their L. C M must be ab

Hence the L C M. of A and B, that is, of aH and bH is abH

Thus $M=abH$

Now $M=abH=aH\,b=Ab$, or $=a\,bH=aB$ (i)

Also $M=abH=aH\,bH\div H=AB\div H$ (ii).

Thus from (i) and (ii), we have the following

RULE —*Divide either polynomial by the* H. C F *of the two, and multiply the quotient by the other* or in other words—*Multiply the polynomials together and divide the product by their* H. C F

COROLLARY. From (ii), it is evident that

$$M\times H=AB\div H\times H=A\times B$$

Thus *the product of two expressions is equal to the product of their* L C M *and* H C F

Ex Find the L C M. of

$$6x^3-5x^2-22x+24 \text{ and } 6x^3-23x^2+29x-12.$$

The H C. F of these expressions will be found to be $6x^2-17x+12$ [Art 169]

And $(6x^3-5x^2-22x+24)\div$the H C F $=x+2$,

$(6x^3-23x^2+29x-12)\div$the H. C F $=x-1$.

Hence L. C M required $=(6x^2-17x+12)(x+2)(x-1)$

$=(2x-3)(3x-4)(x+2)(x-1)$

Examples CVII

Find the Lowest Common Multiple of

1 $3x^2-10x+3$ and $3x^2-19x+6$

2 $2x^2-13x+21$ and $3x^2-23x+42$

3 x^3-3x^2+3x-1 and x^3-x^2-x+1

4 $x^2-(a+b)x+ab$ and $x^2-(a+c)x+ac$

5 $3x^2-5xy+2y^2$ and $4x^3-4x^2y-xy^2+y^3$

6 $x^3-6x^2+11x-6$ and x^3+4x^2+x-6

7 x^3-6x^2+8x and x^2+x-6

8 $mx^2-6mx+5m$ and $nx^2+5nx-6n$

9 $2a^2+ax-3x^2$ and $3a^3-a^2x-ax^2-x^3$

10 $(a^3+a^2b)x^2+a(a^2-b^2)xy-(a^2b+ab^2)y^2$

and $(a^2b-ab^2)x^2-b(a^2-b^2)xy+(ab^2-b^3)y^2$

11 $x^3+3x^2-4x-12$ and x^3+2x^2-x-2

12 $x^5+ax^4+a^2x^3+a^3x^2+a^4x+a^5$ and $x^5-ax^4+a^2x^3-a^3x^2+a^4x-a^5$

13 $2a^4+3a^3x-9a^2x^2$ and $6a^4x-17a^3x^2+14a^2x^3-3ax^4$

14 $x^3+(5a-3)x^2+(6a^2-15a)x-18a^2$

and $x^3+(a-3)x^2-(2a^2+3a)x+6a^2$

15 $x^4-4x^3+2x^2+4x-15$ and $x^4-4x^3+3x^2+2x-12$

16 $2x^4+4x^3-97x^2-2x+48$ and $6x^4+118x^3-43x^2-59x+20$

176 Theorem *Every Common Multiple of two expressions is a multiple of their L C M*

Let A and B be two expressions whose L C M is M

Let μ be any other multiple of A and B Then μ is divisible by M without remainder

If not, if possible, let M be contained q times in μ with a remainder R, then $R=\mu-qM$ [Art 137, *Cor*]

Now A and B divide M and also μ, therefore they divide qM, and $\mu-qM$ or R [Art 167]

But R is of a lower degree than M, the divisor [Art 137, *Def*]

Hence A and B divide an expression which is of lower dimensions than M, their L C M, which is absurd

Therefore there can be no remainder, *i e*, μ is a multiple of M

177 L C M of several Polynomials Let A, B and C be any three expressions, and let M be the L C M of A and B

Now every multiple of M is a common multiple of A and B, therefore every common multiple of M and C is a common multiple of A, B and C.

Also every common multiple of A, B and C is a common multiple of M and C [Art. 176].

Therefore the L. C. M. of M and C is the L. C. M. of A, B and C.

Similarly the reasoning may be extended to the case of any number of polynomials.

We have thus the following

RULE.—*Find the* L. C. M. *of any two expressions; then find the* L. C. M. *of this* L. C. M. *and a third expression; next find the* L. C. M. *of the second* L. C. M. *and a fourth expression, and so on; the last* L. C. M. *will be the* L. C. M. *required.*

Ex. Find the L. C. M. of $6x^2+5x-6$, $12x^2+7x-10$ and $4x^2+x-5$.

The L. C. M. of $6x^2+5x-6$ and $12x^2+7x-10$ will be found to be $(3x-2)(2x+3)(4x+5)$. Therefore the L. C. M. required will be the L. C. M. of this expression and $4x^2+x-5$, which will therefore be

$$(3x-2)(2x+3)(4x+5)(x-1)$$
$$=24x^4+26x^3-49x^2-31x+30.$$

Examples CVIII.

Find the Lowest Common Multiple of

1. $6x^2-x-1$, $3x^2+7x+2$ and $2x^2+3x-2$.
2. $2x^2-7x+3$, $4x^2-7x-15$ and $8x^2+6x-5$.
3. $3x^2-14x-80$, $3x^2+17x-90$ and $x^3-7x^2-80x+576$.
4. $1+4x+8x^2+8x^3$, $1+4x+4x^2-16x^4$ and $1+2x-8x^3-16x^4$.
5. $9x^4-28x^2+3$, $27x^4-12x^2+1$, $27x^4+6x^2-1$ and x^4-6x^2+9.

178. Further applications of the theorems of Chapters XV. and XVI. may be seen from the following examples.

Ex. 1. Shew that the condition that x^2+px+q and $x^2+p'x+q'$ may have a common factor is

$$(q-q')^2=(p-p')(p'q-pq').$$

The common factor is obviously *linear* and of the form $x+a$.

When the first expression is divided by $x+a$, the remainder [Art. 141] is a^2-pa+q, which must vanish, since $x+a$ is a factor of the expression.

Hence $$a^2-pa+q=0 \qquad \text{(i)}$$

Similarly $$a^2-p'a+q'=0 \qquad \text{(ii)}$$

Subtract (i) from (ii), thus

$$(p-p')a-(q-q')=0\ ,$$

whence $$a=\frac{q-q'}{p-p'}$$

Substitute a in (i), thus

$$\left(\frac{q-q'}{p-p'}\right)^2-p\left(\frac{q-q'}{p-p'}\right)+q=0$$

Transpose and multiply by $(p-p')^2$, thus

$$\begin{aligned}(q-q')^2&=p(p-p')(q-q')-q(p-p')^2\\&=(p-p')\{p(q-q')-q(p-p')\}\\&=(p-p')(p'q-pq')\end{aligned}$$

Ex 2 For what value of m will $x^3-(m-6)x^2-2mx+24$ and $x^3-(m+3)x^2-(2m-45)x-30$ have a common factor, and what is that factor?

Let A and B denote the given expressions and F their common factor in x

Thus F is a factor of $A-B$ or $9(x^2-5x+6)$ Evidently F is not a factor of 9, therefore F is a factor $x^2-5x+6=(x-2)(x-3)$ Now $x-2$ and $x-3$ are the *only two* factors of x^2-5x+6 Hence $F=x-2$ or $F=x-3$

If $x-2$ is a factor of the given expressions, then from A by the Remainder Theorem [Art. 141],

$$f(2)=2^3-(m-6)4-4m+24=56-8m=0\ ,$$

whence $$m=7$$

Similarly if $x-3$ is a factor, we have from A

$$f(3)=3^3-(m-6)9-6m+24=105-15m=0\ ,$$

whence $$m=7$$

Thus when $m=7$, both $x-2$ and $x-3$ are factors of A and B, *i e*, $(x-2)(x-3)$ or x^2-5x+6 is the required factor

Ex 3 If $x+f$ be a common factor of x^2+ax+b and $x^2+a'x+b'$ shew that

$$\text{(i)}\ f=\frac{b-b'}{a-a'}\ , \qquad \text{(ii)}\ f=\frac{a'b-ab'}{b-b'}$$

Since $x+f$ is a factor of first expression, we have by the Remainder Theorem

$$\phi(-f)=f^2-af+b=0 \qquad \text{(i)}$$

Similarly from second expression

$$\phi(-f)=f^2-a'f+b'=0. \quad \ldots \ldots (ii)$$

Subtract (i) from (ii); thus $f=\dfrac{b-b'}{a-a'}$

Again multiply (i) by b' and (ii) by b and subtract, thus

$$(b-b')f^2-(a'b-ab')f=0$$

Divide by f and transpose; thus

$$f=\frac{a'b-ab'}{b-b'}$$

Ex 4 If $x+c$ be the H C F of x^2+ax+b and $x^2+a'x+b'$, prove that their L C M will be

$$x^3+(a+a'-c)x^2+(aa'-c^2)x+(a-c)(a'-c)c$$

Divide the given expressions by $x+c$, thus the quotients are $x+a-c$ and $x+a'-c$ respectively Hence, since $M=abH$ [Art 175], the required L C M

$$\begin{aligned}
&=(x+c)(x+a-c)(x+a'-c)\\
&=x^3+(c+a-c+a'-c)x^2\\
&\quad+\{(a-c)(a'-c)+(a'-c)c+c(a-c)\}x+c(a-c)(a'-c)\\
&=x^3+(a+a'-c)x^2+(aa'-c^2)x+c(a-c)(a'-c)
\end{aligned}$$

Miscellaneous Examples CIX.

Find the H C F of

1 $x^4-2ax^3-4a^2x^2+16a^3x+16a^4$

and $x^4-6ax^3-4a^2x^2+16a^3x-16a^4$

2 $2x^4-7x^3+16x^2-17x+12$ and $3x^4-7x^3+13x^2-7x+6$

3. $x^5-3x^4+x^3-3x^2+x-3$ and $7x^4-16x^3-21x^2+9x+27$

4 x^6+11x^3-54 and $x^5+11x+12$

5 $20a^4-3a^2b+b^4$ and $64a^4-3ab^3+5b^4$.

Find the L C M. of

6. x^4+2x^3+6x-9 and $x^4+4x^3+4x^2-9$

7 $3x^3-27ax^2+78a^2x-72a^3$ and $2x^3+10ax^2-4a^2x-48a^3$

8 $x^3-6x^2-37x+210$ and $x^3+4x^2-47x-210$

9 $3x^3-2x^2-x$ and $4x^3-2x^2-3x+1$

10 $7x^3-19x^2+17x-5$ and $2x^4-x^3-9x^2+13x-5$

Find the H. C. F. and the L. C. M. of

11 $ab(x^2+1)+x(a^2+b^2)$ and $ab(x^2-1)+x(a^2-b^2)$

12 $x^2+y^2+2xy-1$ and $x^3+y^3+3xy-1$

13 $24(x^3+x^2y+xy^2+y^3)$ and $16(x^3-x^2y+xy^2-y^3)$

14 Find the H. C. F. of $x^3-7x^2-80x+576$ and $3x^2-14x-80$, and the L. C. M. of these two expressions and $3x^2+17x-90$

15 Find the H. C. F. of $2x^3+x^2y-xy^2-2y^3$ and $x^5-x^3y^2-2x^2y^3+2xy^4$, and shew that its square is a factor of the latter expression

16 Find the H. C. F. of $2x^5-5x^2+3$ and $3x^5-5x^3+2$ and shew that if $x=1$, each of these expressions vanishes

17 Find the H. C. F. of $x^3-7a^2x+6a^3$ and $x^4-3ax^3-2a^2x^2+12a^3x-8a^4$, and shew that each of these expressions vanishes, when $x=a$ or $2a$

18 Find the H. C. F. of $6x^4-2x^3+9x^2+9x-4$ and $9x^4+80x^2-9$ What value of x makes these expressions vanish?

19 For what value of x will the expression $3x^4-5x^3+2x^2+48x+7$ be divisible by x^2+2x-1?

20 What value must be given to a in order that $x^3-ax^2+19x-a-4$ and $x^3-(a+1)x^2+23x-a-7$ may have a common factor?

21. If x^3+px+q and x^3+qx+p have a common factor, then

$$p+q+1=0$$

22 Shew that if x^3+px+q and $x^3+p'x+q'$ have a common factor, then
$$(q'-q)^3+(p'-p)^2(pq'-p'q)=0$$

23 If $x+\alpha$ be a common factor of ax^2+bx+c and $a'x^2+b'x+c'$, shew that

$$\text{(i)}\ \alpha=\frac{ac'-a'c}{ab'-a'b},\ \text{(ii)}\ \alpha=\frac{bc'-b'c}{ac'-a'c},\ \text{(iii)}\ \alpha^2=\frac{bc'-b'c}{ab'-a'b}$$

24 If x^2+ax+b and $x^2+\alpha x+\beta$ have a common factor of the form $x+c$, prove that their L. C. M. is

$$x^3+\frac{ab-\alpha\beta}{b-\beta}x^2+\frac{a\beta^2-\alpha b^2}{a\beta-\alpha b}x+\frac{b\beta(a-\alpha)}{b-\beta}$$

CHAPTER XVII

FRACTIONS

179 *Definitions.* To measure any magnitude we require a unit If a magnitude is less than the unit, it is known as a *fraction* of the unit To measure it, we subdivide the unit into a suitable number of *equal parts* and take one of these parts as the new unit

Thus if we subdivide the unit into *five* equal parts, the required unit is *one fifth* (written $\frac{1}{5}$). Two of these units are written $\frac{2}{5}$ i.e., $\frac{1}{5}\times 2=\frac{2}{5}$ Similarly $\frac{1}{5}\times 3=\frac{3}{5}$ $\frac{1}{5}\times 4=\frac{4}{5}$, &c Evidently *five* of these units make up the unit; thus $\frac{1}{5}\times 5=1$.

Thus generally, if the unit is divided into b equal parts, a of them $=\frac{a}{b}$, i e, $\frac{1}{b}\times a=\frac{a}{b}$; and b of them $=1$, i.e, $\frac{1}{b}\times b=1$

In the fraction $\frac{a}{b}$, a is called the numerator and b the denominator The denominator indicates into how many equal parts, the unit is divided, and the numerator indicates how many of these parts are taken to form the fraction The numerator and denominator are called the terms of the fraction

180 Fraction expresses Quotient We have $\frac{1}{b}\times b=1$

Multiply both sides by a; thus $a\times\frac{1}{b}\times b=1\times a$. i e, $\frac{a}{b}\times b=a$.

Again by the definition of division [Art 56], $a\div b\times b=a$

Hence $\frac{a}{b}\times b=a\div b\times b$. Divide both sides by b, thus $\frac{a}{b}=a\div b$

Thus the fraction $\frac{a}{b}$ expresses the result when a is divided by b

Hence $\frac{a}{b}$ denotes (i) a times $\frac{1}{b}$th part of the unit, or (ii) $\frac{1}{b}$th part of a units. Ex. $\frac{3}{5}$ is either 3 times the fifth part of the unit, or $\frac{1}{5}$ of 3 units

Cor If we divide $-a$ by b and a by $-b$, the result in each case is *negative*. Thus $\frac{-a}{b}=-\frac{a}{b}$ and $\frac{a}{-b}=-\frac{a}{b}$.

Hence *if the sign of the numerator only, or the sign of the denominator only is changed, the sign of the fraction is changed*

Again $a\div b$ and $(-a)\div(-b)$ each gives a *positive* result; thus $\frac{a}{b}=\frac{-a}{-b}$.

Hence *if the signs of both the numerator and denominator are changed the sign of the fraction is not changed*

181 Theorem. *If the numerator and denominator of a fraction be both multiplied, or both divided, by the same quantity, its value is not altered.*

Let $\frac{a}{b}$ be a fraction, we shall prove that for *all values* of m,

$$\text{(i)}\quad \frac{a}{b}=\frac{am}{bm}, \quad \text{and (ii)}\quad \frac{a}{b}=\frac{a-m}{b-m}$$

Let $l=\frac{a}{b}$, thus $l\times b=\frac{a}{b}\times b=a$ [Art 180],

multiply by m, thus $l\times b\times m=a\times m$, or $l\times(bm)=(am)$ [Art 42],

whence $$l=(am)\div(bm)=\frac{am}{bm} \text{ [Art 180]},$$

$$\frac{a}{b}=\frac{am}{bm}$$

Thus (i) is proved

Again, we have $$l\times b=\frac{a}{b}\times b=a,$$

thus $$l\times b-m=a-m, \text{ or } l\times(b-m)=(a-m),$$

whence $$l=(a-m)\div(b-m)=\frac{a-m}{b-m} \text{ [Art 180]},$$

$$\frac{a}{b}=\frac{a-m}{b-m}$$

Thus (ii) is proved

Cor Hence *if the signs of both the numerator and denominator of a fraction be changed, its value is not altered* For this is equivalent to multiplying the numerator and denominator by -1

Thus $\frac{a}{b}=\frac{-a}{-b}$, $\frac{-x}{y}=\frac{x}{-y}$, $\frac{m}{-n}=\frac{-m}{n}$, &c

Examples CX.

1 Shew that $\frac{x}{a}=\frac{x^2}{ax}=\frac{x^2+ax}{ax+a^2}=\frac{ax+bx}{a^2+ab}$

2 Shew that $\frac{1}{a+b}=\frac{4}{4a+4b}=\frac{a-b}{a^2-b^2}=\frac{a+b}{a^2+2ab+b^2}$

3 Shew that $y=\frac{ay}{a}=\frac{y^2-y}{y-1}=\frac{ay+xy}{a+x}=\frac{ay-xy}{a-x}$

4 Shew that $\frac{1+\frac{1}{x}}{x}=\frac{x+1}{x^2}=\frac{x^2+x}{x^3}=\frac{x^2-1}{x^3-x^2}$

5 Shew that $1-\frac{a}{a-b}=1-\frac{-a}{-a+b}=1+\frac{a}{b-a}=1-\frac{-a}{b-a}$

6 Shew that $\frac{x}{x-a}+\frac{a}{a-x}=\frac{x}{x-a}+\frac{-a}{x-a}=\frac{x-a}{x-a}=1$

7 Shew that $\frac{x}{x-a}+\frac{a}{a-x}=1$, by changing the sign of $\frac{x}{x-a}$.

182 Reduction to Lowest Terms A fraction is said to be in its *lowest terms* when its numerator and denominator have no *common* factor A fraction is therefore reduced to its lowest terms by dividing its numerator and denominator their H. C F. Hence to reduce a fraction to its lowest terms, we *strike out their* H C F *from the numerator and denominator*

Ex 1 Reduce $\frac{54a^3bc}{36ab^3c}$ to its lowest terms

$$\text{Given fraction}=\frac{18abc\,3a^2}{18abc\,2b^2}=\frac{3a^2}{2b^2}$$

Ex 2 Reduce $\frac{6ab^3(a^2+ab)}{8a^2b(a^2b-b^3)}$ to its lowest terms

$$\text{Given fraction}=\frac{6a^2b^3(a+b)}{8a^2b^2(a^2-b^2)}=\frac{6a^2b^3(a+b)}{8a^2b^2(a+b)(a-b)}=\frac{3b}{4(a-b)}$$

Note When the factors of numerator and denominator are found by inspection as here, we strike out *all the common factors*, instead of finding the H C F and then striking it out

Ex. 3 Reduce $\frac{2x^2-x-6}{3x^2-8x+4}$ to its lowest terms

$$\text{Given fraction}=\frac{(x-2)(2x+3)}{(x-2)(3x-2)}=\frac{2x+3}{3x-2}$$

Examples CXI

Reduce to lowest terms

1 $\frac{12a^2b}{4ab^2c}$ 2 $\frac{16axy^2}{20ax^2y}$ 3 $\frac{28a^3b^2x^4}{42abx^3y}$ 4 $\frac{63m^4n^5}{405m^6n^3}$

5 $\frac{8x^2-12xy}{16x^4y^2+20xy^3}$ 6 $\frac{a^2xy^2}{a^2xy-axy^2}$ 7 $\frac{8m^2-2n^2}{8am+4an}$

8 $\frac{3mp+3mq}{p^2-q^2}$ 9 $\frac{a^2bc-ab^2c}{a^2bd-ab^2d}$ 10 $\frac{cx+x^2}{a^2c^2-x^2a^2}$

Reduce to lowest terms

11 $\dfrac{x^2-xy}{y^2-xy}$. 12 $\dfrac{24a^2x^2-16ax^3}{24a^2x^2-54a^4}$ 13 $\dfrac{3ab^2(a^2-b^2)}{12a^2b(a^3+b^3)}$

14 $\dfrac{4a^3(x^3-a^3)}{6ax^2(a^4+a^2x^2+x^4)}$ 15 $\dfrac{m^3a^2+n^3a^2}{a(m^2+n^2)-man}$

16 $\dfrac{1-2a+a^2-b^2}{1-a+ab-b^2}$ 17 $\dfrac{a^2+b^2-c^2+2ab}{a^2-b^2-c^2+2bc}$ 18 $\dfrac{4x^2-12ax+9a^2}{8x^3-27a^3}$

19. $\dfrac{ab(x^2+y^2)+xy(a^2+b^2)}{ab(x^2-y^2)+xy(a^2-b^2)}$ 20 $\dfrac{1-a^2b+b-a^2}{1-ab^2+a-b^2}$ 21 $\dfrac{x^2-4x+3}{x^2-2x-3}$

22 $\dfrac{x^2-x-20}{x^2-9x+20}$ 23 $\dfrac{x^2+2x-3}{x^2+5x+6}$ 24 $\dfrac{6a^2+5ax-6x^2}{6a^2+13ax+6x^2}$

25 $\dfrac{6a^2-7ax-3x^2}{6a^2+11ax+3x^2}$ 26 $\dfrac{1+x-12x^2}{3-7x-6x^2}$ 27 $\dfrac{6xy+8x+9y+12}{10xy-8x+15y-12}$

28 $\dfrac{x^2+2axy+(a^2-b^2)y^2}{x^2+2bxy-(a^2-b^2)y^2}$ 29 $\dfrac{a^2x^2-2acxz-b^2y^2+c^2z^2}{a^2x^2+2abxy+b^2y^2-c^2z^2}$

30 $\dfrac{(a+b)x^2-(2a+b)bx+ab^2}{(a-b)x^2-(2a-b)bx+ab^2}$ 31 $\dfrac{a^3+2a^2b-ab^2-2b^3}{a^3-3ab^2+2b^3}$

Ex. 4 Reduce $\dfrac{x^3-39x+70}{x^3+14x^2+39x-70}$ to its lowest terms

As in Art 169, find the H C F of the numerator and denominator, which will be seen to be $x+7$

And
$$(x^3-39x+70)\div(x+7)=x^2-7x+10,$$
$$(x^3+14x^2+39x-70)\div(x+7)=x^2+7x-10,$$
$$\text{reduced fraction}=\frac{x^2-7x+10}{x^2+7x-10}$$

Ex 5 Reduce $\dfrac{2a^4-5a^3b+3a^2b^2-5ab^3-3b^4}{3a^4-7a^3b+a^2b^2-3ab^3-2b^4}$ to its lowest terms

Find the H C F of numerator and denominator as in Art 169, which will thus be $a^2-2ab-b^2$

Now
$$\text{numerator}\div(a^2-2ab-b^2)=2a^2-ab+3b^2,$$
$$\text{denominator}\div(a^2-2ab-b^2)=3a^2-ab+2b^2,$$
$$\text{reduced fraction}=\frac{2a^2-ab+3b^2}{3a^2-ab+2b^2}$$

Examples CXI. *(Continued.)*

Reduce to lowest terms

32 $\dfrac{2y^3+y^2-8y+5}{7y^2-12y+5}$. 33 $\dfrac{x^3+2x^2-2x+3}{x^3-8x+3}$.

34 $\dfrac{2x^3-x^2+x+1}{2x^3+3x^2+3x+1}$. 35 $\dfrac{x^3+3x^2+x+3}{x^3-13x-12}$.

36. $\dfrac{9x^3+6x^2-2x-4}{12x^3-5x^2+4x-4}$. 37 $\dfrac{2x^3-13x+15}{3x^3+9x^2-5x-15}$.

38 $\dfrac{x^3+11x^2+30x}{9x^3+53x^2-9x-18}$. 39. $\dfrac{18x^3-11a^2x-2a^3}{18x^2-6ax-12a^2}$.

40 $\dfrac{3(x^3-y^3)-5xy(x-y)+y^3}{3(x^3+y^3)+xy(x+y)-5y^3}$. 41 $\dfrac{a^2-acx+(ac-b^2+bc)x^2-bcx^3}{a^2+abx+(ac-c^2+bc)x^2+c^2x^3}$.

42 $\dfrac{a^4+a^3b+ab^3+b^4}{a^4+3a^3b+4a^2b^2+3ab^3+b^4}$. 43. $\dfrac{3a^4-a^2b^2-2b^4}{10a^4+15a^3b-10a^2b^2-15ab^3}$.

44 $\dfrac{4x^4+11x^2+25}{4x^4-9x^2+30x-25}$. 45 $\dfrac{x^4-2x^3-25x^2+26x+120}{x^4-4x^3-19x^2+46x+120}$.

46 $\dfrac{3a^4-14a^3x-9ax^3+2x^4}{2a^4-9a^3x-14ax^3+3x^4}$. 47 $\dfrac{3a^5b-27a^4b^2+78a^3b^3-72a^2b^4}{2a^4b^2+10a^3b^3-4a^2b^4-48b^6}$.

48 $\dfrac{6a^2x^6-26a^3x^5+46a^4x^4-42a^5x^3}{18a^3x^6+3a^4x^5-132a^5x^4+63a^6x^3}$.

183 Reduction to Lowest Common Denominator

Let it be required to reduce the fractions $\dfrac{a}{b}, \dfrac{c}{d}, \dfrac{e}{f}$, to a common denominator By Art. 181, we have

$$\frac{a}{b}=\frac{a\times df}{b\times df}=\frac{adf}{bdf}, \quad \frac{c}{d}=\frac{c\times bf}{d\times bf}=\frac{bcf}{bdf}, \quad \frac{e}{f}=\frac{e\times bd}{f\times bd}=\frac{bde}{bdf}.$$

Hence to reduce fractions to a *common denominator*, we *multiply the terms of each fraction by the product of the denominators of all the others* Similarly to reduce them to the *Lowest Common Denominator* (L C D), we find the L C M of the denominators, and then proceed as in the Examples below.

Ex 1 Reduce $\frac{x}{a}$, $\frac{x-a}{a(x+a)}$ and $\frac{3a}{x^2-a^2}$ to the L C D

The L C M of denominators $=a(x^2-a^2)$

Divide the L C M by denr of first fraction, and multiply numr and deni by the quot x^2-a^2,

$$\frac{x}{a}=\frac{x}{a}\times\frac{x^2-a^2}{x^2-a^2}=\frac{x^3-a^2x}{a(x^2-a^2)}$$

Similarly $\frac{x-a}{a(x+a)}=\frac{x-a}{a(x+a)}\times\frac{x-a}{x-a}=\frac{(x-a)^2}{a(x^2-a^2)}$,

and $\frac{3a}{x^2-a^2}=\frac{3a}{x^2-a^2}\times\frac{a}{a}=\frac{3a^2}{a(x^2-a^2)}$

Ex 2 Reduce $\frac{x}{a-1}$, $\frac{2x}{a+1}$ and $\frac{ax-1}{1-a^2}$ to the L C D

The L C M of the denominators of the first and second fractions is a^2-1, which differs from $1-a^2$, the denominator of the third fraction, *only in sign* Changing therefore the sign of the third fraction [Art 180, Cor], the given fractions become

$$\frac{x}{a-1}, \frac{2x}{a+1}, \frac{1-ax}{a^2-1},$$

of which the equivalent forms are

$$\frac{x(a+1)}{a^2-1}, \frac{2x(a-1)}{a^2-1}, \frac{1-ax}{a^2-1}$$

Examples CXII

Reduce to Lowest Common Denominator

1 $\frac{a}{9}, \frac{5a}{12}, \frac{7a}{15}$

2 $\frac{2x}{3}, \frac{3y}{4}, \frac{5z}{18}$

3 $\frac{a^2}{bc}, \frac{b^2}{ca}, \frac{c^2}{ab}$

4 $\frac{5x}{yz}, \frac{4y}{zx}, \frac{6z}{xy}, \frac{x^2+2y^2}{xyz}$

5 $\frac{a+x}{9a}, \frac{a-x}{12x}$

6 $\frac{4x-5}{10}, \frac{2x}{5}, \frac{7x+6}{25}$

7 $\frac{1+a}{5}, \frac{3-a}{6}, \frac{a-8}{10}$

8 $\frac{x^2-ab}{ab}, \frac{y^2-bc}{bc}, \frac{z^2-ac}{ac}$.

9 $\frac{a}{a+b}, \frac{b}{a-b}, \frac{c}{a+b}$

10 $\frac{x}{3}, \frac{5a}{4}, \frac{1+x}{2(1-x)}$

11 $\frac{1}{1-x}, \frac{1}{1+x}, \frac{1}{1-x^2}$

12 $\frac{2}{x-1}, \frac{3}{2-x}, \frac{4}{x^2-4}$

Reduce to Lowest Common Denominator

13 $\frac{x}{2y}, \frac{x}{x-y}, \frac{y-1}{y^2-xy}$. 14. $\frac{a-x}{a^2(a+x)}, \frac{a+x}{x^2(a-x)}, \frac{x^2-1}{ax(x^2-a^2)}$

15 $\frac{ab}{a-b}, \frac{bc}{b-c}, \frac{ca}{c-a}$ 16 $\frac{a}{ab-b^2}, \frac{b}{a^2+ab}, \frac{a+b}{a^2-b^2}$

17 $\frac{3}{2x-1}, \frac{5}{2x+1}, \frac{3x}{4x^2-1}, \frac{4x}{(2x-1)^2}$

184 Addition and Subtraction of Fractions The algebraic sum of two or more fractions is obtained *by reducing them to the lowest common denominator* [Art. 183] *and then finding the algebraic sum of the numerators of the reduced fractions*

Ex 1 Simplify $\frac{a}{9}-\frac{5a}{12}+\frac{7a}{15}$

The L C M of denominators$=180$.

Required value$=\frac{20a}{180}-\frac{75a}{180}+\frac{84a}{180}=\frac{20a-75a+84a}{180}=\frac{29a}{180}$

Ex 2 Find the value of $\frac{a^2}{3bx}+2ax-\frac{5x}{4ab}$.

Required value$=\frac{a^2}{3bx}+\frac{2ax}{1}-\frac{5x}{4ab}$

$$=\frac{4a^3}{12abx}+\frac{24a^2bx^2}{12abx}-\frac{15x^2}{12abx}$$

[for L C M of denominators is $12abx$]

$$=\frac{4a^3+24a^2bx^2-15x^2}{12abx}$$

Ex 3 Simplify $\frac{a}{a-x}+\frac{3a}{a+x}-\frac{2ax}{a^2-x^2}$

The L C M. of denominators$=a^2-x^2$

Required value$=\frac{a(a+x)}{a^2-x^2}+\frac{3a(a-x)}{a^2-x^2}-\frac{2ax}{a^2-x^2}$

$$=\frac{a^2+ax+3a^2-3ax-2ax}{a^2-x^2}$$

$$=\frac{4a^2-4ax}{a^2-x^2}=\frac{4a(a-x)}{a^2-x^2}=\frac{4a}{a+x}.$$

Ex 4 Simplify $\frac{x}{3(a-1)}+\frac{5x}{6(a+1)}+\frac{ax-1}{4(1-a^2)}$

Changing the sign of the third fraction [Art 180, Cor], we have

$$\frac{x}{3(a-1)}+\frac{5x}{6(a+1)}+\frac{1-ax}{4(a^2-1)}$$

The L C M of denominators $=12(a^2-1)$

$$\text{Required sum}=\frac{4x(a+1)}{12(a^2-1)}+\frac{10x(a-1)}{12(a^2-1)}+\frac{3(1-ax)}{12(a^2-1)}$$

$$=\frac{11ax-6x+3}{12(a^2-1)}$$

Ex 5 Simplify $\frac{1}{(b-c)(c-a)}+\frac{1}{(c-a)(a-b)}+\frac{1}{(a-b)(b-c)}$

The L C M of denominators $=(b-c)(c-a)(a-b)$,

$$\therefore \text{given fraction}=\frac{(a-b)+(b-c)+(c-a)}{(b-c)(c-a)(a-b)}=\frac{0}{\text{Denominator}}=0$$

Examples CXIII

Simplify the following expressions

1 $\frac{x}{5}+\frac{5x}{12}+\frac{7x}{20}$ 2 $\frac{ax}{18}-\frac{2ax}{7}+\frac{7ax}{30}$ 3 $\frac{a+x}{15}+\frac{2a}{5}-\frac{4x-3a}{25}$

4 $\frac{x}{2y}+\frac{5x}{6y}+\frac{3x}{8y}$ 5 $\frac{2a}{x}+\frac{3b}{y^2}-\frac{ab}{x^2y}$ 6 $\frac{4x^2}{y}-\frac{3x}{8}+\frac{5y}{12x}$

7 $\frac{a}{bc}+\frac{b}{ca}+\frac{c}{ab}$ 8 $\frac{x-y}{xy}+\frac{y-z}{yz}+\frac{z-x}{zx}$

9 $\frac{a+x}{5ax}-\frac{a-x}{3a^2}+\frac{a}{10x^2}+\frac{a^2+x^2}{6ax^2}$ 10 $\frac{x}{y}-\frac{x+y}{x-y}$

11 $\frac{1}{1+x}+\frac{1}{1-x}$ 12 $\frac{1}{1-x}-\frac{2}{1-x^2}$ 13 $\frac{m}{p+q}-\frac{m}{p-q}$

14 $\frac{x}{a^2}+\frac{a-x}{a(a+x)}$ 15 $\frac{a}{c}-\frac{(ad-bc)x}{c(c+dx)}$ 16 $\frac{3ax}{7}+\frac{13by}{14}-\frac{2ax+9by}{21}$

17 $\frac{15x-1}{18}-\frac{5x-3}{6}+\frac{5}{9}$ 18 $\frac{x}{2a}-\frac{x-a}{2(x+a)}$ 19 $\frac{x}{2x-2y}+\frac{y}{2y-2x}$

20 $\frac{5x-18}{10}+\frac{2x+7}{25}+1\frac{13}{25}$ 21 $\frac{1}{x}+\frac{m-n}{2x^2}+\frac{4a^2}{x^3}$ 22 $\frac{2}{x}+\frac{1+x}{1-x}-\frac{1-x}{1+x}$

Simplify the following expressions

23 $\dfrac{a(a+x)}{a-x}+\dfrac{x(3a-x)}{x-a}$.

24. $\dfrac{m}{m-n}+\dfrac{n}{m+n}+\dfrac{2mn}{n^2-m^2}$

25 $\dfrac{2}{x}-\dfrac{1}{a+x}+\dfrac{1}{a-x}$.

26 $\dfrac{1}{x-1}-\dfrac{1}{2x+2}-\dfrac{x+3}{2x^2+2}$

27 $\dfrac{1}{3(3x+2)}+\dfrac{1}{3x-2}-\dfrac{4}{3(3x-1)}$

28 $\dfrac{1}{x-1}-\dfrac{3}{x+1}+\dfrac{2(x-2)}{x^2+1}$

29 $\dfrac{x+y}{y}-\dfrac{2x}{x+y}+\dfrac{x^3-x^2y}{y^3-x^2y}$

30. $\dfrac{1}{4(1+x)}+\dfrac{1}{4(1-x)}+\dfrac{1}{2(1+x^2)}$

31 $\dfrac{x}{x-3}-\dfrac{x-3}{x}+\dfrac{x}{x+3}-\dfrac{x+3}{x}$

32 $\dfrac{1}{2(x-1)}-\dfrac{4}{x-2}+\dfrac{9}{2(x-3)}$

33 $\dfrac{1}{6m-2n}+\dfrac{1}{3m+2n}-\dfrac{3}{6m+2n}$.

34 $\dfrac{3}{8(1-x)}+\dfrac{1}{8(1+x)}-\dfrac{1-x}{4(1+x^2)}$

35 $\dfrac{4}{x(x-2)}+\dfrac{1}{x^2-5x+6}-\dfrac{3}{x(x-3)}$

36 $\dfrac{1}{8(x-1)}-\dfrac{1}{4(x-3)}+\dfrac{1}{8(x-5)}$

37 $\dfrac{1}{x^2-2}-\dfrac{2}{x^2-1}+\dfrac{2}{x^2+1}-\dfrac{1}{x^2+2}$

38 $\dfrac{x+a}{x-a}+\dfrac{x-a}{x+a}+2\dfrac{x^2-a^2}{x^2+a^2}$.

39 $\dfrac{10x-11}{3(x^3-1)}-\dfrac{10x-1}{3(x^2+x+1)}+\dfrac{x^2-2x+5}{(x^3-1)(x+1)}$

40 $\dfrac{7}{2(x+1)}-\dfrac{1}{6(x-1)}-\dfrac{10x-1}{3(x^2+x+1)}$

41 $\dfrac{a}{b}-\dfrac{(a^2-b^2)x}{b^2}+\dfrac{a(a^2-b^2)x^2}{b^2(b+ax)}$

42 $\dfrac{3}{1-x}-\dfrac{17}{1-2x}+\dfrac{17}{1-3x}$

43 $\dfrac{1}{1-x}-\dfrac{1}{(1-x)^2}+\dfrac{1}{(1-x)^3}-\dfrac{1}{(1-x)^4}$

44 $\dfrac{1}{2(a-b)}+\dfrac{1}{2(a+b)}+\dfrac{a}{a^2+b^2}-\dfrac{a^3}{a^4-b^4}$

45 $\dfrac{1}{(x+1)^3}-\dfrac{3}{2(x+1)^2}+\dfrac{5}{4(x+1)}-\dfrac{5}{4(x+3)}$

46 $\dfrac{a+c}{(a-b)(x-a)}-\dfrac{b+c}{(a-b)(x-b)}$

47 $\dfrac{1+2x}{(3-x)(1+x)}+\dfrac{7}{(2+x)(1-3x)}+\dfrac{x}{(1+x)(2+x)}$

48 $\dfrac{1}{(x+1)(x+2)}-\dfrac{1}{(x+1)(x+2)(x+3)}$

185 Multiplication by an Integer Let $l=\frac{a}{b}$,

thus $$l=a-b\text{[Art. 180]},$$
$$l\times m=a-b\times m=a\times m-b\text{ [Art 57]}=am-b,$$

whence $\frac{a}{b}\times m=\frac{am}{b}$, where m is any integer

Hence *to multiply a fraction by an integer*, we *multiply the numerator by that integer*

186 Multiplication of Fractions Let $l=\frac{a}{b}\times\frac{c}{d}$,

$$l\times b\times d=\frac{a}{b}\times\frac{c}{d}\times b\times d=\frac{a}{b}\times b\times\frac{c}{d}\times d\text{ [Art 43]},$$

but $\frac{a}{b}\times b=a$ and $\frac{c}{d}\times d=c$, therefore

$$l\times b\times d=a\times c,\text{ i e, }l\times(bd)=ac,$$

whence $$l=\frac{ac}{bd}\quad\text{i e, }\frac{a}{b}\times\frac{c}{d}=\frac{ac}{bd}$$

Similarly it may be shewn that

$$\frac{a}{b}\times\frac{c}{d}\times\frac{e}{f}\times\ldots=\frac{ace\ldots}{bdf\ldots}$$

Hence to multiply several fractions together, we *multiply all the numerators for the new numerator, and all the denominators for the new denominator*

Ex 1 $$\frac{ax}{cd}\times\frac{cx}{ab}=\frac{acx^2}{abcd}=\frac{x^2}{bd}$$

REMARK It is always advisable, before we multiply out, according to the Rule, first to *strike out any factor or factors common to both the numerator and denominator* The product will thus be found at once in its *lowest terms* Thus striking out a and c from the numerator and denominator first, we have

$$\frac{x}{d}\times\frac{x}{b}=\frac{x^2}{bd}$$

Ex 2 $$\frac{3(ax+x^2)}{a^2-x^2}\times\frac{bc+cd}{a^2-ax}\times\frac{(a-x)^2}{3b^2-3d^2}$$
$$=\frac{3x(a+x)}{(a+x)(a-x)}\times\frac{c(b+d)}{a(a-x)}\times\frac{(a-x)(a-x)}{3(b+d)(b-d)}=\frac{cx}{a(b-d)}$$

Ex 3 $\left(\frac{x}{a}+\frac{y}{b}+\frac{z}{c}\right)\left(\frac{a}{x}+\frac{b}{y}+\frac{c}{z}\right)$

$$=\left(\frac{x}{a}+\frac{y}{b}+\frac{z}{c}\right)\frac{a}{x}+\left(\frac{x}{a}+\frac{y}{b}+\frac{z}{c}\right)\frac{b}{y}+\left(\frac{x}{a}+\frac{y}{b}+\frac{z}{c}\right)\frac{c}{z}$$

$$=1+\frac{ay}{bx}+\frac{az}{cx}+\frac{bx}{ay}+1+\frac{bz}{cy}+\frac{cx}{az}+\frac{cy}{bz}+1$$

$$=3+\frac{ay}{bx}+\frac{az}{cx}+\frac{bx}{ay}+\frac{bz}{cy}+\frac{cx}{az}+\frac{cy}{bz}$$

Examples CXIV

Simplify

1 $\frac{2ax}{5by}\times\frac{3y^2}{4x^3}$ 2. $\frac{10a^2m}{7m^2z}\times\frac{4mx}{5az}\times\frac{21z^2}{16ax}$ 3 $\frac{ab}{xy}\times\frac{a+x}{a-x}\times\frac{ax}{by}$

4 $\frac{8mn}{5cd}\times\frac{5c+5d}{m^2n-mn^2}$ 5 $\frac{ax}{(a-x)^2}\times\frac{a^2-x^2}{ab}$ 6 $\left(a-\frac{x^2}{a}\right)\left(\frac{a}{x}+\frac{x}{a}\right)$

7 $\left(1+\frac{3x}{a-x}\right)\left(\frac{a-x}{a+2x}\right)^2$ 8 $\frac{a^2-b^2}{5b}\times\frac{15a^2}{a+b}$ 9 $\left(\frac{a}{x}+\frac{b}{y}\right)\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)$

10 $\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)\left(\frac{x}{y}+\frac{y}{z}\right)$ 11 $\left(\frac{a}{x}+\frac{b}{y}\right)\left(\frac{x}{a}-\frac{y}{b}\right)$

12 $\left(\frac{4a}{3x}+\frac{3x}{2b}\right)\left(\frac{2b}{3x}+\frac{3x}{4a}\right)$ 13 $\left(1-\frac{a}{x}\right)\left(1-\frac{a^2}{a^2-x^2}\right)$

14 $\left(a+\frac{ax}{a-x}\right)\left(a-\frac{ax}{a+x}\right)\frac{a^2-x^2}{a^2+x^2}$ 15 $\frac{2ax}{3by}\times\frac{x^2-y^2}{2c^2-2y^2}\times\frac{3bc+3by}{x^2-xy}$

16 $\frac{x+y}{(x-y)^2}\times\frac{x^3-y^3}{x^3+y^3}\times\frac{(x-y)^2+xy}{(x+y)^2-xy}$ 17 $\left(a^4-\frac{a^2}{x^2}\right)\left(\frac{a^2c^2+abx^2}{ax+1}\right)\frac{ax}{a^2-b^2}$

18 $\frac{pr+(pq+qr)x+q^2x^2}{p-qx}\times\frac{ps+(pt-qs)x-qtx^2}{p+qx}$

Multiply

19 x^2+x+1 by $\frac{1}{x^2}-\frac{1}{x}+1$ 20 $1-x+x^2-\frac{x^3}{1+x}$ by $1-x^2$

21 $ax+1+\frac{1}{ax}$ by $ax-1+\frac{1}{ax}$ 22 $\frac{a^2}{c^2}-\frac{ab}{2xy}+\frac{b^2}{y^2}$ by $\frac{3a^2}{x^2}-\frac{ab}{5xy}+\frac{b^2}{y^2}$

187 Division by an Integer Let $l=\frac{a}{b}$, thus

$$l=a\div b \text{ [Art 180]},$$

$$l\div m=a\div b\div m=a\div(bm) \text{ [Art 57]}=\frac{a}{bm},$$

whence $\quad \frac{a}{b}\div m=\frac{a}{bm}$ where m is any integer

Hence *to divide a fraction by an integer, we multiply the denominator by that integer*

188 Division of Fractions Let $q=\frac{a}{b}\div\frac{c}{d}$,

then, since *Divisor* × *Quotient* = *Dividend*, we have $\frac{c}{d}\times q=\frac{a}{b}$,

$$q\times\frac{c}{d}\times b\times d=\frac{a}{b}\times b\times d, \text{ or } q\times b\times\frac{c}{d}\times d=\frac{a}{b}\times b\times d \text{ [Art 43]},$$

but $\quad \frac{c}{d}\times d=c$ and $\frac{a}{b}\times b=a$ therefore

$$q\times b\times c=a\times d, \text{ i e }, q\times bc=ad,$$

whence $\quad q=\frac{ad}{bc}, \qquad \frac{a}{b}\div\frac{c}{d}=\frac{a}{b}\times\frac{d}{c}$

Hence to divide one fraction by another, we *invert the divisor, and proceed as in multiplication*

Ex 1 $\frac{25x^2}{24y^2}\div\frac{5x}{12y}=\frac{25x^2}{24y^2}\times\frac{12y}{5x}=\frac{25\times12\times x^2y}{24\times5\times xy^2}=\frac{5x}{2y}$

Or it may be worked thus—

$\frac{25x^2}{24y^2}\times\frac{12y}{5x}=\frac{5x\times5x}{12y\times2y}\times\frac{12y}{5x}=\frac{5x}{2y}$ (striking out the common factors, *vide* REMARK, Ex. 1, Art 186)

Ex 2 $\frac{4x^2-9}{3xy}\div\frac{2x^2+3x}{6y}=\frac{(2x+3)(2x-3)}{3xy}\times\frac{6y}{x(2x+3)}=\frac{2(2x-3)}{x^2}$

Ex 3 $\left(1-\frac{5x^2}{x^2+y^2}\right)\div\left(1+\frac{2x}{y}\right)=\frac{y^2-4x^2}{x^2+y^2}\div\frac{y+2x}{y}=\frac{y^2-4x^2}{x^2+y^2}\times\frac{y}{y+2x}$

$$=\frac{(y+2x)(y-2x)}{x^2+y^2}\times\frac{y}{y+2x}=\frac{y(y-2x)}{y^2+x^2}$$

Ex 4 Divide $\frac{a^3}{x^3}+a^2-x^2+\frac{x}{a}-\frac{a}{x}-\frac{x^3}{a^3}$ by $\frac{a}{x}-\frac{x}{a}$

Arrange according to powers of a

$$\begin{array}{l}
\frac{a}{x}-\frac{x}{a}\Big)\frac{a^3}{x^3}+a^2-\frac{a}{x}-x^2+\frac{x}{a}-\frac{x^3}{a^3}\Big(\frac{a^2}{x^2}+ax+\frac{x^2}{a^2} \\
\qquad\qquad\underline{\frac{a^3}{x^3}-\frac{a}{x}} \\
\qquad\qquad\qquad a^2-x^2 \\
\qquad\qquad\qquad \underline{a^2-x^2} \\
\qquad\qquad\qquad\qquad \frac{x}{a}-\frac{x^3}{a^3} \\
\qquad\qquad\qquad\qquad \underline{\frac{x}{a}-\frac{x^3}{a^3}}
\end{array}$$

$$\text{required quotient}=\frac{a^2}{x^2}+ax+\frac{x^2}{a^2}$$

Examples CXV

Simplify

1 $\frac{5ax}{8x^2y}\div\frac{30a^2}{12xy}$ 2 $\frac{4xyz}{5a^2b}\div\frac{3yz}{10ax}$ 3 $\frac{ma^3}{nbc}\div\frac{m^2a^2}{nbc^2}$

4 $\left(1-\frac{x^2}{a^2}\right)\div\frac{a+x}{a^2}$ 5 $\frac{a^2+ab}{a-b}\div\frac{a^4-b^4}{(a-b)^2}$ 6 $\frac{2ac-x^2}{c^3-x^3}\div\frac{2a-x}{(c-x)^2}$

7 $\frac{2a(1-x^2)^2}{cy}\div\frac{(1-x)(1+x)^3}{y^3}$ 8 $\left(1+\frac{1}{x}\right)\div\left(x-\frac{1}{x}\right)\div\left(\frac{x}{x-1}\right)^2$.

9 $\frac{a-x}{3x-3}\div\left(1+\frac{2}{x^2-1}\right)\div\left(\frac{a}{x}-1\right)$

Divide

10 $\frac{a}{x}-\frac{y}{b}$ by $\frac{y}{a}-\frac{b}{x}$ 11. $1-\frac{2xy}{x^2+y^2}$ by $\frac{x^3-y^3}{x-y}-3xy$

12 $\frac{a+2x}{a+x}+\frac{a}{x}$ by $\frac{x}{a+x}+\frac{a+x}{x}$. 13 $\frac{x}{y}-\frac{2x^2}{(x+y)^2}$ by $\frac{y}{x}-\frac{2y^2}{(x+y)^2}$

14 $1+\frac{6}{x^2-5x}$ by $1+\frac{3x-5}{x^2-6x+5}$ 15 $\frac{1-2x+x^2}{1-x^6}$ by $\frac{1-3x+3x^2-x^3}{1+x^2+x^4}$.

16 $y^2+1+\frac{1}{y^2}$ by $\frac{1}{y^2}-\frac{1}{y}+1$ 17 $z^2+1+\frac{1}{z^2}$ by $z+1+\frac{1}{z}$

Divide

18 $\frac{3x^2}{8b^2}+\frac{a}{2b}-\frac{4a^2}{x^2}$ by $\frac{2a}{x}+\frac{x}{2b}$ 19 $x^3+\frac{1}{x^3}-\left(x+\frac{1}{x}\right)$ by $\left(x-\frac{1}{x}\right)^2$

20 $x^4-\frac{1}{x^4}-2\left(x^2-\frac{1}{x^2}\right)-\left(x+\frac{1}{x}\right)$ by $x^3-\frac{1}{x^3}-3\left(x-\frac{1}{x}\right)-1$.

21 $\frac{a^2x^2}{bd}+\frac{abx^3}{c^2d}-\frac{acx^2}{d^2}-\frac{b^2x}{cd^2}+\frac{a^2x}{bc}-\frac{a}{d}$ by $\frac{ax}{c}-\frac{b}{d}$

189 Complex Fractions A fraction whose numerator or, denominator, or both, are fractions, is called a *Complex Fraction*

Thus $\frac{\frac{a}{b}}{c}$, $\frac{a}{\frac{b}{c}}$, $\frac{\frac{a}{c}}{\frac{y}{a}}$, are Complex Fractions

Hence in reducing complex fractions, the Rules of the foregoing articles will apply

Ex 1 $\frac{\frac{a}{x}}{\frac{y}{a}}=\frac{a}{x}\div\frac{y}{a}=\frac{a}{x}\times\frac{a}{y}=\frac{a^2}{xy}$

Ex 2 $\frac{1-x^2}{\frac{1}{x}-1}=(1-x^2)\div\left(\frac{1}{x}-1\right)=(1-x^2)\div\frac{1-x}{x}=(1-x^2)\times\frac{x}{1-x}$

$$=(1+x)(1-x)\times\frac{x}{1-x}=x(1+x)$$

Ex 3 $\frac{a+\frac{ab}{a-b}}{\frac{a}{a-b}-\frac{b}{a}}=\left(a+\frac{ab}{a-b}\right)\div\left(\frac{a}{a-b}-\frac{b}{a}\right)$

$$=\frac{a(a-b)+ab}{a-b}\div\frac{a^2-b(a-b)}{a(a-b)}=\frac{a^2}{a-b}\div\frac{a^2-ab+b^2}{a(a-b)}$$

$$=\frac{a^2}{a-b}\times\frac{a(a-b)}{a^2-ab+b^2}=\frac{a^3}{a^2-ab+b^2}$$

Ex 4 $\frac{1}{a+\frac{1}{b+\frac{1}{c}}}=\frac{1}{a+\frac{1}{\frac{bc+1}{c}}}=\frac{1}{a+\frac{c}{bc+1}}=\frac{1}{\frac{abc+a+c}{bc+1}}=\frac{bc+1}{abc+a+c}$

Examples CXVI

Simplify

1. $\dfrac{x-1\frac{1}{3}}{2x}$ 2. $\dfrac{\frac{3}{2}a}{3-\frac{3}{4}a}$ 3. $\dfrac{4m-\frac{1}{m}}{2+\frac{1}{m}}$ 4. $\dfrac{25-\frac{9}{x^2}}{5+\frac{3}{x}}$ 5. $\dfrac{x-\frac{2}{3}}{\frac{3x}{2}-\frac{2}{3x}}$

6. $\dfrac{\frac{a}{x}+\frac{b}{y}}{\frac{a}{x}\times\frac{b}{y}}$ 7. $\dfrac{\frac{m}{a}-\frac{n}{c}}{\frac{x}{a}+\frac{y}{a}}$ 8. $\dfrac{x-\frac{y}{z}}{z-\frac{y}{x}}$ 9. $\dfrac{\frac{a}{b}-\frac{b}{a}}{\frac{a}{b}+\frac{b}{a}}$. 10. $\dfrac{\frac{2a}{a^2-x^2}}{1-\frac{a}{a+x}}$.

11. $\dfrac{\frac{xyz}{a}-mb}{mb-\frac{xyz}{b}}$ 12. $\dfrac{1-\frac{x-y}{x+y}}{2+\frac{2y}{x-y}}$ 13. $\dfrac{x^2-\frac{1}{x}}{x+\frac{1}{x}+1}$. 14. $\dfrac{x+y+\frac{y^2}{x}}{y-\frac{x^3}{y^2}}$.

15. $\dfrac{\frac{1}{1+x}+\frac{x}{1-x}}{\frac{1}{1-x}-\frac{x}{1+x}}$ 16. $\dfrac{1+\frac{2mn}{m^2+n^2}}{\frac{m^3+n^3}{m+n}+3mn}$ 17. $\dfrac{x+\frac{y-x}{1+xy}}{1-x\frac{y-x}{1+xy}}$

18. $\dfrac{\frac{(a+b)^2}{4ab}-1}{\frac{(a-b)^2}{4ab}+1}\times\dfrac{a+b}{a-b}$ 19. $1-\dfrac{x-\frac{1+x^2}{1-2x}}{x+2\frac{1+x^2}{1-2x}}$ 20. $\dfrac{\frac{a+x}{a-x}+\frac{a-x}{a+x}}{\frac{a+x}{a-x}-\frac{a-x}{a+x}}$

21. $\dfrac{\frac{x^2+y^2}{y}-x}{\frac{1}{y}-\frac{1}{x}}\div\dfrac{x^3+y^3}{x^2-y^2}$ 22. $\dfrac{\frac{1}{x}-\frac{1}{y+z}}{\frac{1}{y+z}+\frac{1}{x}}$ 23. $\dfrac{x-\frac{1}{y}}{1-\frac{(1-x)(1+y)}{y-x}}$.

24. $\dfrac{(1-x^2)(1-x^3)}{x(1+x)(1-x)^2}-\dfrac{x^3+\frac{1}{x}}{x^2+\frac{1}{x^2}-1}$ 25. $\dfrac{3}{x+1}-\dfrac{2x-1}{x^2+\frac{x}{2}-\frac{1}{2}}$

26. $\dfrac{a-\frac{ab}{a+b}}{a^2+\frac{a^2b^2}{a^2-b^2}}\times\dfrac{\frac{1}{a^2}-\frac{1}{b^2}}{\frac{1}{a}-\frac{1}{b}}$ 27. $\dfrac{\frac{1}{a}+\frac{1}{b+c}}{\frac{1}{a}-\frac{1}{b+c}}\div\left\{1+\dfrac{b^2+c^2-a^2}{2bc}\right\}$

Simplify

28 $2-\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{x}}}$ 29 $\cfrac{1}{a-b+\cfrac{1}{a-\cfrac{1}{b}}}$ 30 $\cfrac{1}{x-1+\cfrac{1}{1+\cfrac{x}{4-x}}}$

31 $2+\cfrac{3}{2-\cfrac{3}{2+\cfrac{3}{2-\cfrac{3+4x}{2x-1}}}}$

190 Fractions with symmetrical denominators If $a-b$, $b-c$, $c-a$, where a, b, c, follow one another in *cyclic order* [Art 146], enter in pairs the denominators of fractions, it is often seen that the numerators are so arranged that the algebraic sum of the new numerator assumes entirely, or in part, one of these forms

$$a^2(b-c)+b^2(c-a)+c^2(a-b),$$
$$bc(b-c)+ca(c-a)+ab(a-b),$$
$$a(b^2-c^2)+b(c^2-a^2)+c(a^2-b^2),$$

the first two of which are each equal to $-(b-c)(c-a)(a-b)$, and the third to $(b-c)(c-a)(a-b)$ [see Art 154] Hence it is generally advantageous, in simplifying these fractions, to transform the denominators so as to give $(b-c)(c-a)(a-b)$ for their L C M.

Ex 1. Simplify $\dfrac{a^2}{(a-b)(a-c)}+\dfrac{b^2}{(b-c)(b-a)}+\dfrac{c^2}{(c-a)(c-b)}$

Change the signs of the factors $a-c$, $b-a$, $c-b$, thus the given fraction

$$=\frac{a^2}{-(a-b)(c-a)}+\frac{b^2}{-(b-c)(a-b)}+\frac{c^2}{-(c-a)(b-c)}$$

The L C M of the denominators is $-(b-c)(c-a)(a-b)$,

$$\text{given fraction}=\frac{a^2(b-c)+b^2(c-a)+c^2(a-b)}{-(b-c)(c-a)(a-b)}$$

$$=\frac{-(b-c)(c-a)(a-b)}{-(b-c)(c-a)(a-b)}\text{ [Art 154]}=1$$

Ex 2 Simplify $\dfrac{y+z}{yz(x-y)(x-z)}+\dfrac{z+x}{zx(y-z)(y-x)}+\dfrac{x+y}{xy(z-x)(z-y)}$

As in Ex. 1, change the signs of the factors, thus the given

expression $=\dfrac{y+z}{-yz(x-y)(z-x)}+\dfrac{z+x}{-zx(y-z)(x-y)}+\dfrac{x+y}{-xy(z-x)(y-z)}$

$$=\frac{x(y+z)(y-z)+y(z+x)(z-x)+z(x+y)(x-y)}{-xyz(y-z)(z-x)(x-y)}$$

$$=\frac{x(y^2-z^2)+y(z^2-x^2)+z(x^2-y^2)}{-xyz(y-z)(z-x)(x-y)}$$

$$=\frac{(y-z)(z-x)(x-y)}{-xyz(y-z)(z-x)(x-y)}\text{ [Art. 154]}=-\frac{1}{xyz}$$

Ex 3 Simplify $\dfrac{a^2+bc+1}{(a-b)(a-c)}+\dfrac{b^2+ca+1}{(b-c)(b-a)}+\dfrac{c^2+ab+1}{(c-a)(c-b)}$

Given fraction

$$=\frac{a^2+bc+1}{-(a-b)(c-a)}+\frac{b^2+ca+1}{-(b-c)(a-b)}+\frac{c^2+ab+1}{-(c-a)(b-c)}$$

$$=\frac{(a^2+bc+1)(b-c)+(b^2+ca+1)(c-a)+(c^2+ab+1)(a-b)}{-(b-c)(c-a)(a-b)}$$

The numerator

$$=\{(a^2(b-c)+b^2(c-a)+c^2(a-b)\}+\{bc(b-c)+ca(c-a)+ab(a-b)\}$$
$$+\{(b-c)+(c-a)+(a-b)\}$$

$$=-(b-c)(c-a)(a-b)-(b-c)(c-a)(a-b)+0$$

$$=-2(b-c)(c-a)(a-b)$$

value required $=\dfrac{-2(b-c)(c-a)(a-b)}{-(b-c)(c-a)(a-b)}=2$

Examples CXVII

Simplify

1. $\dfrac{x}{(x-y)(x-z)}+\dfrac{y}{(y-z)(y-x)}+\dfrac{z}{(z-x)(z-y)}$

2. $\dfrac{b+c}{(a-b)(a-c)}+\dfrac{c+a}{(b-a)(b-c)}+\dfrac{a+b}{(c-a)(c-b)}$.

3. $\dfrac{a(b+c)}{(c-a)(a-b)}+\dfrac{b(c+a)}{(a-b)(b-c)}+\dfrac{c(a+b)}{(b-c)(c-a)}$.

4. $\dfrac{a+b-c}{(b-c)(c-a)}+\dfrac{b+c-a}{(c-a)(a-b)}+\dfrac{c+a-b}{(a-b)(b-c)}$

5. $\dfrac{a^2-bc}{(c-a)(a-b)}+\dfrac{b^2-ca}{(a-b)(b-c)}+\dfrac{c^2-ab}{(b-c)(c-a)}$

6. $\dfrac{1}{x(x-y)(x-z)}+\dfrac{1}{y(y-x)(y-z)}+\dfrac{1}{z(z-x)(z-y)}$.

Simplify

7 $\dfrac{x^2+k^2}{(x-y)(x-z)}+\dfrac{y^2+k^2}{(y-z)(y-x)}+\dfrac{z^2+k^2}{(z-x)(z-y)}$

8 $\dfrac{a^2+a+1}{(a-b)(a-c)}+\dfrac{b^2+b+1}{(b-c)(b-a)}+\dfrac{c^2+c+1}{(c-a)(c-b)}$

9. $\dfrac{bc(x-a)^2}{(a-b)(a-c)}+\dfrac{ca(x-b)^2}{(b-c)(b-a)}+\dfrac{ab(x-c)^2}{(c-a)(c-b)}$

10 $\dfrac{x+k}{(x-y)(x-z)yz}+\dfrac{y+k}{(y-z)(y-x)zx}+\dfrac{z+k}{(z-x)(z-y)xy}$

11 $\dfrac{x^2+x+1}{(x-y)(x-z)}yz+\dfrac{y^2+y+1}{(y-z)(y-x)}zx+\dfrac{z^2+z+1}{(z-x)(z-y)}xy$

12 $\dfrac{(x-a)(x-b)}{(x-y)(x-z)}+\dfrac{(y-a)(y-b)}{(y-x)(y-z)}+\dfrac{(z-a)(z-b)}{(z-x)(z-y)}$

13 $\dfrac{(1+ab)(1+bc)}{(a-b)(b-c)}+\dfrac{(1+bc)(1+ca)}{(b-c)(c-a)}+\dfrac{(1+ca)(1+ab)}{(c-a)(a-b)}$

14. $\dfrac{(x-b)(x-c)}{(a-b)(a-c)}+\dfrac{(x-c)(x-a)}{(b-c)(b-a)}+\dfrac{(x-a)(x-b)}{(c-a)(c-b)}$

15 $\dfrac{a^2+ah+h^2}{(a-b)(a-c)}+\dfrac{b^2+bh+h^2}{(b-c)(b-a)}+\dfrac{c^2+ch+h^2}{(c-a)(c-b)}$

16 $\dfrac{x^2+x+1}{(x-y)(x-z)}(y+z)+\dfrac{y^2+y+1}{(y-z)(y-x)}(z+x)+\dfrac{z^2+z+1}{(z-x)(z-y)}(x+y).$

17 $\dfrac{(a+b)(a+c)}{(a-b)(a-c)}+\dfrac{(b+c)(b+a)}{(b-c)(b-a)}+\dfrac{(c+a)(c+b)}{(c-a)(c-b)}$

18 $\dfrac{1}{\quad c)(x+a)}+\dfrac{1}{(b-c)(b-a)(x+b)}+\dfrac{1}{(c-a)(c-b)(x+c)}.$

19 $\dfrac{a}{(a-b)(a-c)(x-a)}+\dfrac{b}{(b-c)(b-a)(x-b)}+\dfrac{c}{(c-a)(c-b)(x-c)}$

20 $\dfrac{a^2}{(a-b)(a-c)(x+a)}+\dfrac{b^2}{(b-c)(b-a)(x+b)}+\dfrac{c^2}{(c-a)(c-b)(x+c)}$

21 Prove that $\dfrac{a^2+a+1}{(a-b)(a-c)(x-a)}+\dfrac{b^2+b+1}{(b-c)(b-a)(x-b)}$

$$+\frac{c^2+c+1}{(c-a)(c-b)(x-c)}=\frac{x^2+x+1}{(x-a)(x-b)(x-c)}$$

191 The following examples may with advantage be worked by the method of the last Article

Examples CXVIII.

Simplify

1 $\dfrac{x+a}{(a-b)(a-c)}+\dfrac{x+b}{(b-c)(b-a)}+\dfrac{x+c}{(c-a)(c-b)}$

2 $\dfrac{a-b}{(b+c)(c+a)}+\dfrac{b-c}{(c+a)(a+b)}+\dfrac{c-a}{(a+b)(b+c)}$

3 $\dfrac{x+a}{(x-b)(x-c)}+\dfrac{x+b}{(x-c)(x-a)}+\dfrac{x+c}{(x-a)(x-b)}$

4 $\dfrac{x^2(y-z)}{(x+y)(x+z)}+\dfrac{y^2(z-x)}{(y+z)(y+x)}+\dfrac{z^2(x-y)}{(z+x)(z+y)}$

5 $\dfrac{x^2-yz}{(x+y)(x+z)}+\dfrac{y^2-zx}{(y+z)(y+x)}+\dfrac{z^2-xy}{(z+x)(z+y)}$

6 $\dfrac{x^2-yz}{(x-y)(x-z)}+\dfrac{y^2+zx}{(y+z)(y-x)}+\dfrac{z^2+xy}{(z-x)(z+y)}$

7 $\dfrac{x+y}{(x^2-yz)(y^2-zx)}+\dfrac{y+z}{(y^2-zx)(z^2-xy)}+\dfrac{z+x}{(z^2-xy)(x^2-yz)}$

192 Miscellaneous Examples of Fractions The following examples will further illustrate the principles explained in this chapter

Ex 1 Find the value of $\dfrac{x+2a}{x-2a}+\dfrac{x+2b}{x-2b}$, when $x=\dfrac{4ab}{a+b}$

Given expression $=\dfrac{(x+2a)(x-2b)+(x-2a)(x+2b)}{(x-2a)(x-2b)}$

$$=\frac{x^2+2(a-b)x-4ab+x^2-2(a-b)x-4ab}{x^2-2(a+b)x+4ab}$$

$$=\frac{2(x^2-4ab)}{x^2-2(a+b)x+4ab}$$

$$=\frac{2\{x^2-(a+b)x\}}{x^2-2(a+b)x+(a+b)x}, \qquad (a+b)x=4ab,$$

$$=\frac{2\{x^2-(a+b)x\}}{x^2-(a+b)x}=2$$

Ex 2 Resolve into factors $1-\left\{\frac{a^2+b^2-c^2}{2ab}\right\}^2$ and shew that it is equal to $\frac{4s(s-a)(s-b)(s-c)}{a^2b^2}$, if $2s=a+b+c$

Given expression $=\left\{1+\frac{a^2+b^2-c^2}{2ab}\right\}\left\{1-\frac{a^2+b^2-c^2}{2ab}\right\}$

$=\frac{2ab+a^2+b^2-c^2}{2ab}\times\frac{2ab-a^2-b^2+c^2}{2ab}$

$=\frac{(a+b)^2-c^2}{2ab}\times\frac{c^2-(a-b)^2}{2ab}=\frac{(a+b+c)(a+b-c)}{2ab}\times\frac{(c+a-b)(c-a+b)}{2ab}$

$=\frac{(a+b+c)(b+c-a)(c+a-b)(a+b-c)}{4a^2b^2}$

Again $2s=a+b+c$, $2s-2a=a+b+c-2a$, or $2(s-a)=b+c-a$, similarly $2(s-b)=c+a-b$ and $2(s-c)=a+b-c$,

given expn $=\frac{2s\,2(s-a)\,2(s-b)\,2(s-c)}{4a^2b^2}=\frac{4s(s-a)(s-b)(s-c)}{a^2b^2}$

Ex 3 Reduce to its simplest form, the expression

$$\frac{(a-b)^2}{(b-c)(c-a)}+\frac{(b-c)^2}{(c-a)(a-b)}+\frac{(c-a)^2}{(a-b)(b-c)}$$

The L C M of denominators $=(b-c)(c-a)(a-b)$, thus reduced numerator $=(a-b)^3+(b-c)^3+(c-a)^3=3(b-c)(c-a)(a-b)$ [Art 155, Ex 6]

value required $=3$

Ex 4 Reduce to its simplest form

$$\frac{1}{a-b}+\frac{1}{b-c}+\frac{1}{c-a}+\frac{(a-b)^2+(b-c)^2+(c-a)^2}{2(a-b)(b-c)(c-a)}$$

The L C D $=2(a-b)(b-c)(c-a)$, thus reduced numerator $=2(b-c)(c-a)+2(c-a)(a-b)+2(a-b)(b-c)+(a-b)^2+(b-c)^2+(c-a)^2=\{(a-b)+(b-c)+(c-a)\}^2$ [*For* XVI, Art 148] $=0$,

Examples CXIX.

1 Find the value of $\frac{3\frac{1}{4}-\frac{1}{3}(x-2)}{1\frac{1}{12}+(x-\frac{1}{2})}$ when $x=3\frac{1}{4}$

2 Multiply $a+b+\frac{b^2}{a}+\frac{a^2}{b}$ by $a-b+\frac{b^2}{a}-\frac{a^2}{b}$

3 Reduce $\dfrac{x^2+\left(\dfrac{a}{b}+\dfrac{b}{a}\right)xy+y^2}{x^2+\left(\dfrac{a}{b}-\dfrac{b}{a}\right)xy-y^2}$ to its lowest terms

4 Multiply $\dfrac{a}{a+x}+\dfrac{3a}{a-x}+\dfrac{2ax}{x^2-a^2}$ by $\dfrac{a+x}{2(a-x)}$.

5 Resolve into factors $\dfrac{a^2+b^2-c^2}{2ab}-1$

6 Find the value of $\dfrac{x-a}{b}+\dfrac{x-b}{a}$, when $x=\dfrac{a^2}{a+b}$

7 Divide $\dfrac{1}{a+b}+\dfrac{a-b}{a^2-ab+b^2}+\dfrac{ab-a^2}{a^3+b^3}$ by $\dfrac{a}{a^2-ab+b^2}$

8 Reduce $\dfrac{(a+b)\{(a+b)^2-c^2\}}{4b^2c^2-(a^2-b^2-c^2)^2}$ to its lowest terms.

9 Simplify $\dfrac{a^2-(b-c)^2}{(a+c)^2-b^2}+\dfrac{b^2-(c-a)^2}{(a+b)^2-c^2}+\dfrac{c^2-(a-b)^2}{(b+c)^2-a^2}$

10 Reduce $\dfrac{\left(\dfrac{a^2}{a^2+b^2}+\dfrac{b^2}{a^2-b^2}\right)(a^2+b^2)^2}{\dfrac{a}{a+b}+\dfrac{b}{a-b}}$ to its simplest form

11 Simplify $\dfrac{1}{\left(1-\dfrac{c}{a}\right)\left(1-\dfrac{b}{a}\right)}+\dfrac{1}{\left(1-\dfrac{a}{b}\right)\left(1-\dfrac{c}{b}\right)}+\dfrac{1}{\left(1-\dfrac{b}{c}\right)\left(1-\dfrac{a}{c}\right)}$

12 Reduce $1-\dfrac{a^2-(b+c)^2}{(a+b+c)^2}$ to its simplest form.

13 Simplify $\dfrac{(a+b)(1-ab)}{(1-ab)^2-(a+b)^2}-\dfrac{a(1-b^2)+b(1-a^2)}{(1-a^2)(1-b^2)-4ab}$

14 Reduce $\dfrac{x+y+\dfrac{y^2}{x}}{x-y+\dfrac{y^2}{x}}\times\dfrac{x+\dfrac{y^3}{x^2}}{x-\dfrac{y^3}{x^2}}-\left(\dfrac{x+y}{x-y}\right)^2$

15 Divide $a^2-b^2-c^2-2bc$ by $\dfrac{a-b-c}{a+b-c}$

16 Find the value of $\dfrac{(x+y)^2(x^4-y^4)\{(x^2+y^2)^2-x^2y^2\}}{(x^6-y^6)\{(x^2+y^2)^2+2xy(x^2+y^2)\}}$

17 Reduce $\dfrac{1-a^4}{(1+ax)^2-(a+x)^2}$, and find its value when $x=\dfrac{m-n}{m+n}$.

18 Multiply $\dfrac{x^2-7xy+12y^2}{x^2+5xy+6y^2}$ by $\dfrac{x^2+xy-2y^2}{x^2-5xy+4y^2}$

19 Resolve into factors $1-\left\{\dfrac{a^2+b^2-c^2-d^2}{2(ab+cd)}\right\}^2$

20 Simplify $\dfrac{1}{x}\left(x+\dfrac{1}{1-\dfrac{1}{x}}\right)-\dfrac{1}{x+1}-\dfrac{x}{x^2-1}$

21 Reduce $\dfrac{6x^3-19x^2y+18xy^2-5y^3}{2x^2-3xy+y^2}$

22 Find the value of $\dfrac{x^2-y^2+x}{y^2-x^2+y}$, when $x=\dfrac{a-b}{a+b}$ and $y=\dfrac{a+b}{a-b}$

23 Simplify $\dfrac{\left(\dfrac{x}{a}+1\right)^2}{\dfrac{x}{a}-\dfrac{a}{x}}\times\dfrac{\dfrac{x}{a}+\dfrac{a}{x}-1}{\dfrac{x^3}{a^3}+1}\div\dfrac{\dfrac{x^2}{a^2}+\dfrac{x}{a}+1}{\dfrac{x^3}{a^3}-1}$

24 Divide $a^6+\dfrac{1}{a^6}+a^4+\dfrac{1}{a^4}+a^2+\dfrac{1}{a^2}+2$ by $a^3+\dfrac{1}{a^3}+a+\dfrac{1}{a}$

25 Find the value of $\dfrac{b-c}{b+c-2a}+\dfrac{c-a}{c+a-2b}+\dfrac{a-b}{a+b-2c}$

26 Simplify $\dfrac{ab^2-(ab-b^2)x+(a-b)x^2+x^3}{ab^2+(ab+b^2)x+(a+b)x^2+x^3}$

27 Find the value of

$$\frac{b+c}{bc}(b^2+c^2-a^2)+\frac{c+a}{ca}(c^2+a^2-b^2)+\frac{a+b}{ab}(a^2+b^2-c^2)$$

28 Reduce $\dfrac{\dfrac{x-y}{1+xy}+\dfrac{y-z}{1+yz}}{1-\dfrac{(x-y)(y-z)}{(1+xy)(1+yz)}}$ to a simple form

29 Simplify $\dfrac{a^3}{(a-b)(a-c)}+\dfrac{b^3}{(b-c)(b-a)}+\dfrac{c^3}{(c-a)(c-b)}$

30 Reduce $\dfrac{3x^2-(4a+2b)x+2ab+a^2}{x^3-(2a+b)x^2+(2ab+a^2)x-a^2b}$

31 Simplify $\dfrac{x^2+1}{x^2-1}-1\div\left(\dfrac{2x}{2x-1}-\dfrac{2}{2-x}\right)$

32 Simplify $\dfrac{(1-ab)^2+(a+b-2)(a+b-2ab)}{(1+ab)^2-(a+b)^2}$

33 Find the value of the expression

$$\frac{1-x}{1+x}+\frac{(1-x)(1-x^2)}{(1+x)(1+x^2)}+\frac{(1-x)(1-x^2)(1-x^3)}{(1+x)(1+x^2)(1+x^3)}$$

34 Simplify $\dfrac{\dfrac{a-bx}{a-bx}+\dfrac{b+ax}{b-ax}}{\dfrac{a+bx}{a-bx}-\dfrac{b+ax}{b-ax}}-1-\dfrac{2\left(\dfrac{1}{a}+\dfrac{1}{b}\right)}{\dfrac{x}{a-b}}$

35 Simplify $\dfrac{3xyz}{yz+zx+xy}-\dfrac{\dfrac{x-1}{x}+\dfrac{y-1}{y}+\dfrac{z-1}{z}}{\dfrac{1}{x}+\dfrac{1}{y}+\dfrac{1}{z}}$.

36 Reduce $\dfrac{x}{x-y}-\dfrac{x}{x+y}-\dfrac{\dfrac{x+y}{x-y}-\dfrac{x-y}{x+y}}{\dfrac{x+y}{x-y}+\dfrac{x-y}{x+y}}$

37 If $x^2=\dfrac{ab(c^2+d^2)-cd(a^2+b^2)}{ab-cd}$, decompose

$1-\left\{\dfrac{a^2+b^2-x^2}{2ab}\right\}^2$ into simple factors.

193. Harder Identities The examples below will be found very useful.

Ex. 1. Prove that $a(a+d)(a+2d)(a+3d)+d^4=(a^2+3ad+d^2)^2$

The left side $=a(a+3d)\times(a+d)(a+2d)+d^4$

$$=(a^2+3ad)(a^2+3ad+2d^2)+d^4$$
$$=(a^2+3ad)^2+2d^2(a^2+3ad)+d^4$$
$$=(a^2+3ad+d^2)^2$$

Note If a, b, c, d be 4 consecutive numbers, then it follows from this example that $abcd+1=(ad+1)^2$, that is, *the product of 4 consecutive numbers increased by 1, is a square number.*

Ex 2 If $2s=a+b+c$, prove that

$$s^2+(s-a)^2+(s-b)^2+(s-c)^2=a^2+b^2+c^2$$

The left side $=s^2+(s^2-2sa+a^2)+(s^2-2sb+b^2)+(s^2-2sc+c^2)$

$$=4s^2-2s(a+b+c)+a^2+b^2+c^2$$
$$=4s^2-2s\,2s+a^2+b^2+c^2, \qquad 2s=a+b+c,$$
$$=4s^2-4s^2+a^2+b^2+c^2=a^2+b^2+c^2$$

Note When we see an expression involving a symbol *that stands for an expression*, as s here, it is *generally* advisable to effect reduction in terms of it, and when the work is thus simplified, to substitute its value

Ex 3 If $(a+b-c-d)x=cd-ab$, shew that

$$(a+x)(b+x)=(c+x)(d+x)$$

We have $(a+b)x-(c+d)x=cd-ab$,

$(a+b)x+ab=(c+d)x+cd$, by transp ,

add x^2 to both sides , thus

$$x^2+(a+b)x+ab=x^2+(c+d)x+cd\,,$$
$$(x+a)(x+b)=(x+c)(x+d)$$

Ex 4 Prove the identity $(ax+by+cz)^2+(ay-bx)^2+(bz-cy)^2$
$+(cx-az)^2=(a^2+b^2+c^2)(x^2+y^2+z^2)$

Expn $=(ax+by)^2+2cz(ax+by)+c^2z^2+(ay-bx)^2+(bz-cy)^2+(cx-az)^2$

$$=(ax+by)^2+(ay-bx)^2+2acxz+2bcyz+c^2z^2+b^2z^2+c^2y^2-2bcyz$$
$$+c^2x^2+a^2z^2-2acxz$$
$$=(a^2+b^2)(x^2+y^2)+c^2(x^2+y^2)+(a^2+b^2+c^2)z^2$$
$$=(a^2+b^2+c^2)(x^2+y^2)+(a^2+b^2+c^2)z^2$$
$$=(a^2+b^2+c^2)(x^2+y^2+z^2)$$

Ex 5 Prove the identity $(x^2-yz)^3+(y^2-zx)^3+(z^2-xy)^3$
$-3(x^2-yz)(y^2-zx)(z^2-xy)=(x^3+y^3+z^3-3xyz)^2$.

Left side

$$=\tfrac{1}{2}(x^2-yz+y^2-zx+z^2-xy)\times$$
$$\{[y^2-z^2+x(y-z)]^2+[z^2-x^2+y(z-x)]^2+[x^2-y^2+z(x-y)]^2\}$$
$$=\tfrac{1}{2}(x^2+y^2+z^2-yz-zx-xy)\times$$
$$\{(x+y+z)^2(y-z)^2+(x+y+z)^2(z-x)^2+(x+y+z)^2(x-y)^2\}$$
$$=\tfrac{1}{2}(x^2+y^2+z^2-yz-zx-xy)\times$$
$$\{(x+y+z)^2\times 2(x^2+y^2+z^2-yz-zx-xy)\}$$
$$=(x+y+z)^2(x^2+y^2+z^2-yz-zx-xy)^2=(x^3+y^3+z^3-3xyz)^2$$

Ex 6 If $2s=a+b+c$, prove that

$$a(s-b)(s-c)+b(s-c)(s-a)+c(s-a)(s-b)$$
$$=a(s-a)^2+b(s-b)^2+c(s-c)^2$$

From the given relation, we have

$$a=(s-b)+(s-c),$$
$$b=(s-c)+(s-a),$$
$$c=(s-a)+(s-b),$$
$$a(s-b)(s-c)=(s-b)^2(s-c)+(s-b)(s-c)^2,$$
$$b(s-c)(s-a)=(s-c)^2(s-a)+(s-c)(s-a)^2,$$
$$c(s-a)(s-b)=(s-a)^2(s-b)+(s-a)(s-b)^2,$$

whence by addition

$$a(s-b)(s-c)+b(s-c)(s-a)+c(s-a)(s-b)$$
$$=(s-a)^2(2s-b-c)+(s-b)^2(2s-c-a)+(s-c)^2(2s-a-b)$$
$$=a(s-a)^2+b(s-b)^2+c(s-c)^2$$

Examples CXX

Prove the identities

1 $(ax+by-cz)^2-2\{b^2y^2-(cz-ax)^2\}+(by+cz-ax)^2=4(ax-cz)^2$

2 $x^3+y^3+z^3-3xyz=(x+y+z)^3-3(x+y+z)(yz+zx+xy)$

3 $(ax+by)^2+(ay-bx)^2=(a^2+b^2)(x^2+y^2)$

4 $(ax-by)(ax-by+1)^2-ax+by=(ax-by)^2(ax-by+2)$

5 $(x-ay)^3-(y-ax)^3=(a+1)^3(x-y)^3+3(a+1)(x-y)(x-ay)(y-ax)$

6 $n(n-1)(n-2)-p(p-1)(p-2)=(n-p)\{(n+p-1)(n+p-2)-np\}$

7 $(x^2-ay^2)^3=(x^3+3axy^2)^2-a(3x^2y+ay^3)^2$

8 $(a+b)^4=2(a^2+b^2)(a+b)^2-(a^2-b^2)^2$

9 $(x-a)^2+(y-b)^2+(a^2+b^2-1)(x^2+y^2-1)$
$$=(ax+by-1)^2+(ay-bx)^2$$

10 $(x-y)(x+1)(y+1)-x(y+1)^2+y(x+1)^2=(x-y)(x+y+2xy)$

11 $(a+b)^2(x-y)^2+4xy(a^2+b^2)-4ab(x^2+y^2)=(a-b)^2(x+y)^2$

12 $\{(ac-bd)x+(ad+bc)y\}^2+\{(ac-bd)y-(ad+bc)x\}^2$
$$=(a^2+b^2)(c^2+d^2)(x^2+y^2)$$

13 $a(b+c)(b^2+c^2-a^2)+b(c+a)(c^2+a^2-b^2)+c(a+b)(a^2+b^2-c^2)$
$=2abc(a+b+c)$

14 If $a+b+c=0$, shew that $a^2-bc=b^2-ca=c^2-ab$

15 If $x=b+c$, $y=c+a$, $z=a+b$, prove that $x^2+y^2+z^2-yz-zx-xy$
$=a^2+b^2+c^2-bc-ca-ab$

16 If $bz-cy=\alpha$, $cx-az=\beta$, $ay-bx=\gamma$,
prove that $$a\alpha+b\beta+c\gamma=0$$

17 If $x^2-yz=a$, $y^2-zx=b$, $z^2-xy=c$,
shew that $$(a+b+c)(x+y+z)=ax+by+cz$$

18 If $x=a-b$, $y=b-c$, $z=c-a$, prove that
$$2(ax+by+cz)=x^2+y^2+z^2$$

19 If $a^2+b^2=1=c^2+d^2$, shew that $(ac+bd)(ac-bd)=(a+d)(a-d)$

20 If $ax+by=1$, then will
$$ab(x^2+y^2)+(a^2+b^2)xy+(a-b)(x-y)=1.$$

21 Prove that $x(x-1)(x-2)(x-3)+1=(x^2-3x+1)^2$

22 If $2s=a+b+c$, shew that $s^2+(s-a)(s-b)+(s-b)(s-c)$
$+(s-c)(s-a)=bc+ca+ab$

23 If $2s=a+b+c$, shew that
$$\{(s-a)+(s-b)\}^3=(s-a)^3+(s-b)^3+3(s-a)(s-b)c$$

24 If $2s=a+b+c$, shew that $2(s-a)(s-b)(s-c)+a(s-b)(s-c)$
$+b(s-c)(s-a)+c(s-a)(s-b)=abc$

25 If $s^2=bc+ca+ab$, shew that
$$(s-a)(s-b)(s-c)+a(s-b)(s-c)+b(s-c)(s-a)+c(s-a)(s-b)=2abc$$

26 Prove that
$$\{(y-z)^2+(z-x)^2+(x-y)^2\}^2=2\{(y-z)^4+(z-x)^4+(x-y)^4\}$$

27 If $A=x^2+y^2+z^2$ and $B=yz+zx+xy$, shew that
$$A^3-3AB^2+2B^3=(x^3+y^3+z^3-3xyz)^2$$

28 Shew that $(x+y+z)(y+z-x)+(x+y-z)(x-y+z)$
$+(y+z-x)(x+y-z)+(x+y+z)(x-y+z)=4(x+y)z$

29 Prove that $a(b+c-a)^2+b(c+a-b)^2+c(a+b-c)^2$
$+(b+c-a)(c+a-b)(a+b-c)=4abc$

30 If $A=a^2-bc$, $B=b^2-ca$, $C=c^2-ab$, prove that
$$\frac{A^2-BC}{a}=\frac{B^2-CA}{b}=\frac{C^2-AB}{c}=(A+B+C)(a+b+c)$$

31. Prove that
$$a^3(b^2-ca)^3+b^3(c^2-ab)^3+c^3(a^2-bc)^3=3abc(b^2-ca)(c^2-ab)(a^2-bc)$$

32 Shew that $a^3+b^3+c^3+6abc-a(a-b)(a-c)-b(b-c)(b-a)$
$$-c(c-a)(c-b)=(a+b+c)(bc+ca+ab)$$

33 Shew that $(a^2-bc)(b^3-c^3)+(b^2-ca)(c^3-a^3)+(c^2-ab)(a^3-b^3)$
$$=(b-c)(c-a)(a-b)(a+b+c)^2$$

34 Shew that $(a+1)^3(b-c)^3+(b+1)^3(c-a)^3+(c+1)^3(a-b)^3$
$$=3(a+1)(b+1)(c+1)(b-c)(c-a)(a-b)$$

35 Prove that
$$x^8+y^8+(x+y)^8=2(x^2+xy+y^2)^4+8x^2y^2(x+y)^2(x^2+xy+y^2)$$

36 If $a^2=y+z$, $b^2=z+x$, $c^2=x+y$ and $2s=a+b+c$, shew that
$$s(s-a)(s-b)(s-c)=\tfrac{1}{4}(yz+zx+xy)$$

37 If $(by-cx)^2=(b^2-ac)(y^2-cz)$, prove that
$$(bx-ay)^2=(b^2-ac)(x^2-az) \quad \text{[See } App \text{]}$$

Ex 7 If $y=x+\frac{1}{x}$, shew that $x^3+\frac{1}{x^3}=y^3-3y$

We have $x^3+\frac{1}{x^3}=\left(x+\frac{1}{x}\right)^3-3\,x\,\frac{1}{x}\left(x+\frac{1}{x}\right)$ [Art 144] $=y^3-3y$

Ex 8 Shew that $\frac{1}{(a-b)^2}+\frac{1}{(b-c)^2}+\frac{1}{(c-a)^2}=\left(\frac{1}{a-b}+\frac{1}{b-c}+\frac{1}{c-a}\right)^2$.

Right side $=\frac{1}{(a-b)^2}+\frac{1}{(b-c)^2}+\frac{1}{(c-a)^2}$
$$+2\left\{\frac{1}{(a-b)(b-c)}+\frac{1}{(b-c)(c-a)}+\frac{1}{(c-a)(a-b)}\right\}$$

$=\frac{1}{(a-b)^2}+\frac{1}{(b-c)^2}+\frac{1}{(c-a)^2}$, the expression within the braces $=0$ [see Art 184, Ex 5]

Ex 9 Prove the identity

$$\left(\frac{b}{c}+\frac{c}{b}\right)^2+\left(\frac{c}{a}+\frac{a}{c}\right)^2+\left(\frac{a}{b}+\frac{b}{a}\right)^2=4+\left(\frac{b}{c}+\frac{c}{b}\right)\left(\frac{c}{a}+\frac{a}{c}\right)\left(\frac{a}{b}+\frac{b}{a}\right)$$

$$\text{Left side}=\left(\frac{b}{c}+\frac{c}{b}\right)^2+\frac{c^2}{a^2}+2+\frac{a^2}{c^2}+\frac{a^2}{b^2}+2+\frac{b^2}{a^2}$$

$$=4+\left(\frac{b}{c}+\frac{c}{b}\right)^2+\frac{1}{a^2}(b^2+c^2)+a^2\left(\frac{1}{c^2}+\frac{1}{b^2}\right)$$

$$=4+\left(\frac{b}{c}+\frac{c}{b}\right)^2+\frac{bc}{a^2}\left(\frac{b^2+c^2}{bc}\right)+\frac{a^2}{bc}\left(\frac{bc}{c^2}+\frac{bc}{b^2}\right)$$

$$=4+\left(\frac{b}{c}+\frac{c}{b}\right)^2+\frac{bc}{a^2}\left(\frac{b}{c}+\frac{c}{b}\right)+\frac{a^2}{bc}\left(\frac{b}{c}+\frac{c}{b}\right)$$

$$=4+\left(\frac{b}{c}+\frac{c}{b}\right)\left\{\frac{b}{c}+\frac{c}{b}+\frac{bc}{a^2}+\frac{a^2}{bc}\right\}$$

$$=4+\left(\frac{b}{c}+\frac{c}{b}\right)\left\{\left(\frac{ac}{ab}+\frac{a^2}{bc}\right)+\left(\frac{bc}{a^2}+\frac{ab}{ac}\right)\right\}$$

$$=4+\left(\frac{b}{c}+\frac{c}{b}\right)\left\{\frac{a}{b}\left(\frac{c}{a}+\frac{a}{c}\right)+\frac{b}{a}\left(\frac{c}{a}+\frac{a}{c}\right)\right\}$$

$$=4+\left(\frac{b}{c}+\frac{c}{b}\right)\left(\frac{c}{a}+\frac{a}{c}\right)\left(\frac{a}{b}+\frac{b}{a}\right).$$

Ex 10 If $x=\dfrac{ab-cd}{(a-b)-(c-d)}$, prove that $(x+a)(x-b)=(x+c)(x-d)$,

and for this value of x, shew that $\dfrac{x+a}{x-b}=\dfrac{(a-c)(a+d)}{(b+c)(b-d)}$

We have $\quad\{(a-b)-(c-d)\}x=ab-cd,$

whence as in Ex 3, we get

$$(x+a)(x-b)=(x+c)(x-d)$$

$$\text{Again } \frac{x+a}{x-b}=\frac{\dfrac{ab-cd}{(a-b)-(c-d)}+a}{\dfrac{ab-cd}{(a-b)-(c-d)}-b}=\frac{a^2-ac+ad-cd}{b^2+bc-bd-cd}=\frac{(a-c)(a+d)}{(b+c)(b-d)}$$

Ex 11 Prove that

$$\frac{a^2}{(a-b)(a-c)(b+c)}+\frac{b^2}{(b-c)(b-a)(c+a)}+\frac{c^2}{(c-a)(c-b)(a+b)}$$
$$=\frac{(a+b+c)^2}{(b+c)(c+a)(a+b)}$$

Change the sign of each term, thus

L.C.M. $=-(b^2-c^2)(c^2-a^2)(a^2-b^2)$, and therefore reduced numerator

$$=a^2(b-c)(c+a)(a+b)+b^2(c-a)(a+b)(b+c)+c^2(a-b)(b+c)(c+a)$$

$$=a^2(b-c)\{a^2+a(b+c)+bc\}+b^2(c-a)\{b^2+b(c+a)+ca\}$$
$$+c^2(a-b)\{c^2+c(a+b)+ab\}$$

$$=a^2(b-c)\{a^2+a(b+c)\}+a^2bc(b-c)+b^2(c-a)\{b^2+b(c+a)\}$$
$$+ab^2c(c-a)+c^2(a-b)\{c^2+c(a+b)\}+abc^2(a-b)$$

$$=a^3(b-c)(a+b+c)+b^3(c-a)(a+b+c)+c^3(a-b)(a+b+c)$$
$$+abc\{a(b-c)+b(c-a)+c(a-b)\}$$

$$=(a+b+c)\{a^3(b-c)+b^3(c-a)+c^3(a-b)\}+abc\times 0 \text{ [Art. 149]}$$

$$=(a+b+c)\times -(a+b+c)(b-c)(c-a)(a-b) \text{ [Art. 154, Ex. 2.]}$$

$$=-(a+b+c)^2(b-c)(c-a)(a-b),$$

$$\text{left side}=\frac{-(a+b+c)^2(b-c)(c-a)(a-b)}{-(b^2-c^2)(c^2-a^2)(a^2-b^2)}=\frac{(a+b+c)^2}{(b+c)(c+a)(a+b)}.$$

Ex. 12. Prove that

$$\frac{x}{x^2-1}+\frac{x^2}{x^4-1}+\frac{x^4}{x^8-1}+\frac{x^8}{x^{16}-1}=\frac{1}{2}\left(\frac{x+1}{x-1}-\frac{x^{16}+1}{x^{16}-1}\right).$$

We have $\dfrac{x}{x^2-1}=\dfrac{1}{2}\times\dfrac{2x}{x^2-1}=\dfrac{1}{2}\times\dfrac{(x+1)^2-(x^2+1)}{x^2-1}$

$$=\frac{1}{2}\left\{\frac{(x+1)^2}{x^2-1}-\frac{x^2+1}{x^2-1}\right\}$$

$$=\frac{1}{2}\left(\frac{x+1}{x-1}-\frac{x^2+1}{x^2-1}\right),$$

similarly
$$\frac{x^2}{x^4-1}=\frac{1}{2}\left(\frac{x^2+1}{x^2-1}-\frac{x^4+1}{x^4-1}\right),$$

$$\frac{x^4}{x^8-1}=\frac{1}{2}\left(\frac{x^4+1}{x^4-1}-\frac{x^8+1}{x^8-1}\right),$$

$$\frac{x^8}{x^{16}-1}=\frac{1}{2}\left(\frac{x^8+1}{x^8-1}-\frac{x^{16}+1}{x^{16}-1}\right),$$

therefore by addition, the left side of the proposed expression

$$=\frac{1}{2}\left(\frac{x+1}{x-1}-\frac{x^{16}+1}{x^{16}-1}\right).$$

Examples CXX (*Continued*)

38 If $x=b+c$, $y=c+a$, $z=a+b$, shew that

$$\frac{x^3+y^3+z^3-3xyz}{a^3+b^3+c^3-3abc}=2$$

39 Prove that $\frac{a+b}{ab}\left(\frac{1}{a}-\frac{1}{b}\right)+\frac{b+c}{bc}\left(\frac{1}{b}-\frac{1}{c}\right)=\frac{a+c}{ac}\left(\frac{1}{a}-\frac{1}{c}\right)$

40 Having given $\frac{yz}{x^2}=y+z$, $\frac{zx}{y^2}=z+x$, $\frac{xy}{z^2}=x+y$,

shew that $$yz+zx+xy+2xyz=1$$

41 If $a+b+c=0$, prove that

$$3+\frac{b^2+c^2-a^2}{2bc}+\frac{c^2+a^2-b^2}{2ca}+\frac{a^2+b^2-c^2}{2ab}=0$$

42 Shew that

$$\left\{\frac{2bc}{b+c}-b\right\}-\left\{\frac{1}{c}+\frac{1}{b-2c}\right\}+\left\{\frac{2bc}{b+c}-c\right\}-\left\{\frac{1}{b}+\frac{1}{c-2b}\right\}=bc$$

43 If $\frac{x^2}{y^2}=\frac{2a^2}{b^2}+1$, prove that $\frac{2y^2}{x^2+y^2}+\frac{a^2}{a^2+b^2}=1$

44 If $y+\frac{1}{z}=1$, $z+\frac{1}{x}=1$, then $x+\frac{1}{y}=1$

45 If $x+y+z=0$, prove that

$$\frac{1}{y^2+z^2-x^2}+\frac{1}{z^2+x^2-y^2}+\frac{1}{x^2+y^2-z^2}=0$$

46 If $x=a+b+\frac{(a-b)^2}{4(a+b)}$, $y=\frac{a+b}{4}+\frac{ab}{a+b}$, shew that

$$(x-a)^2-(y-b)^2=b^2$$

47 If $a^2\frac{a'-b}{a-b'}=a'^2\frac{a-b}{a'-b'}$, then $\frac{1}{a}+\frac{1}{a'}=\frac{1}{b}+\frac{1}{b'}$

48 If $x=\frac{b-c}{a}$, $y=\frac{c-a}{b}$, $z=\frac{a-b}{c}$ shew that

$$xyz+x+y+z=0$$

49 If $x=\frac{a+b-c}{a+b+c}$, then $\frac{a+bx^2}{b+ax^2}=\frac{(a-b+c)^2+4ab}{(b+c-a)^2+4ab}$

50 Shew that $\dfrac{a}{b-c}+\dfrac{b}{c-a}+\dfrac{c}{a-b}+\dfrac{abc}{(b-c)(c-a)(a-b)}$

$$=\left\{1+\frac{a}{b-c}\right\}\left\{1+\frac{b}{c-a}\right\}\left\{1+\frac{c}{a-b}\right\}$$

51 If $(x+a)(x+b)=(a+b)^2$, shew that $\dfrac{ax-b^2}{x-a}=\dfrac{bx-a^2}{x-b}$

52 Prove that

$$\frac{(a^2-b^2)^3+(b^2-c^2)^3+(c^2-a^2)^3}{(a-b)^3+(b-c)^3+(c-a)^3}=(a+b)(b+c)(c+a)$$

53 If $x+y+z=0$, and $\dfrac{a}{x}+\dfrac{b}{y}+\dfrac{c}{z}=0$, shew that

$$bx^2+(a+b-c)xy+ay^2=0$$

54 Prove that $\dfrac{(a+b)^3-c^3}{a+b-c}+\dfrac{(b+c)^3-a^3}{b+c-a}+\dfrac{(c+a)^3-b^3}{c+a-b}$

$$=2(a+b+c)^2+a^2+b^2+c^2$$

55 If $x+y+z=0$, shew that $\dfrac{x(y^3-z^3)}{y-z}+\dfrac{y(z^3-x^3)}{z-x}+\dfrac{z(x^3-y^3)}{x-y}=0$

56 Prove that

$$\frac{1}{x(x-a)(x-b)}=\frac{1}{abx}+\frac{1}{a(a-b)(x-a)}+\frac{1}{b(b-a)(x-b)}$$

57 Shew that $\dfrac{(2x-y-z)^2+(2y-z-x)^2+(2z-x-y)^2}{(y-z)^2+(z-x)^2+(x-y)^2}=3$

58 If $x=\dfrac{2ab+b^2}{a^2+ab+b^2}$, $y=\dfrac{a^2-b^2}{a^2+ab+b^2}$, then $x^3+y=y^3+x$

59 If $2s=a+b+c$, shew that

$$\frac{1}{s-a}+\frac{1}{s-b}+\frac{1}{s-c}-\frac{1}{s}=\frac{abc}{s(s-a)(s-b)(s-c)}$$

60 Prove that

$$\frac{la^2+ma+n}{(a-b)(a-c)(x+a)}+\frac{lb^2+mb+n}{(b-c)(b-a)(x+b)}+\frac{lc^2+mc+n}{(c-a)(c-b)(x+c)}$$
$$=\frac{lx^2-mx+n}{(x+a)(x+b)(x+c)}$$

61 If $ax=y+z+u$, $by=z+u+x$, $cz=u+x+y$, $du=x+y+z$,

shew that $\dfrac{1}{a+1}+\dfrac{1}{b+1}+\dfrac{1}{c+1}+\dfrac{1}{d+1}=1$

CHAPTER XVIII

INVOLUTION AND EVOLUTION

Involution

194 *Definition* **Involution** is the process of finding any power of a given quantity Hence this operation is nothing but repeated multiplication, where the multiplicand and the multiplier are the *same* quantity

195 Powers of Quantities We have

$$(x^2)^3 = x^2 \times x^2 \times x^2 = x^{2+2+2} = x^6,$$

$$(x^3)^2 = x^3 \times x^3 = x^{3+3} = x^6,$$

$$(a^m)^n = a^{m+m+\cdots \text{ to } n \text{ terms}} = a^{mn},$$

$$\begin{aligned}(ab)^n &= ab \times ab \times ab \times \cdots \text{ to } n \text{ factors} \\ &= (a \times a \times a \times \cdots \text{ to } n \text{ factors}) \\ &\quad \times (b \times b \times b \times \cdots \text{ to } n \text{ factors}) \\ &= a^n \times b^n,\end{aligned}$$

$$(3x^3)^5 = (3)^5(x^3)^5 = 243x^{15},$$

and so on Hence the

Rule —*Any power of a power of a given quantity is obtained by* **multiplying** *together the indices of the two powers*

196 Signs of Powers It is plain that if the quantity involved has a *positive* sign, the sign of any power will be *positive* But if it has a *negative* sign, the sign of the *even* powers will be *positive* and that of the *odd* powers will be *negative* For $(-a)(-a) = a^2$, $(-a)(-a)(-a) = -a^3$, $(-a)(-a)(-a)(-a) = a^4$, and generally $(-a)(-a)(-a) \cdots$ to m factors $= (-a)^m = \pm a^m$, according as m is *even* or *odd*

The *general* expression for $(-a)^m$ however is $(-1)^m a^m$, for $(-a)^m = (-1 \times a)^m = (-1)^m a^m$

REMARK Observe that $2m$ is *always even* and $2m+1$ is *always odd*, whatever number m may represent Hence

$$(-1)^{2m} = 1 \text{ and } (-1)^{2m+1} = -1$$

Hence we have the following corollaries

Cor 1 The square, or any *even power*, of a quantity, whether *positive* or *negative*, is always *positive*

Cor 2 Any *odd power* of a quantity has the *same sign* as the quantity itself

197 Involution of Monomials The examples here given depend on the two preceding articles

Ex 1 $(-3a^2)^3 = -27a^6$, the sign is $-$, for the index is *odd*

Ex 2 $(-2a^2x)^4 = 16a^8x^4$, the sign is $+$, the index being *even*

Ex 3 $\left(\dfrac{a}{b}\right)^2 = \dfrac{a}{b} \times \dfrac{a}{b} = \dfrac{aa}{bb} = \dfrac{a^2}{b^2}$ Ex 4 $\left(-\dfrac{2a^3b^2}{x^5yz^4}\right)^9 = -\dfrac{512a^{27}b^{18}}{x^{45}y^9z^{36}}$

REMARK The last two examples shew that when the quantity involved is a *fraction*, both of its terms are raised to the proposed power

Examples CXXI

Find the value of

1 $(-a)^0$, $(a^3)^7$, $(x^3y^3)^8$, $(-3a^3x^2)^5$, $(-a^2x^4y^5)^0$

2 $(-3a^2xy^3)^8$ $(-4x^3yz^4)^4$, $(2a^3b^3x^4)^5$, $(-xy^3z^0)^7$

3 $\left(\dfrac{x}{y^2}\right)^4$, $\left(-\dfrac{3a}{x}\right)^3$, $\left(-\dfrac{a^2b^3c}{2}\right)^6$, $\left(-\dfrac{x^3y^4}{a^3cz^5}\right)^8$

4 $(-z^2)^3(4y^3)^4$, $(-2ax^3)^6(-3a^4bx^3)^2$, $(-2a)^2(4ab)^8(-b^6c^3)^5$

5 $(a^2)^5(-b)^8(-3ac)^4$, $(x^{2m}y^{3n}z^p)^p$, $(-ax^my^p)^n$

6. $(-a)^{2m}$, $(-a)^{m+1}$, $\left(-\dfrac{a}{b}\right)^{2m-1}$; $\left(-\dfrac{a}{b}\right)^{2m+2}$

198 Square of Polynomials By actual multiplication we have

$$(a+b)^2 = a^2 + 2ab + b^2$$

$$\begin{aligned}(a+b+c)^2 &= a^2 + 2a(b+c) + (b+c)^2 \\ &= a^2 + 2a(b+c) + b^2 + 2bc + c^2 && (\alpha) \\ &= a^2 + b^2 + c^2 + 2ab + 2ac + 2bc && (\beta)\end{aligned}$$

$$\begin{aligned}(a+b+c+d)^2 &= a^2 + 2a(b+c+d) + b^2 + 2b(c+d) + c^2 + 2cd + d^2 && (\alpha) \\ &= a^2 + b^2 + c^2 + d^2 + 2ab + 2ac + 2ad + 2bc + 2bd + 2cd && (\beta).\end{aligned}$$

And so on

We have thus *two forms*, (α) and (β), of the results which enable us to state the **Rule** in two ways —

(i) *The square of any polynomial is equal to the sum of the squares of all the terms together with twice the product of each term into the sum of all the others which follow it*

(ii) *The square of any polynomial is equal to the sum of the squares of all the terms together with twice the product of every two of them*

REMARK Any one or more of the quantities a, b, c, may be negative, and then the sign of each term of the result will be determined by the Law of Signs [Art 39]

We have given several examples before [Art 64] The following are other examples most of which involve fractions

Examples CXXII

Find the value of

1 $\left(\frac{x}{2}+\frac{y}{3}\right)^2$ 2 $\left(\frac{a}{b}-\frac{5b}{a}\right)^2$ 3 $\left(\frac{x}{2}-\frac{2}{3x^2}\right)^2$

4 $\left(1+\frac{x}{2}-\frac{x^2}{3}\right)^2$ 5 $\left(\frac{x}{a}+\frac{y}{b}-\frac{z}{c}\right)^2$ 6 $\left(\frac{a^2}{x}-\frac{2b^2}{y}+\frac{3c^2}{z}\right)^2$

7 $(ax-2by+3cz)^2+(ax+2by-3cz)^2$

8 $(a^m+b^n)^2$ 9 $(a^m-2b^n+c)^2$ 10 $(e^{2x}+e^x+1)^2$

11 $\{(a+b)x-(a-b)y\}^2$ 12 $\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)^2$ 13 $\left(\frac{ax}{m^2}+\frac{by}{n^2}-1\right)^2$

14 $(x^3-2x^2+x-2)^2$ 15 $(a+bx+cx^2+dx^3)^2$

199 Cubes of Polynomials In Art 66, we have shewn how to find the cube of a binomial The cube of a trinomial can be found thus

$$(a+b+c)^3=\{a+(b+c)\}^3=a^3+3a^2(b+c)+3a(b+c)^2+(b+c)^3$$
$$=a^3+3a^2b+3a^2c+3a(b^2+2bc+c^2)+(b^3+3b^2c+3bc^2+c^3)$$
$$=a^3+b^3+c^3+3a^2b+3ab^2+3a^2c+3ac^2+3b^2c+3bc^2+6abc$$

Thus by the repeated application of the Formulæ of Art 66, we can find the cube of any polynomial

Several examples have been given before [See Art 66] We shall here give a few more examples, some involving fractions

Examples CXXIII

Find the value of

1 $\left(\frac{2a}{b}-1\right)^3$ 2 $\left(\frac{4}{x}-\frac{3}{y}\right)^3$ 3 $(a^2-2b^2+3c^2)^3$

4 $(1-2x+3x^2)^3$ 5 $(a+bx+cx^2)^3$. 6 $\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)^3$

7 $\left(\frac{3x^4}{2y^4}-\frac{2y^3}{3x^3}\right)^3$ 8. $\left(\frac{x^m}{a^n}+\frac{y^n}{b^m}\right)^3$ 9 $\left(x^4-\frac{2ax^2}{c}+\frac{a^2}{c^2}\right)^3$.

200 Other Powers of Polynomials Remembering that $a^4=(a^2)^2$, $a^6=(a^2)^3=(a^3)^2$, $a^8=(a^4)^2$, $a^9=(a^3)^3$, $a^8=\{(a^2)^2\}^2$, &c, we can find other powers of polynomials as in the examples below.

Ex 1 $(3x+1)^4=\{(3x+1)^2\}^2=(9x^2+6x+1)^2$
$=81x^4+108x^3+54x^2+12x+1$

Ex 2 $\left(\frac{a}{x}-\frac{x}{a}\right)^6=\left\{\left(\frac{a}{x}-\frac{x}{a}\right)^3\right\}^2=\left(\frac{a^3}{x^3}-\frac{3a}{x}+\frac{3x}{a}-\frac{x^3}{a^3}\right)^2$

$=\frac{a^6}{x^6}+\frac{9a^2}{x^2}+\frac{9x^2}{a^2}+\frac{x^6}{a^6}-\frac{6a^4}{x^4}+\frac{6a^2}{x^2}-20+\frac{6x^2}{a^2}-\frac{6x^4}{a^4}$

$=\frac{a^6}{x^6}-\frac{6a^4}{x^4}+\frac{15a^2}{x^2}-20+\frac{15x^2}{a^2}-\frac{6x^4}{a^4}+\frac{x^6}{a^6}$

Ex 3 $(a+b-c)^4=\{(a+b-c)^2\}^2$
$=(a^2+b^2+c^2+2ab-2ac-2bc)^2$
$=a^4+b^4+c^4+4a^2b^2+4a^2c^2+$&c

201 Expansion of any power of a Binomial The method of the last article is tedious We shall therefore give a rule for expanding a Binomial raised to any power, which is practically very useful This rule is obtained from the BINOMIAL THEOREM

The value of an expression raised to a certain power is called its **development or expansion** Thus the development or expansion of $(a+b)^2$ is $a^2+2ab+b^2$ The general **Rule** by which such expansions are obtained without multiplication is as follows —

In the expansion of $A+B$ raised to any power—

The first and last terms are A and B each raised to that power and in each successive term the index of A is less by 1 and that of B greater by 1, than those of A and B in the next preceding term

The coefficient of the first term is unity and that of each successive term is obtained by multiplying the coefficient of the next preceding term by the index of A in that term and then dividing the product by the number of terms preceding the term whose coefficient is sought

The same theorem also furnishes us with the following tests to examine the accuracy of our work —

(1) *The number of terms in a binomial expansion is one greater than the index of its power*

(2) *Each term is homogeneous and of a degree denoted by the index of the power*

(3) *The coefficients of terms equidistant from the beginning and the end are equal*

Ex 1. Expand $(a+b)^4$

The expansion will contain $4+1$ or 5 terms, of which the first and last are a^4 and b^4 respectively,

in the *second* term, the index of a is $4-1$ or 3, and that of b is 1,

third	$3-1$ or 2,	$1+1$ or 2,
fourth	$2-1$ or 1,	$2+1$ or 3,

The coefficient of *second* term $=\frac{1\times 4}{1}=4$,

third $=\frac{4\times 3}{2}=6$,

fourth $=\frac{6\times 2}{3}=4$

The *fifth* or last term has already been found Thus

$$(a+b)^4=a^4+4a^3b+6a^2b^2+4ab^3+b^4.$$

The above work may be more concisely shown, thus —

$$(a+b)^4=a^4+\frac{1\times 4}{1}a^{4-1}b+\frac{4\times 3}{2}a^{3-1}b^2+\frac{6\times 2}{3}a^{2-1}b^3+b^4$$
$$=a^4+4a^3b\qquad+6a^2b^2\qquad+4ab^3\qquad+b^4,$$

each term of the *second* line being put down in a *simplified form* to facilitate calculation as soon as the *corresponding* term has been put down in the *first* line according to the Rule.

Note *Every term* in the expansion of $(a+b)^4$ is *homogeneous* and of the *fourth* degree.

Ex 2 Develop $(a-b)^4$

Change the sign of b in Ex 1, thus

$$(a-b)^4=a^4+4a^3(-b)+6a^2(-b)^2+4a(-b)^3+(-b)^4$$
$$=a^4-4a^3b+6a^2b^2-4ab^3+b^4$$

Thus *the terms of the expansion of any power of a binomial of the form $A-B$ beginning from the first, are* alternately *+ and −*

Ex. 3 Expand $(2x-3y)^6$

$$(2x-3y)^6=(2x)^6-6(2x)^{6-1}(3y)+\frac{6\times 5}{2}(2x)^{5-1}(3y)^2$$
$$-\frac{15\times 4}{3}(2x)^{4-1}(3y)^3+\frac{20\times 3}{4}(2x)^{3-1}(3y)^4-\frac{15\times 2}{5}(2x)^{2-1}(3y)^5+(3y)^6$$
$$=64x^6-576x^5y+2160x^4y^2-4320x^3y^3+4860x^2y^4-2916xy^5+729y^6$$

Examples CXXIV

Develop

1 $(a-b)^6$ 2 $(a-x)^8$ 3 $(1-2x)^6$

4 $(3x-1)^7$ 5 $\left(\frac{x}{a}+\frac{a}{x}\right)^6$ 6 $\left(\frac{2a}{b}-1\right)^6$

Evolution

202 *Definitions* The **root*** of a quantity is that quantity which being raised to the power denoted by the index of the root gives the proposed quantity Thus the m^{th} root of a quantity is that quantity whose m^{th} power gives the proposed quantity, as a is an m^{th} root of a^m [See the definitions of *square root* and *cubic root* in Art 15]

To express a given root of a quantity, place before the quantity, the radical sign $\sqrt{}$ with the number that denotes the index of the root. Thus $\sqrt[5]{a}$ denotes a *fifth* root of a, $\sqrt[n]{x}$ denotes an n^{th} root of x, and so on.

Evolution is the process of finding the root of a given quantity. Involution and Evolution are thus two inverse processes.

203 Signs of Roots We have

$\sqrt{a^2}=\pm a$, for $aa=a^2$, and $(-a)(-a)=a^2$,

$\sqrt[4]{a^4}=\pm a$, for $aaaa=a^4$, and $(-a)(-a)(-a)(-a)=a^4$, and so on.

Hence *if an* **even root**† *of a positive quantity be extracted, the root may be either positive or negative*

Again $\sqrt[3]{a^3}=a$ for $aaa=a^3$, $\sqrt[3]{-a^3}=-a$, for $(-a)(-a)(-a)=-a^3$

Similarly $\sqrt[5]{a^5}=a$, and $\sqrt[5]{-a^5}=-a$ And so on

Hence *if an* **odd root**† *of a quantity be extracted, the sign of the root will be the same as the sign of the quantity itself*

When a proposed quantity is *negative*, no *even root* can be extracted, for *no quantity raised to an even power can give a negative quantity* Hence such quantities as $\sqrt{-1}$, $\sqrt{-4}$, $\sqrt{-5}$ cannot have any arithmetical meaning and are called **imaginary** or **impossible** quantities Thus $\sqrt{-a}$ is the type of an Imaginary Quantity

* The student should notice the different senses in which the word '*root*' has been used [Art 85].

† A root expressed by an *odd* number, may be called an *odd root*, and that expressed by an *even* number, may be called an *even root*

204 Extraction of Roots We have $(a^2)^3=a^6$,

$\sqrt[3]{a^6}=a^2=a^{6\div 3}$, $(a^m)^n=a^{mn}$, $\sqrt[n]{a^{mn}}=a^m=a^{mn\div n}$, and so on.

Thus *any root of a given quantity is obtained by dividing the index of the power of that quantity by the index of the required root*

Hence we can extract *any root* of a monomial

Thus

(i) $\sqrt{(a^6b^4)}=\pm a^3b^2$, for the root being *even*, it must have the double sign [Art. 203], and $6\div 2=3$, $4\div 2=2$

(ii) $\sqrt[3]{(-27a^3b^9)}=-3ab^3$, for the root must be negative, as $-27a^3b^9$ is negative [Art 203], and $\sqrt[3]{27}=3$, $3\div 3=1$, $9\div 3=3$

(iii) $\sqrt[5]{(32a^{10}x^{15})}=2a^2x^3$, for the root being *odd*, it must have the sign of $32a^{10}x^{15}$, i e, + [Art 203], and $\sqrt[5]{32}=2$, $10\div 5=2$, $15\div 5=3$

(iv) $\sqrt{\left(\frac{9x^{12}}{16a^4b^6}\right)}=\pm\frac{3x^{\frac{12}{2}}}{4a^{\frac{4}{2}}b^{\frac{6}{2}}}=\pm\frac{3x^6}{4a^2b^3}$

(v) $\sqrt[m]{(a^{2m}b^mc^{3m})}=a^{\frac{2m}{m}}b^{\frac{m}{m}}c^{\frac{3m}{m}}=a^2bc^3$

Examples CXXV

Find the square root of

1 a^4b^8 2 $25x^2y^4z^6$ 3 $49m^8n^4z^2$ 4 $144x^{12}y^{10}z^4$

5 $64x^{2m}y^{4n}$ 6 $81a^{2x+2}b^{2x-2}$ 7 $169x^{4m}y^{8n}z^{2m}$ 8 a^2b^8

9 $\frac{25a^2x^4}{9y^2z^6}$ 10 $\frac{36x^{2m}y^2}{49y^{n}z^4}$ 11 $\frac{75a^4x^2}{48y^4z^6}$ 12 $\frac{63a^2}{175b^4}$

Find the value of

13 $\sqrt[3]{a^3x^3y^6}$ 14 $\sqrt[3]{-8x^3y^{12}}$ 15 $\sqrt[4]{x^{4m}y^{8n}}$

16 $\sqrt[5]{-a^5x^{10}y^{15}}$ 17. $\sqrt[3]{\left(\frac{125x^6y^3}{8a^3b^6}\right)}$ 18 $\sqrt[4]{\frac{16m^4n^8}{81p^{12}}}$

19 $\sqrt[m]{5x^{am}y^{3m}}$ 20 $\sqrt[n]{a^{mn}x^{2n+an}}$

205 Square root of a Trinomial From Art 53, we see that when two of the terms of a *trinomial* are the squares of any two quantities, and the remaining term is twice their product with the sign + or −, the trinomial is the square of the sum or difference of those two quantities, according as twice their product has the sign, + or −

Thus we can at once express the trinomial as a square quantity and hence find its square root

For example in the trinomial $4x^2-20xy+25y^2$, the first term $=(2x)^2$, the third term $=(5y)^2$, and the second term $=2(2x)(5y)$ with the $-$ sign Hence the trinomial $=(2x-5y)^2$, and its square root

$$=\pm(2x-5y)$$

REMARK The square root is not generally written with the *double* sign, thus the above root is written $2x-5y$

Ex 1 Find the square root of $9a^2-42ax+49x^2$

Given expression $=(3a)^2-2(3a)(7x)+(7x)^2=(3a-7x)^2$,

square root required $=3a-7x$

Ex 2 Find the square root of $a^2-ax+\frac{x^2}{4}$

Given expression $=a^2-2a\frac{x}{2}+\left(\frac{x}{2}\right)^2=\left(a-\frac{x}{2}\right)^2$,

square root required $=a-\frac{x}{2}$

Examples CXXVI.

Extract the square root of

1 $9a^2+6ab+b^2$ 2 $16x^4-24x^2+9$ 3 $x^4-8ax^3+16a^2x^2$

4 $64a^2x^2+48ab^2x+9b^4$ 5 $36a^4x^4-24a^3bx^3+4a^2b^2x^2$

6 $25x^4+30x^3y+9x^2y^2$ 7 $4a^2b^2-12abc+9c^2$

8 $9x^4y^2+66x^2yz+121z^2$ 9 $9a^2-102abc+289b^2c^2$

10 $x^2+xy+\frac{y^2}{4}$ 11 $x^2-x+\frac{1}{4}$ 12 $9x^2-2xy+\frac{y^2}{9}$

13 $4a^2-ax+\frac{x^2}{16}$ 14 $x^2y^4+xy^2z^3-\frac{z^4}{4}$

15 $\frac{a^2}{c^2}+\frac{2ax}{c}+x^2$ 16 $\frac{x^2}{y^2}-2+\frac{y^2}{x^2}$ 17 $\frac{x^2}{a^2}-\frac{x}{y}+\frac{a^2}{4y^2}$

18 $a^6+2+\frac{1}{a^6}$ 19 $\frac{25}{4x^4}+\frac{x^2}{25}-1$ 20 $9a^8+\frac{a^4b^4}{36}-a^6b^2$

206 Square root found by inspection If a polynomial is a complete square and contains the *first and second powers only* of some one letter, it will be at once seen to be a perfect square when arranged according to the powers of that letter, and thus its square root can be found

Ex 1 Extract the square root of $x^2+4xz-12yz+9y^2-6xy+4z^2$

Arrange according to the descending powers of x, thus

given expression $=x^2-2x(3y-2z)+(9y^2-12yz+4z^2)$

$$=x^2-2x(3y-2z)+(3y-2z)^2$$
$$=(x-\overline{3y-2z})^2=(x-3y+2z)^2,$$

square root required $=x-3y+2z$

[Try by arranging according to the powers of y and z]

Ex 2 Find the square root of $x^4-2bx^3-(2a^2-b^2)x^2+2a^2bx+a^4$

Put $m=a^2$, thus m (or a^2) occurs in the first and second powers only

Arrange according to powers of a, thus

given expression $=a^4+2a^2(bx-x^2)+b^2x^2-2bx^3+x^4$

$$=a^4+2a^2(bx-x^2)+(bx-x^2)^2=\{a^2+(bx-x^2)\}^2,$$

square root required $=a^2+bx-x^2$

[Try this example by arranging according to powers of b]

Ex 3 Find the square root of $4-4c+2b+c^2-bc+\dfrac{b^2}{4}$

Arrange according to powers of c, thus

given expression $=c^2-(4+b)c+\left(4+2b+\dfrac{b^2}{4}\right)$

$$=c^2-2\left(2+\frac{b}{2}\right)c+\left(2+\frac{b}{2}\right)^2=\left\{c-\left(2+\frac{b}{2}\right)\right\}^2,$$

square root required $=c-2-\dfrac{b}{2}$

Ex. 4 Extract the square root of $(bc+ca+ab)^2-4abc(a+c)$

Given expression when developed and arranged in powers of a

$$=a^2(b^2-2bc+c^2)+2a(b^2c-bc^2)+b^2c^2$$
$$=a^2(b-c)^2+2a(b-c)bc+(bc)^2=\{a(b-c)+bc\}^2,$$

square root required $=ab-ac+bc$

Examples CXXVII

Extract the square root of

1. $a^2+b^2+c^2-2ab+2ac-2bc$
2. $9x^2-12xy+4y^2+6xz-4yz+z^2$
3. $x^2-2ax+a^2+2xy-2ay+y^2$
4. $x^2+\dfrac{2ax}{3}-bx+\dfrac{a^2}{9}-\dfrac{ab}{3}+\dfrac{b^2}{4}$
5. $4x(1+x)-2a(1+2x)+a^2+1$
6. $b^2x^4-2abx^3+(a^2+2b)x^2-2ax+1$

Extract the square root of

7 $x^3(x-2a)+a^2b(b-2x)+(a^2+2ab)x^2$

8 $\dfrac{a^2}{x^2}+\dfrac{ab}{xy}+\dfrac{b^2}{4y^2}-\dfrac{6ac}{xz}-\dfrac{3bc}{yz}+\dfrac{9c^2}{z^2}$ 9 $x^4+(x^2-a)x-2(a-\frac{1}{8})x^2+a^2$

10 $a^2+b^2+c^2+d^2-2a(b-c+d)-2b(c-d)-2cd$

11 $4x^4+8ax^3+4a^2x^2+16b^2x^2+16ab^2 \ +16b^4$ [*Cal*, 1870]

207 General Rule We know that $(a+b+c)^2$

$$=a^2+b^2+c^2+2bc+2ca+2ab \quad \ldots \quad (\text{i})$$

The expansion contains 6 terms and may be arranged thus

$$a^2+(2a+b)b+\{2(a+b)+c\}c \quad \ldots (\text{ii})$$

We see that the square root of (ii), i.e., of (i) is $a+b+c$. How to obtain the terms a, b and c?

(1) To obtain a, we notice that a is the square root of a^2, the first term of (ii) Subtract a^2 from (ii), and we get

$$(2a+b)b+\{2(a+b)+c\}c, \text{ the remainder} \quad (\text{iii})$$

(2) To obtain b, we divide $(2a+b)b$ by $2a+b$, which we call the *first* divisor Thus the first divisor is *twice* the first term a + the second term b of the root. [*Observe that b can be obtained practically by dividing* 2ab *by* 2a.]

Subtracting $(2a+b)b$ from (ii), we have

$$\{2(a+b)+c\}c, \text{ the } \textit{second} \text{ remainder} \quad . \quad (\text{iv})$$

(3) To obtain c, we divide (iv), the second remainder, by $2(a+b)+c$, which we call the *second* divisor Thus the second divisor is *twice* the sum of the first *two* terms a and b + the third term c [*Note that* c *can be practically obtained by dividing* 2ac, the *first term* of the second remainder, *by* 2a, the *first term* of the second divisor]

Now the product of the second divisor and c is equal to the second remainder, and therefore there is *no more* remainder

Thus all the terms a, b and c of the root are obtained

We may arrange the work thus

$$\begin{array}{r|l}
 & a^2+(2a+b)b+\{2(a+b)+c\}c \ (a+b+c \\
 & a^2 \\ \hline
2a+b & (2a+b)b \\
 & (2a+b)b \\ \hline
2(a+b)+c & \{2(a+b)+c\}c \\
 & \{2(a+b)+c\}c \\ \hline
\end{array}$$

Similarly we can find the square root of any other expression.

Hence the following

Rule —Arrange the proposed expression according to the powers of some *one* letter ,

Take the square root of the first term and set it down as the *first term* of the root , subtract the square of the first term from the given expression and bring down the next two terms to form the *first* remainder ,

Take *twice* the first term of the root and put this down as the first term of the *first* divisor , divide by this term the first term of the first remainder and put the quotient as the *second* term of the root and also as the last term of the first divisor , multiply the first divisor by the second term of the root and subtract to get the *second* remainder ,

Take *twice* the sum of the first two terms of the root, and set it down as the first two terms of the *second* divisor , and repeat the process to obtain the *third* term of the root And so on

In practice we proceed according to the rule and then the above work is shewn as below, when (1) is arranged in powers of a

$$\begin{array}{r|l}
 & a^2+2ab+2ac+b^2+2bc+c^2\,(a+b+c \\
 & a^2 \\ \hline
2a+b & 2ab+2ac \\
 & 2ab+b^2 \\ \hline
2a+2b+c & 2ac+2bc+c^2 \\
 & 2ac+2bc+c^2 \\ \hline
\end{array}$$

Ex 1 Extract the square root of

$$4x^6-4x^5+13x^4-22x^3+17x^2-24x+16$$

$$\begin{array}{r|l}
 & 4x^6-4x^5+13x^4-22x^3+17x^2-24x+16\,(2x^3-x^2+3x-4 \\
 & 4x^6 \\ \hline
4x^3-x^2 & -4x^5+13x^4 \\
 & -4x^5+x^4 \\ \hline
4x^3-2x^2+3x & 12x^4-22x^3+17x^2 \\
 & 12x^4-6x^3+9x^2 \\ \hline
4x^3-2x^2+6x-4 & -16x^3+8x^2-24x+16 \\
 & -16x^3+8x^2-24x+16 \\ \hline
\end{array}$$

Find the square root by arranging the expression in ascending powers of x

Ex 2 Find the square root of $16(a^4+1)-24a(a^2+1)+41a^2$

Remove the brackets and arrange in descending powers of a Thus the given expression $=16a^4-24a^3+41a^2-24a+16$

The square root will be found to be $4a^2-3a+4$

Otherwise —Given expn $=16a^4-24a^3+32a^2+9a^2-24a+16$

$$=16a^4-8a^2(3a-4)+(3a-4)^2$$
$$=(4a^2)^2-2(4a^2)(3a-4)+(3a-4)^2$$
$$=\{4a^2-(3a-4)\}^2,$$

square root required $=4a^2-3a+4$

Ex 3 Extract the square root of $\frac{x^2}{4}+4x+\frac{4}{x^2}-2+x^4-x^3$

Arrange in descending powers of x

$$x^4-x^3+\frac{x^2}{4}+4x-2+\frac{4}{x^2}\left(x^2-\frac{x}{2}+\frac{2}{x}\right.$$
$$x^4$$
$$2x^2-\frac{x}{2}\,\Big|\,-x^3+\frac{x^2}{4}$$
$$-x^3+\frac{x^2}{4}$$
$$2x^2-x+\frac{2}{x}\,\Big|\,4x-2+\frac{4}{x^2}$$
$$4x-2+\frac{4}{x^2}$$

Otherwise —Given expn $=x^4-x^3+\frac{x^2}{4}+4\left(x-\frac{1}{2}\right)+\frac{4}{x^2}$

$$=\left(x^2-\frac{x}{2}\right)^2+2\frac{2}{x}\left(x^2-\frac{x}{2}\right)+\left(\frac{2}{x}\right)^2$$
$$=\left\{\left(x^2-\frac{x}{2}\right)+\frac{2}{x}\right\}^2,$$

square root required $=x^2-\frac{x}{2}+\frac{2}{x}$

Ex 4. Extract the square root of $\left(x^2+\frac{1}{x^2}\right)^2-4\left(x+\frac{1}{x}\right)^2-12$

[*Cal*, 1866]

Simplify and arrange the expression and then proceed as in the above examples.

Otherwise —Given expn

$$=\left(x^2+\frac{1}{x^2}\right)^2-4\left(x^2+\frac{1}{x^2}\right)-8+12$$
$$=\left(x^2+\frac{1}{x^2}\right)^2-4\left(x^2+\frac{1}{x^2}\right)+4=\left\{\left(x^2+\frac{1}{x^2}\right)-2\right\}^2$$

square root required $=x^2+\frac{1}{x^2}-2$

Ex 5 Extract the square root of $\dfrac{(a^2+b^2)^2}{a^4+b^4-2a^2b^2}+4\times\dfrac{a}{a+b}\times\dfrac{b}{a-b}$

[*Cal*, 1886]

Simplify and find the square root of the numerator and denominator

Otherwise —Given expn $=\dfrac{(a^2-b^2)^2+4a^2b^2}{(a^2-b^2)^2}+\dfrac{4ab}{a^2-b^2}$

$$=1+\frac{4a^2b^2}{(a^2-b^2)^2}+\frac{4ab}{a^2-b^2}$$

$$=1+2\times\frac{2ab}{a^2-b^2}+\left(\frac{2ab}{a^2-b^2}\right)^2=\left(1+\frac{2ab}{a^2-b^2}\right)^2,$$

$$\therefore \text{ square root required}=1+\frac{2ab}{a^2-b^2}$$

Examples CXXVIII

Extract the square root of

1 $9x^4+6x^3+7x^2+2x+1$

2 $x^4+4x^3-2x^2-12x+9.$

3 $x^4-2x^3+3x^2-2x+1$

4 $4x^4-12x^3+25x^2-24x+16$

5 $1-2x+5x^2-4x^3+4x^4$

6 $16x^4-24x^3-31x^2+30x+25$

7 $25x^4-50x^3+85x^2-60x+36$

8 $81x^4+108x^3-24x+4$

9 $9-24a-68a^2+112a^3+196a^4$

10 $a^{12}-8a^9+18a^6-8a^3+1$

11 $x^4+4x^3y+12x^2y^2+16xy^3+16y^4$

12 $x^6+8x^4-2x^3+16x^2-8x+1$

13 $4+12ax-6ax^3+9a^2x^2+4x^3+x^4$

14 $64p^4-48p^3q+41p^2q^2-12pq^3+4q^4$

15 $x^4-4x^3+8x+4.$

16 $4x^4-16x^3-8x^2+48x+36$

17 $4a^2x^4-12a^3x^3+13a^4x^2-6a^5x+a^6$

18 $x^4+2x^3-x+\frac{1}{4}$

19 $x^2+\dfrac{1}{x^2}+2\left(x-\dfrac{1}{x}\right)-1$

20 $\left(x+\dfrac{1}{x}\right)^2-4\left(x-\dfrac{1}{x}\right)$

21 $x^2+8x-\dfrac{64}{x}+\dfrac{64}{x^2}$

22 $x^2-2+\dfrac{2}{x}+\dfrac{1}{x^2}-\dfrac{2}{x^3}+\dfrac{1}{x^4}$

23 $x^4-3x^3+\dfrac{9x^2}{4}+\dfrac{x}{2}-\dfrac{3}{4}+\dfrac{1}{16x^2}$

24 $\dfrac{x^2}{y^2}\left(\dfrac{x^2}{4y^2}+1\right)+\dfrac{4y^2}{x^2}\left(\dfrac{y^2}{x^2}+1\right)+3$

25 $\dfrac{x^2}{y^2}+\dfrac{y^2}{4x^2}-\dfrac{x}{y}+\dfrac{y}{2x}-\dfrac{3}{4}$

26 $16a^2x^2-16a^2bx+4a^2b^2-24abxy+12ab^2y+9b^2y^2$

Extract the square root of

27 $x^6+4x^5+10x^4+10x^3+5x^2-6x+1$

28 $16a^6-24a^5+25a^4-20a^3+10a^2-4a+1$

29 $a^4-a^2b+\frac{2a^2c^2}{3}-\frac{a^2d^3}{2}+\frac{b^2}{4}-\frac{bc^2}{3}+\frac{bd^3}{4}+\frac{c^4}{9}-\frac{c^2d^3}{6}+\frac{d^6}{16}$

30 $x(x+1)(x+2)(x+3)+1$ 31 $\left(\frac{x}{x-1}\right)^2-\frac{2x^2}{x^2-1}+\left(\frac{x}{x+1}\right)^2$

32 $(a-b)^4-2(a^2+b^2)(a-b)^2+2(a^4+b^4)$

33 $(a^2+b^2)(x^2+y^2)-2(a^2-b^2)xy+2ab(x^2-y^2)$

208 Cube Root found by inspection From Art. 66, we see that the cube root of $a^3+3a^2b+3ab^2+b^3$ is $a+b$, and that of $a^3-3a^2b+3ab^2-b^3$ is $a-b$

Thus if an expression is a *complete cube* and consists of only *four* terms, its cube root is the *sum* of *two* terms which are the *cube root* of the two cube terms.

Ex 1. Find the cube root of $a^3+15a^2+75a+125$

Here $\sqrt[3]{a^3}=a$ and $\sqrt[3]{125}=5$ Thus reqd root $=a+5$.

Ex 2 Extract the cube root of $8x^3-36x^2y+54xy^2-27y^3$.

Here $\sqrt[3]{(8x^3)}=2x$ and $\sqrt[3]{(-27y^3)}=-3y$

Thus required cube root $=2x-3y$

From Art 199, we find that the expansion of a *trinomial* cannot contain more than 10 terms Thus when the cube root of an expression is a trinomial, we can find the cube root as above

Ex 3 Extract the cube root of

$$27a^6-27a^5-45a^4+35a^3+30a^2-12a-8$$

Here $\sqrt[3]{(27a^6)}=3a^2$ and $\sqrt[3]{(-8)}=-2$, also $-27a^5\div3(3a^2)^2=-a$

cube root required $=3a^2-a-2$

Examples CXXIX

Extract the cube root of

1 $8x^3-12ax^2+6a^2x-a^3$ 2 $x^3+9x^2y+27xy^2+27y^3$

3 $8a^3-60a^2b+150ab^2-125b^3$ 4 $x^6-12a^2x^4+48a^4x^2-64a^6$

5 $x^3+ax^2+\frac{a^2x}{3}+\frac{a^3}{27}$ 6 $\frac{a^6}{27}-a^4b+9a^2b^2-27b^3$

7 $x^6-3x^5+6x^4-7x^3+6x^2-3x+1$

8 $1+6a+9a^2-4a^3+9a^4+6a^5-a^6$

Extract the cube root of

9 $8a^6+12a^5b-30a^4b^2-35a^3b^3+45a^2b^4+27ab^5-27b^6$

10 $x^3-9x^2y+6x^2z+27xy^2-36xyz+12xz^2-27y^3+54y^2z-36yz^2+8z^3$

11 $\frac{8x^3}{y^3}-\frac{12x^2}{y^2}+\frac{12x}{y}+\frac{3y}{x}-\frac{3y^2}{4x^2}+\frac{y^3}{8x^3}-7$

209 Fourth, Sixth, Eighth, &c Roots We can now find the *fourth*, *sixth*, *eighth*, *sixteenth*, &c, roots of polynomials, for the fourth root is the square root of the square root, the sixth root is the square root of the cube root, the eighth root is the square root of the fourth root, and so on

Ex Find the fourth root of $1-12x+54x^2-108x^3+81x^4$

The square root of this expression will be found to be $1-6x+9x^2$, and the square root of this latter expression will be seen to be $1-3x$

Examples CXXX

1 Find the fourth root of $x^4+4x^3+6x^2+4x+1$,

2 Find the fourth root of $a^4-8a^3+24a^2-32a+16$

3 Find the fourth root of $4a^2b^2+(a^2+b^2)^2-4ab(a^2+b^2)$

4 Find the fourth root of $x^4-4x^2+6-\frac{4}{x^2}+\frac{1}{x^4}$

5 Find the sixth root of $x^{12}-6x^{10}+15x^8-20x^6+15x^4-6x^2+1$.

6 Find the eighth root of $256a^8-1024a^7x^2+1792a^6x^4-1792a^5x^6$
$+1120a^4x^8-448a^3x^{10}+112a^2x^{12}-16ax^{14}+x^{16}$

CHAPTER XIX

INDICES

210 The Index Law In Art 44, it has been proved that

$$a^m\times a^n=a^{m+n}$$

when m and n are positive integers This result is called the *Index Law*, for this is the *fundamental law* from which all other laws regulating Indices are derived

211 As consequences of the above law, we have, when the indices are all *positive integers*,

(i) $a^m\times a^n\times a^p\times \dots = a^{m+n+p+\dots}$

(ii) $a^m\div a^n=a^{m-n}$, where $m>n$ [Art 61]

(iii) $(a^n)^m = a^{mn} = (a^m)^n$ [Art 195]

(iv) $(ab)^n = a^n b^n$ [Art 195], and *generally* $(abc \ldots)^n = a^n b^n c^n \ldots$

(v) $(a^m b^n)^r = a^{mr} b^{nr}$, and *generally* $(a^m b^n c^p \ldots)^r = a^{mr} b^{nr} c^{pr} \ldots$

(vi) $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$

$$\left[\text{For } \left(\frac{a}{b}\right)^n = \frac{a}{b} \times \frac{a}{b} \times \frac{a}{b} \ldots \text{ to } n \text{ factors}\right.$$

$$\left. = \frac{a \times a \times a \times \ldots \text{ to } n \text{ factors}}{b \times b \times b \times \ldots \text{ to } n \text{ factors}} = \frac{a^n}{b^n}\right]$$

212 Fractional and Negative Indices The definition of Art 13, is intelligible so long as the Index is a *positive integer*, but has no meaning when such quantities as $a^{\frac{1}{2}}$, a^{-1}, &c, are considered We have therefore to assign *new meanings* to fractional and negative indices In doing so we must bear in mind that algebraical symbols, whatever their values may be, must always be subject to the same laws Hence *our meaning of a^n must be such that the fundamental Index Law $a^m \times a^n = a^{m+n}$ shall in every case be obeyed*

213 Meaning of $a^{\frac{1}{n}}$, where n is a positive integer

If the product $a^{\frac{1}{2}} \times a^{\frac{1}{2}}$ obeys the Index Law, we must have

$$a^{\frac{1}{2}} \times a^{\frac{1}{2}} = a^{\frac{1}{2}+\frac{1}{2}} = a^1 = a$$

Thus $a^{\frac{1}{2}}$ is a quantity whose *square* is a, i e, $a^{\frac{1}{2}} = \sqrt{a}$

Hence $a^{\frac{1}{2}}$ denotes the square root of a

Similarly $a^{\frac{1}{3}} \times a^{\frac{1}{3}} \times a^{\frac{1}{3}} = a^{\frac{1}{3}+\frac{1}{3}+\frac{1}{3}} = a^1 = a$

Thus $a^{\frac{1}{3}}$ is a quantity whose *cube* is a, i e, $a^{\frac{1}{3}} = \sqrt[3]{a}$

Hence $a^{\frac{1}{3}}$ denotes the cube root of a

And generally

$$a^{\frac{1}{n}} \times a^{\frac{1}{n}} \times a^{\frac{1}{n}} \times \ldots \text{ to } n \text{ factors}$$

$$= a^{\frac{1}{n}+\frac{1}{n}+\ldots \text{ to } n \text{ terms}} = a^{\frac{1}{n} \times n} = a^1 = a$$

Thus $a^{\frac{1}{n}}$ is a quantity whose n^{th} power is a, i e, $a^{\frac{1}{n}} = \sqrt[n]{a}$

Hence $a^{\frac{1}{n}}$ denotes the n^{th} root of a

Ex $8^{\frac{1}{3}}=2$, $(32)^{\frac{1}{5}}=2$, $(81)^{\frac{1}{4}}=3$, &c

214 Meaning of $a^{\frac{m}{n}}$, where m and n are both positive integers
If the Index Law is to be obeyed, we must have

$$a^{\frac{m}{n}}\times a^{\frac{m}{n}}\times a^{\frac{m}{n}}\times \qquad \text{to } n \text{ factors}$$

$$=a^{\frac{m}{n}+\frac{m}{n}+ \quad \text{to } n \text{ terms}}=a^{\frac{m}{n}\times n}=a^{m}$$

Thus $a^{\frac{m}{n}}$ is a quantity whose n^{th} power is a^m, i e, it is the n^{th} root of a^m, thus $a^{\frac{m}{n}}=\sqrt[n]{a^m}$

Also

$$a^{\frac{1}{n}}\times a^{\frac{1}{n}}\times a^{\frac{1}{n}}\times \qquad \text{to } m \text{ factors}$$

$$=a^{\frac{1}{n}+\frac{1}{n}+ \quad \text{to } m \text{ terms}}=\left(a^{\frac{1}{n}}\right)^m=a^{\frac{1}{n}\times m}=a^{\frac{m}{n}}$$

Thus $a^{\frac{m}{n}}$ is the m^{th} power of the n^{th} root of a, i e, $a^{\frac{m}{n}}=(\sqrt[n]{a})^m$

Therefore $a^{\frac{m}{n}}=\sqrt[n]{a^m}=(\sqrt[n]{a})^m$

Hence $a^{\frac{m}{n}}$ has two meanings —(i) **it is the nth root of the mth power of a, and (ii) it is the mth power of the nth root of a**

Thus *when the index is a fraction, the numerator denotes a power and the denominator a root*

Cor Hence $a^{\frac{m}{n}}=a^{\frac{mp}{np}}$ For if $x=a^{\frac{m}{n}}$, we have $x^n=a^m$

or $(x^n)^p=(a^m)^p$, that is, $x^{np}=a^{mp}$, thus $x=a^{\frac{mp}{np}}$, or $a^{\frac{m}{n}}=a^{\frac{mp}{np}}$

215 Meaning of a^0, where a may have any value
If a^0 is subject to the Index Law, we must have

$$a^0\times a^p=a^{0+p}=a^p\text{, therefore } a^0=a^p\div a^p=1$$

Hence $a^0=1$, i e, any quantity raised to a 0-power is 1.

216 Meaning of a^{-p}, where p is any positive quantity
By the Index Law

$$a^{-p}\times a^p=a^{-p+p}=a^0=1 \text{ [Art 215]}$$

$$a^{-p}=1\div a^p=\frac{1}{a^p}\text{, also } a^p=1\div a^{-p}=\frac{1}{a^{-p}}$$

Thus the negative index indicates the reciprocal of the power denoted by the positive index

Hence $\qquad a^{-1}=\frac{1}{a}, \quad a^{-2}=\frac{1}{a^2}, \quad a^{-3}=\frac{1}{a^3}, \ldots$

also $\qquad a=\frac{1}{a^{-1}}, \quad a^2=\frac{1}{a^{-2}}, \quad a^3=\frac{1}{a^{-3}},$

Cor We have seen [Art 50] how an expression may be arranged according to the powers of a letter having *positive integral* indices We may now arrange an expression in which the *symbol of reference* has *fractional* and *negative* indices. Thus

$$x^2+x+x^{\frac{1}{2}}+x^{\frac{1}{4}}+1$$

is arranged in *descending* powers of x, and the same in a *reversed* order will be arranged in *ascending* powers of x

Again the expression

$$x^3+2x^2+5x+3+4x^{-1}+6x^{-2}+x^{-5}$$
$$=x^3+2x^2+5x+3x^0+4x^{-1}+6x^{-2}+x^{-5}$$

is arranged in *descending* powers of x, as 3, 2, 1, 0, -1, -2, -5 are in descending order of magnitude [Art 26] And since $x^{-1}=\frac{1}{x}$, $x^{-2}=\frac{1}{x^2}$ and $x^{-5}=\frac{1}{x^5}$, the same expression is equivalent to

$$x^3+2x^2+5x+3+\frac{4}{x}+\frac{6}{x^2}+\frac{1}{x^5},$$

this therefore is also arranged in descending powers of x

Examples CXXXI

1 Find the value of x^{-0}, $\frac{1}{x^{-0}}$, x^3y^0, $x^{-0}y^{-0}$ and $\frac{1-x^0}{1+x^0}$

2. Express with *fractional* indices

$$\sqrt{x^3}, \ \sqrt[3]{x^6}, \ (\sqrt[3]{x^2})^4, \ \sqrt{(\sqrt[6]{a^3b^2})^3}, \ \{\sqrt[4]{(\sqrt[3]{x^4})^2}\}^5$$

3 Express with *negative* indices

$$\frac{2}{x^3}, \ 4a^3x^3, \ \frac{3x^2}{yz^3}, \ \frac{4z}{x\sqrt{y}}, \ \frac{5}{x}\sqrt{\left(\frac{y}{z}\right)^5}$$

4 Express with *radical* signs $5a^{\frac{3}{2}}$, $3x^{\frac{5}{3}}y^{\frac{2}{3}}z^{\frac{4}{3}}$ $\ \frac{a^{\frac{3}{5}}x^{\frac{1}{5}}}{3y^{\frac{4}{5}}}$

5 Express with *positive* indices $3ab^{-1}$, $a^{-1}x^2y^{-2}$, $\dfrac{x^{-\frac{2}{5}}c}{b^{-\frac{1}{3}}c^{-2}}$

Simplify

6 $\dfrac{8}{3^{-3}}$ 7 $\dfrac{20}{(-5)^3}$ 8 $\dfrac{18}{(-4)^{-3}}$ 9 $\left(\dfrac{1}{8}\right)^{-2}-\left(\dfrac{1}{5}\right)^{-2}$

Simplify and express the result with *positive* indices

10 $c^{-\frac{3}{2}}\left\{\dfrac{x^{-8}}{c^2y^{-4}}\right\}^{-\frac{1}{4}}$ 11 $\{(-3x^2)^{-1}\}^3$ 12 $\left\{\dfrac{a^mb^{-n}}{cd^{-2}}\right\}^{-x}$

13 $\left\{-\dfrac{a^mb^{-x}}{b^{-y}c^x}\right\}^{-3}$ 14 $\left(\dfrac{1}{a^{-m}}\right)^2-\left(\dfrac{1}{a}\right)^{-2m}$ 15 $\left\{\dfrac{1}{x^{-2}}\right\}^2\left\{\dfrac{m^{-1}}{x^{-3}}\right\}^{-1}$

217 Bearing in mind the meanings that have been found for *fractional* and *negative* indices in Arts 213—216, we can prove the Index Law to hold in all cases

To prove that $a^m\times a^n=a^{m+n}$ *for all values of* m *and* n

This has already been proved for positive integers [Art 44]

Now let $m=\dfrac{p}{q}$, $n=\dfrac{r}{s}$, where p, q, r and s are all positive integers

Thus $a^m\times a^n=a^{\frac{p}{q}}\times a^{\frac{r}{s}}=a^{\frac{ps}{qs}}\times a^{\frac{qr}{qs}}$ [Art 214, *Cor*]

$$=\sqrt[qs]{(a^{ps})}\times\sqrt[qs]{(a^{qr})}=\sqrt[qs]{(a^{ps}\times a^{qr})}$$

$$=\sqrt[qs]{a^{ps+qr}}=a^{\frac{ps+qr}{qs}}=a^{\frac{p}{q}+\frac{r}{s}}$$

Next let *both* m and n be negative, and let $m=-p$, $n=-q$, then

$$a^m\times a^n=a^{-p}\times a^{-q}=\frac{1}{a^p}\times\frac{1}{a^q}=\frac{1}{a^{p+q}}=a^{-p-q}=a^{m+n}$$

Secondly, let *one* of m and n be negative, and let $m=p$ and $n=-q$ Here we have to consider two cases according as $p>$ or $<q$

(1) Let $p>q$, i e, let $p-q$ be positive, then

$$a^{p-q}\times a^q=a^{p-q+q}=a^p,$$

or $a^{p-q}=a^p\div a^q$ [Art 56] $=a^p\times\dfrac{1}{a^q}$ [Art 58] $=a^p\times a^{-q}$;

$$a^m\times a^n=a^p\times a^{-q}=a^{p-q}=a^{m+n}$$

(ii) Let $p<q$, i.e., let $p-q$ be negative, then

$$a^m \times a^n = a^p \times a^{-q} = \frac{1}{a^{-p}} \times \frac{1}{a^q} = \frac{1}{a^{q-p}} = a^{p-q} = a^{m+n}$$

Thus the Law is proved for all values of m and n

218 As in Art. 217, the following index laws may be shewn to be true for *all values* of m and n

(i) $a^m \div a^n = a^{m-n}$, (ii) $(a^m)^n = a^{mn}$, (iii) $(ab)^n = a^n b^n$

The proofs of these are left as exercise for the student

219 From Arts 217 and 218, we see that fractional and negative indices are subject to the same laws as positive integral indices Hence quantities with fractional and negative indices can be dealt with exactly in the same way as those having positive integral indices

Ex 1 Simplify $\dfrac{(a^{-1}b^{\frac{3}{4}}c^3)^{-\frac{2}{3}}}{(ab^{-\frac{1}{3}}c^{-2})^{\frac{3}{2}}}$.

$$\text{Given expression} = \left(a^{-1}b^{\frac{3}{4}}c^3\right)^{-\frac{2}{3}} \times \left(ab^{-\frac{1}{3}}c^{-2}\right)^{-\frac{3}{2}}$$

$$= a^{\frac{2}{3}}b^{-\frac{1}{2}}c^{-2} \times a^{-\frac{3}{2}}b^{\frac{1}{2}}c^3$$

$$= a^{-\frac{5}{6}}b^0c = a^{-\frac{5}{6}}$$

The result expressed with positive indices is $\dfrac{c}{a^{\frac{5}{6}}}$, and with radical sign is $\dfrac{c}{\sqrt[6]{(a^5)}}$

Ex 2 Multiply $3x^{\frac{2}{3}} - x^{\frac{1}{3}}y^{\frac{1}{3}} + 2y^{\frac{2}{3}}$

$$\begin{array}{rlllll}
\text{Multiply} & 3x^{\frac{2}{3}} - x^{\frac{1}{3}}y^{\frac{1}{3}} + 2y^{\frac{2}{3}} & & & & \\
\text{by} & 3x^{\frac{2}{3}} + x^{\frac{1}{3}}y^{\frac{1}{3}} + 2y^{\frac{2}{3}} & & & & \\
\hline
 & 9x^{\frac{4}{3}} & -3xy^{\frac{1}{3}} & +6x^{\frac{2}{3}}y^{\frac{2}{3}} & & \\
 & & +3xy^{\frac{1}{3}} & -x^{\frac{2}{3}}y^{\frac{2}{3}} & +2x^{\frac{1}{3}}y & \\
 & & & +6x^{\frac{2}{3}}y^{\frac{2}{3}} & -2x^{\frac{1}{3}}y & +4y^{\frac{4}{3}} \\
\hline
\text{Product} & 9x^{\frac{4}{3}} & & +11x^{\frac{2}{3}}y^{\frac{2}{3}} & & +4y^{\frac{4}{3}}
\end{array}$$

Ex. 3 Divide $a+b$ by $a^{\frac{1}{3}}+b^{\frac{1}{3}}$

$$
\begin{array}{r|l}
a^{\frac{1}{3}}+b^{\frac{1}{3}} & a+b \quad \left(a^{\frac{2}{3}}-a^{\frac{1}{3}}b^{\frac{1}{3}}+b^{\frac{2}{3}}\right. \\
 & \underline{a+a^{\frac{2}{3}}b^{\frac{1}{3}}} \\
 & -a^{\frac{2}{3}}b^{\frac{1}{3}}+b \\
 & \underline{-a^{\frac{2}{3}}b^{\frac{1}{3}}-a^{\frac{1}{3}}b^{\frac{2}{3}}} \\
 & a^{\frac{1}{3}}b^{\frac{2}{3}}+b \\
 & \underline{a^{\frac{1}{3}}b^{\frac{2}{3}}+b}
\end{array}
$$

Examples CXXXII

1. Add together
$4a^{\frac{2}{5}}-5a^{\frac{3}{5}}+6a^{\frac{1}{5}}$, $3a^{\frac{3}{5}}-8a^{\frac{2}{5}}-4a^{\frac{1}{5}}$, $a^{\frac{1}{5}}-6a^{\frac{3}{5}}+7a^{\frac{2}{5}}$, and $4a^{\frac{3}{5}}-3a^{\frac{1}{5}}$

2 From $5x^{-\frac{2}{3}}-3x^{-\frac{1}{3}}a^{-\frac{1}{2}}+4$ take $2a^{-\frac{1}{2}}x^{-\frac{1}{3}}-x^{-\frac{2}{3}}-2\frac{1}{2}$

Multiply

3 $x^{\frac{2}{3}}-x^{\frac{1}{3}}y^{\frac{1}{3}}+y^{\frac{2}{3}}$ by $x^{\frac{1}{3}}+y^{\frac{1}{3}}$ 4 $x^{\frac{3}{2}}-4x^{\frac{1}{2}}+2$ by $x-x^{\frac{1}{2}}$

5 $2a^{-4}b^{-1}-3a^{-5}$ by $3a^{-4}b^{2}-2a^{-5}$

6 $\frac{3}{4}a^{-1}x^{-3}-\frac{2}{5}b^{-2}y$ by $\frac{2}{3}a^{-1}x^{-3}+\frac{1}{3}b^{-2}y$

7 $a^{\frac{m}{2}}-a^{\frac{m}{4}}b^{\frac{n}{4}}+b^{\frac{n}{2}}$ by $a^{\frac{m}{2}}+a^{\frac{m}{4}}b^{\frac{n}{4}}+b^{\frac{n}{2}}$

Divide

8 $a^{\frac{3}{2}}+b^{\frac{3}{2}}$ by $a^{\frac{1}{2}}+b^{\frac{1}{2}}$ 9 $27-a^{-\frac{3}{4}}$ by $3-a^{-\frac{1}{4}}$

10 $x-2x^{\frac{1}{2}}+1$ by $x^{\frac{1}{3}}-2x^{\frac{1}{6}}+1$ 11 $x^{4}+x^{-4}+1$ by $x^{2}+x^{-2}+1$

12 $x+y-z+3x^{\frac{1}{3}}y^{\frac{1}{3}}z^{\frac{1}{3}}$ by $x^{\frac{1}{3}}+y^{\frac{1}{3}}-z^{\frac{1}{3}}$

Simplify

13 $\left(\frac{x}{y}\right)^{2m}\left(-\frac{y}{x}\right)^{2m-1}$. 14 $\left(\frac{ay}{x}\right)^{\frac{1}{2}}\left(\frac{bx}{y^2}\right)^{\frac{1}{3}}\left(\frac{y^2}{a^3b^2}\right)^{\frac{1}{6}}$

15 $\dfrac{\left\{(a^m)^{\frac{1}{r}}(a^q)^{\frac{1}{n}}\right\}^{nr}}{\left(b^{\frac{n}{q}}b^{\frac{r}{m}}\right)^{mq}}$ 16 $\left\{(x^a)^{1-\frac{1}{a}}\right\}^{\frac{1}{a-1}}$. 17 $\left\{(ax^x)^{a+\frac{1}{x}}\right\}^{1-\frac{ax}{1+ax}}$

Simplify

18 $\left(-\frac{x}{a}\right)^{2m-1}\left(-\frac{a}{y}\right)^{2m+1}$ 19 $\left(\frac{3x}{a}+\frac{2y^{-1}}{b^{-2}}\right)^{-2}\left(\frac{2y^{-1}}{b^{-2}}-\frac{3x}{a}\right)^{-2}$

20 $\left(\frac{x-y}{a-b}\right)^{2m}\left(\frac{b-a}{x-y}\right)^{2m+1}$ 21 $\left(\frac{z-y}{x-a}\right)^{2m+1}\left(\frac{3y}{y-z}\right)^{2m}\left(\frac{a-x}{z-y}\right)^{2}$

22 $\frac{x^{a+b}}{x^{2c}}\times\frac{x^{a+c}}{x^{2b}}\times\frac{x^{b+c}}{x^{2a}}$

Resolve into factors

23 $x^{2m}-y^{2n}$ 24. $x^{-6}-\left(\frac{1}{y^{-1}}\right)^{-2}$.

25 Extract the square root of $a^{-2}x^{-2}-2a^{-1}x^{-1}+3-2ax+a^2x^2$

Reduce to a simple form

26 $\dfrac{\frac{1}{4^{-3}}-\frac{2}{10^{-2}}}{\frac{1}{2^{-2}}+\frac{1}{4^{-1}}}$ 27. $\left\{\frac{a^{-1}+b^{-1}}{a^{-2}-b^{-2}}\right\}^{-2}-\left\{\frac{a^{-2}+b^{-2}}{a^{-4}-b^{-4}}\right\}^{-3}$.

28 $\dfrac{x^{\frac{4}{3}}+x^{\frac{2}{3}}y^{\frac{2}{3}}+y^{\frac{4}{3}}}{x^{\frac{2}{3}}-x^{\frac{1}{3}}y^{\frac{1}{3}}+y^{\frac{2}{3}}}$ 29 $\dfrac{e^{2x}+e^{-x}-e^{x}-1}{e^{2x}-e^{-x}+e^{x}-1}$

30 $\dfrac{a^2b^{-2}+(xy^{-1}+yx^{-1})ab^{-1}+1}{a^2b^{-2}+(xy^{-1}-yx^{-1})ab^{-1}-1}$

31 If $x^2=y^3$, shew that $\left(\frac{x}{y}\right)^{\frac{3}{2}}+\left(\frac{y}{x}\right)^{\frac{2}{3}}=x^{\frac{1}{2}}+y^{-\frac{1}{3}}$.

32 Evaluate $\dfrac{(a^{n+x}-a^n)\times(a^n-a^{n-x})}{(a^{n+x}-a^n)-(a^n-a^{n-x})}$

CHAPTER XX

SURDS

220 *Definition* When any proposed root of a quantity cannot be obtained exactly (*i e*, in any finite number of terms), it is called an **Irrational Quantity**, or a **Surd** Thus $\sqrt{2}$, $\sqrt[3]{3}$, $\sqrt{a}$, $\sqrt[3]{a^3+b^3}$, &c, are Surds

A surd is expressed either by means of the radical sign as above or by means of its equivalent the fractional index [Art 213] Thus

$$\sqrt{a^2+b^2}=(a^2+b^2)^{\frac{1}{2}}, \quad \sqrt{(a-b)^3}=(a-b)^{\frac{3}{2}}, \quad \&c.$$

Hence *surds are subject to all the Laws of Indices* [Arts. 210 and 211]

221 We have

(i) $x = x^{\frac{2}{2}} = \sqrt{x^2} = x^{\frac{m}{m}} = \sqrt[m]{x^m} = \&c$

(ii) $a - x = (a-x)^{\frac{3}{3}} = \sqrt[3]{(a-x)^3} = \sqrt[n]{(a-x)^n} = \&c$

Thus *a rational quantity may be reduced to the form of a given surd, by raising it to the power whose root the surd expresses*

222 We have

(i) $3\sqrt{3} = \sqrt{3^2}\sqrt{3} = \sqrt{3^2 \times 3} = \sqrt{27}$ or $= 3^{\frac{2}{2}} \times 3^{\frac{1}{2}} = 3^{\frac{3}{2}} = \sqrt{3^3}$

(ii) $a\sqrt{x} = a^{\frac{2}{2}} \times x^{\frac{1}{2}} = (a^2)^{\frac{1}{2}} x^{\frac{1}{2}} = (a^2 x)^{\frac{1}{2}}$ [Art 211] $= \sqrt{a^2 x}$

(iii) $x\sqrt[m]{(a-x)^n} = x^{\frac{m}{m}}(a-x)^{\frac{n}{m}} = \{x^m(a-x)^n\}^{\frac{1}{m}}$

Thus *to introduce a coefficient of a surd under the radical, reduce the coefficient to the form of the surd and then multiply the quantity so reduced by that under the radical*

Examples CXXXIII

Introduce under the radical, the coefficient of

1 $5\sqrt{2}$. 2 $3\sqrt[3]{4}$ 3 $10\sqrt{ax^3}$ 4 $a^3x\sqrt[3]{ax^4}$ 5 $q^2r^3\sqrt[m]{q^3r^2}$

223 It is easy to see that

(i) $\sqrt{18} = \sqrt{9 \times 2} = \sqrt{3^2 \times 2} = 3\sqrt{2}$

(ii) $\sqrt{(a^3x^4)} = \sqrt{(a^2ax^2x^2)} = ax^2\sqrt{a}$

(iii) $(a^2 - x^2)^{\frac{1}{2}} = \left\{a^2\left(1 - \frac{x^2}{a^2}\right)\right\}^{\frac{1}{2}} = (a^2)^{\frac{1}{2}}\left(1 - \frac{x^2}{a^2}\right)^{\frac{1}{2}} = a\sqrt{1 - \frac{x^2}{a^2}}$

or $= \left\{x^2\left(\frac{a^2}{x^2} - 1\right)\right\}^{\frac{1}{2}} = (x^2)^{\frac{1}{2}}\left(\frac{a^2}{x^2} - 1\right)^{\frac{1}{2}} = x\sqrt{\frac{a^2}{x^2} - 1}$

Thus *any quantity may be made the coefficient of a surd, if every part under the radical be divided by this quantity raised to the power whose root the radical expresses*

COR Hence *a given surd may be reduced to its simplest form by first resolving into factors the quantity under the radical and then removing from under the sign those whose indices are equal to, or are multiples of, the denominator of the surd-index*

Thus $\sqrt{252} = \sqrt{9 \times 4 \times 7} = \sqrt{3^2 \times 2^2 \times 7} = 3 \times 2\sqrt{7} = 6\sqrt{7}$

Examples CXXXIV

Simplify

1 $\sqrt{72}$ 2 $\sqrt[3]{1080}$ 3 $\sqrt{10368}$ 4 $\sqrt[5]{3a^{10}b^3c^4}$

5 $\sqrt{2a^4x+4a^3x^2+2a^2x^3}$

224 Addition and subtraction of Surds It is easily seen that

(i) $3\sqrt{2}+5\sqrt{2}=(3+5)\sqrt{2}=8\sqrt{2}$

(ii) $8\sqrt{3}-3\sqrt{3}=(8-3)\sqrt{3}=5\sqrt{3}$

(iii). $a\sqrt{x}\pm\sqrt{x} =(a\pm1)\sqrt{x}$

(iv). $5a\sqrt{x}-3a\sqrt{x}+a\sqrt{x}=(5a-3a+a)\sqrt{x}=3a\sqrt{x}$

Thus *the sum of similar surds is found by prefixing the algebraic sum of the coefficients to the irrational part*

Definition **Similar or Like** surds are those which have the *same irrational part* Thus $\sqrt{x}$ and $3\sqrt{x}$ are similar surds, so are $a\sqrt[3]{x}$ and $2\sqrt[3]{m^3x}$, for $2\sqrt[3]{m^3x}=2m\sqrt[3]{x}$; &c

225 Multiplication and Division of Surds It is evident that

(i) $\sqrt{5}\times\sqrt{3}=5^{\frac{1}{2}}\times3^{\frac{1}{2}}=(5\times3)^{\frac{1}{2}}$ [Art. 211] $=(15)^{\frac{1}{2}}=\sqrt{15}$

(ii) $\sqrt[m]{a}\times\sqrt[m]{c}=(a)^{\frac{1}{m}}(c)^{\frac{1}{m}}=(ac)^{\frac{1}{m}}=\sqrt[m]{ac}$

Thus *if the surds have the same index, their product is found by taking the product of the quantities under the radical and retaining the common index*

(iii) $x^{\frac{3}{m}}\times y^{\frac{2}{m}}=(x^3)^{\frac{1}{m}}(y^2)^{\frac{1}{m}}=(x^3y^2)^{\frac{1}{m}}=\sqrt[m]{x^3y^2}.$

Thus *if the indices of the surds have the* same *denominator, their product is found by taking the product of the powers expressed by the numerators and affixing the radical expressed by the denominator*

(iv) $\sqrt[m]{x}\sqrt[n]{y}=x^{\frac{1}{m}}y^{\frac{1}{n}}=x^{\frac{n}{mn}}y^{\frac{m}{mn}}=(x^n)^{\frac{1}{mn}}(y^m)^{\frac{1}{nm}}=(x^ny^m)^{\frac{1}{nm}}$

(v) $(a+x)^{\frac{1}{2}}(a-x)^{\frac{1}{3}}=(a+x)^{\frac{3}{6}}(a-x)^{\frac{2}{6}}=\{(a+x)^3\}^{\frac{1}{6}}\{(a-x)^2\}^{\frac{1}{6}}$

$=\{(a+x)^3(a-x)^2\}^{\frac{1}{6}}=\sqrt[6]{(a+x)^3(a-x)^2}$

Thus *if the indices have* different *denominators, reduce them to a common denominator and proceed as in* (iii)

REMARK Since to divide a by b is the same as to multiply a by $\frac{1}{b}$ [Art 58], *i e*, by b^{-1} [Art 216], *the rules of this Article will likewise apply to examples of division of one surd by another*

Examples CXXXV

Simplify

1 $\sqrt{15}\times\sqrt{3}$ 2 $(3+2\sqrt{5})(2-\sqrt{5})$ 3. $\sqrt{ab^2}\times\sqrt[4]{a^2b^3}$

4 $\sqrt[4]{\frac{ab^2}{x^3}}\times\sqrt[6]{\frac{cx^5}{a^2b}}$ 5 $\sqrt{x^3\sqrt{x^2}}\times\sqrt{\sqrt[3]{x}}$

6 $\sqrt{ab+bc}\times\sqrt{a^2+ac}$ 7 $\sqrt{ax+x^2}\times\sqrt{ab-bx}$

8 $\sqrt[3]{54a^7}-\sqrt[3]{2a}$ 9 $\sqrt{a^3x}-\sqrt{ax^5}$

10 $\left(\frac{x}{a^2}-y\right)-\left(\frac{1}{a}\sqrt{x}+\sqrt{y}\right)$ 11 $\sqrt{\frac{a^3}{b^6}}-\sqrt{\frac{a}{b^3}}$

12 $\sqrt{\frac{54a^3}{x}}-\sqrt[4]{\frac{8x^5}{9a}}$ 13 $(x+y)-\frac{1}{3}\sqrt{x^2-y^2}$

226 *Definition* A **Quadratic Surd** is that of which the index is $\frac{1}{2}$, as $\sqrt{a}$, $\sqrt{(a+x)}$, $\sqrt{a^3+b^3+c^3}$, &c

A **Simple Quadratic Surd** is that which consists of a *single* term, $\sqrt{x}$, $10\sqrt{y}$, $5a\sqrt{x^3}$, &c

A **Binomial Quadratic Surd** is that which consists of *two* terms, one or both of which are simple quadratic surds, as $a+\sqrt{b}$, $\sqrt{x}-3\sqrt{y}$, &c

Two binomial quadratic surds are said to be **conjugate** or **complementary** when they have the same terms connected respectively by the signs + and −, as $\sqrt{x}+\sqrt{y}$ and $\sqrt{x}-\sqrt{y}$

227 Rationalisation of Surds To rationalise a surd is to multiply it by a quantity so as to give a *rational product*

Thus $\sqrt{a}$ is rationalised when it is multiplied by $\sqrt{a}$, for $\sqrt{a}\sqrt{a}=a$, $\sqrt[3]{a}$ is rationalised by multiplying it by $\sqrt[3]{a^2}$, for $\sqrt[3]{a}\sqrt[3]{a^2}=\sqrt[3]{a^3}=a$, &c

Again $\sqrt{x}+\sqrt{y}$ is rationalised when it is multiplied by $\sqrt{x}-\sqrt{y}$, for $(\sqrt{x}+\sqrt{y})(\sqrt{x}-\sqrt{y})=x-y$ Hence *a binomial quadratic surd is rationalised by multiplying by its conjugate*

To rationalise $a^{\frac{1}{3}}+b^{\frac{1}{3}}$, the multiplier is evidently $a^{\frac{2}{3}}-a^{\frac{1}{3}}b^{\frac{1}{3}}+b^{\frac{2}{3}}$, for $(a^{\frac{1}{3}}+b^{\frac{1}{3}})(a^{\frac{2}{3}}-a^{\frac{1}{3}}b^{\frac{1}{3}}+b^{\frac{2}{3}})=a+b$

To rationalise $\sqrt{a}+\sqrt{b}+\sqrt{c}$

Multiply first by $\sqrt{a}+\sqrt{b}-\sqrt{c}$, the product is $(\sqrt{a}+\sqrt{b})^2-(\sqrt{c})^2$ $=a+b-c+2\sqrt{(ab)}$, next multiply by $a+b-c-2\sqrt{(ab)}$, the product is $(a+b-c)^2-(2\sqrt{ab})^2=(a+b-c)^2-4ab$, thus the required multiplier is $(\sqrt{a}+\sqrt{b}-\sqrt{c})(a+b-c-2\sqrt{ab})$

We proceed in this way in the case of any other surd

228 Fractions with surd denominators The principal use of the last article is to find without much labour the values of fractions having *irrational denominators* This will be seen from the following examples

Ex 1 Find the value of $\dfrac{2}{\sqrt{3}}$ to 5 decimal places

$$\frac{2}{\sqrt{3}}=\frac{2\sqrt{3}}{\sqrt{3}\sqrt{3}}\text{ [Art 181]}=\frac{2\sqrt{3}}{3}=1\ 15470.$$

REMARK If, instead of rationalising the denominator, we had found the square root of 3 first, and then divided 2 by it, the operation would have been long and tedious

Ex 2 Find the value of $\dfrac{\sqrt{3}}{\sqrt{2}+1}$ to 5 places of decimals.

Multiply the numerator and denominator by the conjugate surd $\sqrt{2}-1$, thus

$$\frac{\sqrt{3}}{\sqrt{2}+1}=\frac{\sqrt{3}(\sqrt{2}-1)}{(\sqrt{2}+1)(\sqrt{2}-1)}=\frac{\sqrt{6}-\sqrt{3}}{2-1}=\sqrt{6}-\sqrt{3}=\ 71743$$

Ex 3 Simplify $\dfrac{2+\sqrt{3}}{2-\sqrt{3}}+\dfrac{2-\sqrt{3}}{2+\sqrt{3}}$

$$\text{Given surd}=\frac{(2+\sqrt{3})^2}{(2-\sqrt{3})(2+\sqrt{3})}+\frac{(2-\sqrt{3})^2}{(2+\sqrt{3})(2-\sqrt{3})}$$
$$=\frac{(2+\sqrt{3})^2}{4-3}+\frac{(2-\sqrt{3})^2}{4-3}$$
$$=4+4\sqrt{3}+3+4-4\sqrt{3}+3=14.$$

Ex 4 Simplify $\dfrac{2}{\sqrt{(xy)}-y}-\dfrac{1}{x+\sqrt{(xy)}}-\dfrac{\sqrt{x}+\sqrt{y}}{x\sqrt{y}-y\sqrt{x}}$

$$\text{Given expn}=\frac{2}{\sqrt{y}(\sqrt{x}-\sqrt{y})}-\frac{1}{\sqrt{x}(\sqrt{x}+\sqrt{y})}-\frac{\sqrt{x}+\sqrt{y}}{\sqrt{(xy)}(\sqrt{x}-\sqrt{y})}$$
$$=\frac{2\sqrt{x}(\sqrt{x}+\sqrt{y})-\sqrt{y}(\sqrt{x}-\sqrt{y})-(\sqrt{x}+\sqrt{y})^2}{\sqrt{(xy)}(\sqrt{x}+\sqrt{y})(\sqrt{x}-\sqrt{y})}$$
$$=\frac{2x+2\sqrt{(xy)}-\sqrt{(xy)}+y-x-y-2\sqrt{(xy)}}{\sqrt{(xy)}(\sqrt{x}+\sqrt{y})(\sqrt{x}-\sqrt{y})}$$
$$=\frac{x-\sqrt{(xy)}}{\text{Dnr}}=\frac{\sqrt{x}(\sqrt{x}-\sqrt{y})}{\sqrt{x}\sqrt{y}(\sqrt{x}+\sqrt{y})(\sqrt{x}-\sqrt{y})}=\frac{1}{\sqrt{(xy)}+y}$$

Ex 5 Prove that $\sqrt{a+\sqrt{b}}+\sqrt{a-\sqrt{b}}=\sqrt{2a+2\sqrt{a^2-b}}$

Let $x=\sqrt{a+\sqrt{b}}+\sqrt{a-\sqrt{b}}$,

$x^2=a+\sqrt{b}+a-\sqrt{b}+2\sqrt{a^2-b}=2a+2\sqrt{a^2-b}$,

whence $x=\sqrt{2a+2\sqrt{a^2-b}}$, &c

Examples CXXXVI

Rationalise the denominator of

1 $\frac{2\sqrt{3}}{\sqrt{3}+1}$ 2 $\frac{\sqrt{7}+\sqrt{3}}{5-\sqrt{21}}$ 3 $\frac{1+2\sqrt{3}}{3-\sqrt{2}}$ 4 $\frac{2-\sqrt{3}}{2+\sqrt{3}}$

5 Rationalise the numerator of $\frac{\sqrt{a+x}+\sqrt{a-x}}{\sqrt{a+x}-\sqrt{a-x}}$

Simplify

6 $\frac{1}{2-\sqrt{3}}+\frac{1}{3-2\sqrt{2}}$

7 $\frac{1}{a-\sqrt{b}}-\frac{1}{a+\sqrt{b}}$.

8. $\frac{\sqrt{x}+\sqrt{y}}{\sqrt{x}-\sqrt{y}}+\frac{\sqrt{x}-\sqrt{y}}{\sqrt{x}+\sqrt{y}}$

9 $\frac{\sqrt{5}-\sqrt{3}}{\sqrt{5}+\sqrt{3}}\times(8+2\sqrt{15})$

10 $\frac{\sqrt{2}(2+\sqrt{3})}{\sqrt{3}(1+\sqrt{3})}-\frac{\sqrt{2}(2-\sqrt{3})}{\sqrt{3}(\sqrt{3}-1)}$

11. $\frac{\sqrt{18}}{\sqrt{3}+\sqrt{2}}+\frac{\sqrt{12}}{\sqrt{3}-\sqrt{2}}$

12 $\sqrt{\frac{a+x}{a-x}}+\sqrt{\frac{a-x}{a+x}}$

13 $\frac{1}{1-\sqrt{1-x^2}}\pm\frac{1}{1+\sqrt{1-x^2}}$

14 $\frac{x+\sqrt{x^2-a^2}}{x-\sqrt{x^2-a^2}}-\frac{x-\sqrt{x^2-a^2}}{x+\sqrt{x^2-a^2}}$.

15 $\frac{\sqrt{x^2+1}+\sqrt{x^2-1}}{\sqrt{x^2+1}-\sqrt{x^2-1}}+\frac{\sqrt{x^2+1}-\sqrt{x^2-1}}{\sqrt{x^2+1}+\sqrt{x^2-1}}$.

16 $\sqrt{\left\{\frac{1-x-\sqrt{2x+x^2}}{1-x+\sqrt{2x+x^2}}\right\}}+\sqrt{\left\{\frac{1-x+\sqrt{2x+x^2}}{1-x-\sqrt{2x+x^2}}\right\}}$.

17 Simplify $\frac{a}{a-\sqrt{a^2-x^2}}-\frac{a}{a+\sqrt{a^2-x^2}}$ and find its value, when $x=\frac{\sqrt{3}}{2}a$

Find the value of

18 $\dfrac{1+\sqrt{3}}{1-\sqrt{3}}+\dfrac{2+\sqrt{3}}{2-\sqrt{3}}-\dfrac{2\sqrt{3}+1}{2+\sqrt{3}}$. 19 $\dfrac{1}{2(a+\sqrt{x})}+\dfrac{1}{2(a-\sqrt{x})}+\dfrac{a}{a^2+x}$

20 x^2+ax+b, when $x=\sqrt{\left(\dfrac{a^2}{4}-b\right)}-\dfrac{a}{2}$.

21. $\dfrac{x}{y}-\sqrt{\dfrac{1+x}{1-y}}$, when $x=\frac{1}{4}, y=\frac{1}{5}$

22 $\sqrt{\frac{3}{4}-x}+\sqrt{2x}-\frac{3}{2}\sqrt{1-4x}$, when $x=\frac{1}{12}$.

23 If $y=\dfrac{1}{x+\sqrt{x^2-1}}$, shew that $2x=y+y^{-1}$.

24 Find the value of $x^3+y^3+z^3-xyz$, when $x=\sqrt{q}-\sqrt{r}$, $y=\sqrt{r}-\sqrt{p}$, $z=\sqrt{p}-\sqrt{q}$.

229 Extraction of the Square Root. In the present Article we shall shew how to find the square root of binomial quadratic surds by *inspection*.

Ex 1. Extract the root of $5+2\sqrt{6}$

Here $2\sqrt{6}=2\sqrt{3\times 2}=2\sqrt{3}\sqrt{2}$ (i),

and $5=3+2=(\sqrt{3})^2+(\sqrt{2})^2$ (ii)

Now (i) is *twice the product* of two numbers, the *sum of whose squares* is (ii), therefore by Art. 64, we have

$$5+2\sqrt{6}=(\sqrt{3})^2+(\sqrt{2})^2+2\sqrt{3}\sqrt{2}=(\sqrt{3}+\sqrt{2})^2,$$

$$\therefore \text{ square root required}=\sqrt{3}+\sqrt{2},$$

Ex. 2. Extract the square root of $4-2\sqrt{3}$

Here $2\sqrt{3}=2\times\sqrt{3}\times 1$, and $4=3+1$,

$$4-2\sqrt{3}=(\sqrt{3})^2+1^2-2\sqrt{3}\times 1=(\sqrt{3}-1)^2;$$

whence required square root $=\sqrt{3}-1$

Ex 3 Extract the square root of $2a\pm 2\sqrt{a^2-x^2}$

$$\text{Given expression}=(a+x)+(a-x)\pm 2\sqrt{(a+x)(a-x)}$$
$$=\{\sqrt{(a+x)}\pm\sqrt{(a-x)}\}^2, \quad \&c$$

REMARK Hence it appears that the sign before the radical determines whether the required root will be the *sum* or *difference* of the two quantities obtained by factorizing the quantity under the radical

Examples CXXXVII

Find the square root of

1 $11+6\sqrt{2}$ 2 $14+6\sqrt{5}$ 3 $8+2\sqrt{15}$

4 $19-8\sqrt{3}$ 5 $9-4\sqrt{5}$ 6 $30+12\sqrt{6}$

7 $9-4\sqrt{2}$ 8 $4\frac{1}{3}-\frac{4}{3}\sqrt{3}$ 9 $2+\sqrt{3}$

10 $2(a+b+\sqrt{a^2+2ab})$. 11 $x+y-2\sqrt{(x+1)(y-1)}$

12. $1+\sqrt{1-x^2}$ 13 $x+y+z-2\sqrt{xy+xz}$

14 $ax-2a\sqrt{ax-a^2}$ 15 $a^2+2x\sqrt{a^2-x^2}$

16 Find the value of $\dfrac{\sqrt{1+x}-1}{\sqrt{1-x}+1}+\dfrac{\sqrt{1-x}+1}{\sqrt{1+x}-1}$, when $x=\dfrac{\sqrt{3}}{2}$

17 Express $\alpha\beta+\sqrt{(\alpha^2-1)(\beta^2-1)}$ in terms of x and y, when $2\alpha=x+x^{-1}$, $2\beta=y+y^{-1}$ [See *App*]

Examples for Revision (D)

1 Put $(x^3+3x+15)^2-(x^3-3x+15)^2$ in its simplest form, and find its value when $2x=-5$

2 By how much does $(x^2-3x+1)^2$ exceed $x(x-1)(x-2)(x-3)$?

3 Simplify $(2a+3b)^3-(3a+2b)^3+3(a-b)(3a+2b)(2a+3b)$

4 If $ax-by=c$, find the value of $a^3x^3-3abcxy-b^3y^3$.

5 Divide $(a+1)^2x^3+(a+1)x^2+a^2(a-1)x-a^5$ by $(a+1)x-a^2$

6 Resolve $(a^2-b^2)(x^2-y^2)+4abxy$ into factors

7 Extract the square root of $4(x-1)(x^3-1)+9x^2$

8 Simplify $(8a^{-1}b^{\frac{1}{4}})(a^{\frac{1}{2}}b^{-1})(-2ab^{\frac{3}{4}})(a^{\frac{1}{2}}b^{-\frac{1}{2}})$

---

9 Find the value of $a^2x^2-b^2y^2$, when $x=2a-b$, $y=a-2b$

10 Simplify $(x^2+xy+y^2)(x+y)-(x^2-xy+y^2)(x-y)$.

11 Divide $(b-c)(x-a)^3+(c-a)(x-b)^3+(a-b)(x-c)^3$ by $(b-c)(c-a)(a-b)$

12 Find the value of $x^2(-3x^{-\frac{1}{3}})(-x^{\frac{4}{3}})\times 5x^{-1}$

13 Prove that $(x-y)^2-(x-y)(x-z)+(x-z)^2$
$$=\tfrac{1}{2}\{(y-z)^2+(z-x)^2+(x-y)^2\}$$

14 Given $p-q=a$ and $p^3-q^3=b^3$, find pq in terms of a and b

15 Find the square root of $x^{12}(x^6-2)^2+4(x^6-1)^2$

16 Find the relation between a and b, when

$$\frac{3}{x-a}-\frac{2}{x+a}=\frac{x+b}{x^2-a^2}$$

---

17 If $a=1$, $b=2$, $c=-\frac{1}{2}$, $d=0$, find the value of

$$\frac{a-b+c}{a-b-c}-\frac{ad-bc}{bd+ac}-\sqrt{\frac{b^3}{a^3}-\frac{a^3}{c^3}}$$

18 Multiply $a(x+y)+b(x-y)$ by $a(x-y)-b(x+y)$

19 If $x+y=a$ and $xy=b^2$, express $x-y$ in terms of a and b.

20 Factorize (i) $(ax+by)^4-(bx-ay)^4$,

(ii) $(x^2-z^2)^2-y^2(2x+y)^2$.

21 Express $(x+3a)(x+5a)(x+7a)(x+9a)$ as the difference of two squares

22 If $a+b+c+d=0$, show that

$$a^3+b^3+c^3+d^3+3(b+c)(c+a)(a+b)=0$$

23 Find the coefficient of x^3 in $(x-1)(x^2-x-1)(x^3-2x-2)$

24 Shew that $(21)^n-1$ is divisible by 20 or 22, if n be even

---

25 Find the value of $a^2+b^2-c^2+2ab$, when $a=\cdot03$, $b=\cdot07$, $c=\cdot10$

26 If $x=b+c-2a$, $y=c+a-2b$, $z=a+b-2c$, find the value of $\frac{1}{2}(a+b+c)(ax+by+cz)$

27 Divide $a^8-b^8+a^2b^2(a^4-b^4)$ by the product of a^2+b^2, a^2-ab+b^2, and a^2+ab+b^2

28 Find the H. C. F. of $3x^2-(4a+2b)x+2ab+a^2$ and

$$x^3-(2a+b)x^2+(2ab+a^2)x-a^2b.$$

29 If $x-y=2$ and $x^3-y^3=14$, find the value of x^2-xy+y^2

30 Factorize (i) $(ax+by)^2+(bx-ay)^2$,

(ii) $x^2(x^2+3a^2)^2-a^2(3x^2+a^2)^2$

31 Simplify $\left(\frac{x^q}{x^r}\right)^{q+r}\times\left(\frac{x^r}{x^p}\right)^{r+p}\times\left(\frac{x^p}{x^q}\right)^{p+q}$

32 Shew that $8^{2n}-3^{2n}$ is always divisible by 55

---

33 Multiply $3+\cdot01x$ by $1-x$, and find the value of the product when $x=\cdot1$

34 Resolve into factors (i) $x^3(x+2y)-y^3(2x+y)$,

(ii) $a^3(x^6-1)-x^3(a^6-1)$

35 Extract the square root of $x^2+4(1-x)(1+4x)+8x^2(x-1)(2x-1)$

36 If $x+\frac{1}{x}=2a$, $x-\frac{1}{x}=2b$, $y+\frac{1}{y}=2c$, $y-\frac{1}{y}=2d$, find the value of $xy+\frac{1}{xy}$

37 Simplify $\dfrac{\frac{x}{x+1}+\frac{x}{x-1}}{\frac{2}{x^2-1}}-\dfrac{4x-\frac{1}{x}}{2+\frac{1}{x}}$

38 Divide $x^{-3}+y$ by $x^{-2}-x^{-1}y^{\frac{1}{3}}+y^{\frac{2}{3}}$

39 Find the L C M of $2ab+a^2+b^2-c^2$, $3abc-a^3-b^3-c^3$ and $(b+c)(c+a)(a+b)+abc$

40 If $a+b+c=\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=1$, prove that one of the quantities a, b and c must be equal to unity

---

41 Find the value of

$$\sqrt{2a+1}-\left(a+\frac{6}{\sqrt{a}}\right)-\left(3-\frac{a^2}{4-\sqrt[3]{2a}}\right), \text{ if } a=4$$

42 Resolve $(ab+1)^4-4ab(ab+1)^2-(a^2-b^2)^2$ into factors

43 Extract the square root of $\frac{x}{y}\left(2+\frac{x}{y}\right)-\frac{y}{x}\left(2-\frac{y}{c}\right)-1$

44. Multiply $ab^{-1}+1+ba^{-1}$ by $ab^{-1}-1+ba^{-1}$

45 Reduce $\dfrac{e^{2x}x^3+e^{2x}-x^3-1}{e^{2x}x^2+2e^{x}x^2-e^{2x}-2e^{x}+x^2-1}$ to its lowest terms

46 If h be the H C F of a and b whose L C M is l, prove that $hl=ab$

47 If $s=a+b+c$, prove that

$$s(s-2a)(s-2b)+s(s-2b)(s-2c)+s(s-2c)(s-2a)$$
$$=(s-2a)(s-2b)(s-2c)+8abc$$

48 If $f(x)=ax^2+bx+c$ and $\phi(x)=a-bx+cx^2$ find the value of $f(0)-\phi(1)$

---

49 Factorize $\frac{1}{x^2}-\frac{1}{x}+\frac{1}{4}$ and $\left(a^2+\frac{1}{a}\right)-\left(b^2+\frac{1}{b}\right)$

50 Multiply $a+a^{\frac{1}{2}}b^{-\frac{1}{3}}+b^{-\frac{2}{3}}$ by $a^{\frac{1}{2}}-b^{-\frac{1}{3}}$

51 Divide $(4x^3-3a^2x)^2+(4y^3-3a^2y)^2-a^6$ by $x^2+y^2-a^2$

52 If $bc+ca+ab=0$, prove that $(a+b+c)^3=a^3+b^3+c^3-3abc$

53 Find the H. C. F. of $(a^2+a-2)x^2+(2a^2+a+3)x+a^2-1$

and $(a^2+4a+4)x^2+(2a^2+a-6)x+a^2-3a+2$

54 Extract the square root of $x^2+\frac{1}{x^2}+4a\left(x+\frac{1}{x}\right)+2(1+2a^2)$

55 Shew that

$$\frac{x^2}{a^2}+\left(\frac{z-x}{b}\right)^2 \text{ is identical with } \frac{z^2}{a^2+b^2}+\frac{a^2+b^2}{a^2b^2}\left(x-\frac{za^2}{a^2+b^2}\right)^2.$$

56 Find the cube root of $x^3-\frac{1}{x^3}+3\left(x^2+\frac{1}{x^2}\right)-5$

---

57 Multiply $a^3-a^2b+2ab^2$ by $a^2+ab+2b^2$

58 Divide $(ax^2-ay^2+2bxy)^2+(by^2-bx^2+2axy)^2$ by $(x^2+y^2)^2$

59 What term is wanting to make $x^4-2x^2-\frac{2}{x^2}+\frac{1}{x^4}$ a perfect square?

60 Find the product of $a^{m-1}b^{n-1}-1$ and $a^{1-m}b$.

61 Factorize $x^2-2(a+b)x-ab(a-2)(b+2)$

62 If $x+y+z=a$, $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=\frac{1}{b}$, $(y+z)(z+x)(x+y)=c$,

prove that $xyz=\frac{bc^2}{a-b}$

63 Extract the square root of $21-6\sqrt{6}$

64 Simplify $\left\{\frac{(x^m)^{\frac{1}{r}}\times(x^n)^{\frac{1}{p}}}{\sqrt[n]{x^p}\times\sqrt[m]{x^r}}\right\}^{pr}$

---

65 Multiply $(4a^2-4ab+b^2)+(2a-b)(a+b)+(a+b)^2$ by $a-2b$

66 Factorize (i) $lm(x+y)(x-y)-xy(l+m)(l-m)$,

(ii) $(1+x)^2(1+y^2)-(1+x^2)(1+y)^2$.

67 Divide $a^3(b-c)^3+b^3(c-a)^3+c^3(a-b)^3$

by $(b-c)^3+(c-a)^3+(a-b)^3$.

68 Extract the square root of

$$x^6+\frac{1}{x^6}+6\left(x^4+\frac{1}{x^4}\right)+15\left(x^2+\frac{1}{x^2}\right)+20$$

69 Simplify $\left(\frac{x^p}{x^q}\right)^{p+q}-\left(\frac{x^{p+q}}{x^{p-q}}\right)^{\frac{p^2}{q}}$

70 If $a^2-bc=0$, prove that

$$b(c^2-ab)^2+c(b^2-ca)^2+2a(b^2-ca)(c^2-ab)=0$$

71 Simplify $\frac{x^{3n}}{x^n-1}-\frac{x^{2n}}{x^n+1}-\frac{1}{x^n-1}+\frac{1}{x^n+1}$

72 Find the square root of $a^2+2b\sqrt{a^2-b^2}$

---

73 If $x=\frac{\sqrt{3}-1}{\sqrt{3}+1}$ and $y=\frac{\sqrt{3}+1}{\sqrt{3}-1}$, find the value of x^3+y^3

74 Multiply $x^{\frac{1}{2}}+x^{-\frac{1}{2}}-1$ by $x^{\frac{1}{2}}+x^{-\frac{1}{2}}+1$

75 Factorize $ab(x+y)^2-(a+b)(x^2-y^2)+(x-y)^2$

76 Divide $x^4-x^3y-xy^3+y^4$ by x^2+xy+y^2, and find the value of the quotient, when $x=a^2-ab+b^2$, $y=a^2+ab+b^2$

77 Prove that $\left(\frac{1}{a}-\frac{1}{c}\right)^2+\frac{4}{(a+c)^2}=\left(\frac{a+c}{ac}-\frac{2}{a+c}\right)^2$

78 Simplify $\frac{1}{a-\sqrt{a^2-x^2}}-\frac{1}{a+\sqrt{a^2-x^2}}$.

79. Shew that b^2-ac is a factor of

$$(2b^2+a^2-ac)(2b^2+c^2-ac)-b^2(a+c)^2$$

and that the other factor is positive for all values of a, b and c

80 If $f(x)=3x^2+5x-4$ and $\phi(x)=1-2x+3x^2$, find the value of $f(x+1)-\phi(x-1)$

---

81 If $x+y=1$, prove that $x^3(y+1)-y^3(x+1)-x+y=0$

82 Factorize $x^3(a+1)-xy(x-y)(a-b)+y^3(b+1)$

83 If $a+b+c=0$, shew that

$$a^4+b^4+c^4+(bc+ca+ab)(a^2+b^2+c^2)=0$$

84 Divide 1 by $1-x$ to 5 terms What is the remainder ?

85 Prove that

$$\left\{\frac{(mn+1)^2}{m^2n^2}-\frac{4}{mn}\right\}x^2+2\left\{\frac{(mn+1)^2}{mn}-2mn-\frac{2}{mn}\right\}x+(mn+1)^2-4mn$$

is a perfect square, and find its square root

86 Find the value of $\frac{3+\sqrt{7}}{3-\sqrt{7}}-\frac{3-\sqrt{7}}{3+\sqrt{7}}$, correct to 5 decimal places.

87 Simplify $\frac{5^{n+2}}{6^{-n-1}}-\frac{15^{n+3}}{2^{2-n}}$.

88 If $f(x)=2x^2-3x+5$, find the value of $f(x+1)+f(0)-f(x-1)$

CHAPTER XXI

HARDER EQUATIONS

Equations in one Variable

230 Literal Equations The method of solution is the same as that for equations having numerical coefficients and will be seen from the following examples

Ex 1 Solve $ax-b=cx+d$

Transpose, thus $ax-cx=b+d$,

or $(a-c)x=b+d$,

divide by $a-c$; thus $x=\frac{b+d}{a-c}$

Ex. 2 Solve $\frac{ax}{b}-\frac{cx}{d}=1$

Here the L C D is bd, multiply therefore by bd; thus

$$adx-bcx=bd,$$

$$(ad-bc)x=bd,$$

$$x=\frac{bd}{ad-bc}.$$

Ex 3 Solve $\frac{a-x}{a}+\frac{2a-x}{2a}+\frac{x-3a}{3a}=0$

Multiply by $6a$, the L C D, thus

$$6(a-x)+3(2a-x)+2(x-3a)=0$$

$$7x=6a, \text{ or } x=\frac{6a}{7}$$

Ex 4 Solve $\frac{a^2x}{bc}-\frac{d^2}{a}+bx=\frac{ex}{f}-b+(d+b)x$

Multiply by $abcf$, thus

$$a^3fx-bcd^2f+ab^2cfx=abcex-ab^2cf+(d+b)abcfx,$$
$$(a^3f-abce-abcdf)x=bcd^2f-ab^2cf,$$
$$x=\frac{bcf(d^2-ab)}{a^3f-abce-abcdf}$$

Examples CXXXVIII

Solve the equations

1 $cx+d=a-3x$

2. $ax+bx-c=x+bx-d$

3 $(2m-1)x-bx=bx+m(2x-a)$

4 $(a-b)x-mx=c-mx$

5. $2ax+(2+x)(a-3)+4=0$

6 $2(a+1)x-5=(2a-3)x+20.$

7 $(m+n)(m-x)=m(n-x)$

8 $(ax+3)^2-(ax-3)^2=12$

9 $x=\frac{ax-b^2}{c}$

10. $\frac{a}{x}+\frac{b}{c}-\frac{d}{c}=0$

11 $\frac{a}{bx}+\frac{b}{ax}=a^2+b^2$

12 $\frac{a(d^2+x^2)}{dx}=ac+\frac{ax}{d}$

13 $\frac{x-a}{b}+\frac{x-b}{a}=0$

14 $\frac{a+bx}{a+b}=\frac{c-dx}{c-d}$

15 $\frac{ax}{b}+g=qx+h$

16 $a+\frac{bx}{a}=\frac{ax-b^2}{b}$

17 $\frac{b}{x}+\frac{2}{x}-a=\frac{3}{x}+2c$

18 $a+\frac{c-x}{x}=b-1+\frac{d}{x}$

19 $\frac{5}{6}ab+\frac{4}{5}ac-\frac{2}{3}cx=\frac{3}{4}ac+2ab-6cx$

20 $\frac{ax}{b}+\frac{cx}{f}+g=qx+\frac{1}{f}(fh-cx)$

21 $\frac{x}{bc}-\frac{3x-1}{ca}=\frac{x+3a}{ab}$

22 $\frac{a}{b}\left(1-\frac{a}{x}\right)+\frac{b}{a}\left(1-\frac{b}{x}\right)=1$

23 $\frac{ax-b}{m}-\frac{bx-c}{n}+\frac{dx}{2m}=\frac{3}{p}.$

24 $(a+x)(b+x)-a(b+c)=\frac{a^2c}{b}+x^2$

25 $ax-\frac{a^2-3bx}{a}-ab^2=bx+\frac{6bx-5a^2}{2a}-\frac{bx+4a}{4}$

26 $\frac{1}{ab-ax}+\frac{1}{bc-bx}=\frac{1}{ac-ax}.$

27 $\frac{x+a}{a}-\frac{2x}{x+a}=3-\frac{x^3-x^2a}{a^3-ax^2}$

28. $\frac{3bx}{2a^2}-\frac{x-b}{a+b}-\frac{bx-a^2}{a^2-b^2}+\frac{x}{4a}=0$

231 In solving equations, much labour is sometimes saved by a **suitable transposition of terms**, as for instance by transposing all the terms having *monomial* denominators to one side

Ex Solve $\dfrac{4x-5}{3}+\dfrac{10x-3}{20x+17}=\dfrac{12x-11}{9}$.

By transp,
$$\frac{10x-3}{20x+17}=\frac{12x-11}{9}-\frac{4x-5}{3}$$
$$=\frac{12x-11}{9}-\frac{12x-15}{9}=\frac{4}{9},$$

whence $9(10x-3)=4(20x+17)$,

or $10x=95$, $x=9\frac{1}{2}$

Examples CXXXIX

Solve the following equations

1 $\dfrac{7x+16}{21}-\dfrac{x+8}{4x-11}=\dfrac{x}{3}$

2 $\dfrac{2x}{5}+\dfrac{3x+5}{5x-25}=\dfrac{6x+13}{15}$

3 $\dfrac{8x+5}{14}+\dfrac{7x-3}{6x+2}=\dfrac{4x+6}{7}$

4 $\dfrac{10x+17}{18}-\dfrac{12x+2}{11x-8}=\dfrac{5x-4}{9}$

5 $\dfrac{1}{x-1}-\dfrac{2}{x+7}=\dfrac{1}{7(x-1)}$.

6 $\dfrac{9x+20}{36}=\dfrac{4x-12}{5x-4}+\dfrac{x}{4}$

7 $\dfrac{2x+8\frac{1}{2}}{9}+\dfrac{x}{3}+\dfrac{x+16}{36}=\dfrac{7x}{12}-\dfrac{2-13x}{17x-32}$.

8 $\dfrac{6x-7\frac{1}{3}}{15-2x}+2x+\dfrac{1+16x}{24}=4\frac{5}{12}-\dfrac{12\frac{5}{8}-8x}{3}$

232 In certain cases, solution is very easily effected by the actual **division of numerator by denominator**

Ex. 1 Solve $\dfrac{2x^2+3x+1}{x+2}=\dfrac{2x^2-7x+5}{x-3}$

By actual division,
$$2x-1+\frac{3}{x+2}=2x-1+\frac{2}{x-3}\ \text{[Art 137]},$$
$$\frac{3}{x+2}=\frac{2}{x-3},\ \text{whence } x=13$$

Ex 2 Solve $\dfrac{x-1}{x-2}-\dfrac{x-2}{x-3}=\dfrac{x-5}{x-6}-\dfrac{x-6}{x-7}$

By division, $1+\dfrac{1}{x-2}-1-\dfrac{1}{x-3}=1+\dfrac{1}{x-6}-1-\dfrac{1}{x-7}$ [Art 137],

or
$$\frac{1}{x-2}-\frac{1}{x-3}=\frac{1}{x-6}-\frac{1}{x-7},$$
$$\frac{x-3-x+2}{(x-2)(x-3)}=\frac{x-7-x+6}{(x-6)(x-7)},$$
$$\frac{-1}{(x-2)(x-3)}=\frac{-1}{(x-6)(x-7)},$$
$$(x-2)(x-3)=(x-6)(x-7),$$
whence
$$x=4\tfrac{1}{2}$$

Examples CXXXIX. *(Continued)*

Solve the following equations

9 $\dfrac{15x+8}{3x-7}=\dfrac{25x-62}{5x-21}$ 10 $\dfrac{mx-a}{nx-b}=\dfrac{mx-c}{nx-d}$

11 $\dfrac{a^2x^2+b^2}{ax-b}=\dfrac{a^2x^2+2abx-2b^2}{ax+b}$ 12 $\dfrac{ax^2+bx+c}{px^2+qx+r}=\dfrac{ax+b}{px+q}$

13 $\dfrac{x^3+2x^2+1}{x^2+2x}=x+\dfrac{3}{x+2}$ 14 $\dfrac{x-1}{x-3}+\dfrac{x-9}{x-11}=\dfrac{x-3}{x-5}+\dfrac{x-7}{x-9}$

15 $\dfrac{ax-2}{ax-3}-\dfrac{ax-3}{ax-4}=\dfrac{ax-5}{ax-6}-\dfrac{ax-6}{ax-7}$

16 $\dfrac{7x-26}{x-4}-\dfrac{4x-21}{x-6}=\dfrac{9x-68}{x-8}-\dfrac{6x-55}{x-10}$

17 $\dfrac{x+2}{x-2}+\dfrac{x+5}{x-5}=\dfrac{x+3}{x-3}+\dfrac{x+4}{x-4}$

18 $\dfrac{4x-17}{x-4}+\dfrac{10x-13}{2x-3}=\dfrac{8x-30}{2x-7}+\dfrac{5x-4}{x-1}$

19 $\dfrac{x^2+5x+4}{x+3}+\dfrac{x^2+5x-3}{x+4}=\dfrac{2x^2+7x-3}{x+2}$

20 $\dfrac{x^2+2x+2}{x+1}+\dfrac{x^2+8x+20}{x+4}=\dfrac{x^2+4x+6}{x+2}+\dfrac{x^2+6x+12}{x+3}$

233 Fractional equations are sometimes easily solved by splitting a term into partial fractions.

Ex. Solve $\dfrac{a}{x-3a}-\dfrac{b}{x+3b}=\dfrac{2c}{x-3c}$

Since $$\frac{2c}{x-2c}=\frac{c}{x-3c}+\frac{c}{x-3c},$$

we have $$\frac{a}{x-3a}-\frac{c}{x-3c}=\frac{c}{x-3c}+\frac{b}{x+3b},$$

or $$\frac{(a-c)x}{(x-3a)(x-3c)}=\frac{(b+c)x}{(x-3c)(x+3b)}$$

dividing by $\frac{x}{x-3c}$, $$\frac{a-c}{x-3a}=\frac{b+c}{x+3b},$$

or $$(a-c)x+3b(a-c)=(b+c)x-3a(b+c),$$

i e, $$x(a-b-2c)=3(bc-ca-2ab),$$

$$x=\frac{3(bc-ca-2ab)}{a-b-2c}.$$

Examples CXXXIX. *(Continued.)*

Solve the following equations

21 $\frac{5}{x+15}+\frac{8}{x-12}=\frac{3}{x-9}$ 22 $\frac{11}{x-33}-\frac{13}{x+39}=\frac{20}{x-30}$.

23 $\frac{m}{mx+a}+\frac{n}{nx+a}=\frac{2p}{px+a}$. 24 $\frac{x-1}{x+1}+\frac{x+3}{x-3}=2\frac{x+2}{x-2}$

25 $\frac{a}{x+a}-\frac{b}{x+b}=\frac{a-b}{x-c}$ 26 $\frac{a}{b^2+abx}=\frac{2a}{1+abx}-\frac{1}{a+bx}$

27 $\frac{m+n}{x+m+n}+\frac{m-n}{x+m-n}=\frac{2m}{x+n}$

234 In the following examples two or more of the preceding methods are combined

Ex 1. Solve $\frac{7x+1}{x-1}=\frac{35}{9}\cdot\frac{x+4}{x+2}+3\frac{1}{9}$

We may proceed in the usual way, or perhaps thus —

By division, $$7+\frac{8}{x-1}=\frac{35}{9}\,\frac{x+4}{x+2}+3\tfrac{1}{9},$$

or $$\frac{8}{x-1}=\frac{35}{9}\,\frac{x+4}{x+2}+3\tfrac{1}{9}-7=\frac{35}{9}\,\frac{x+4}{x+2}-\frac{35}{9}=\frac{35}{9}\cdot\frac{2}{x+2},$$

or $$\frac{4}{x-1}=\frac{35}{9(x+2)},$$

$$36(x+2)=35x-35, \text{ or } x=-107$$

Ex 2 Solve $\dfrac{m(x+a)}{x+b}+\dfrac{n(x+b)}{x+a}=m+n$

From the given equations, we have

$$\frac{m(x+a)}{x+b}-m+\frac{n(x+b)}{x+a}-n=0,$$

or
$$\frac{m(a-b)}{x+b}+\frac{n(b-a)}{x+a}=0,$$

dividing both sides by $a-b$, we have

$$\frac{m}{x+b}-\frac{n}{x+a}=0$$

or
$$\frac{m}{x+b}=\frac{n}{x+a}, \text{ whence } x=\frac{nb-ma}{m-n}.$$

Ex 3 Solve $\dfrac{x+3}{x+1}-\dfrac{x+4}{x+2}+\dfrac{x-6}{x-4}=\dfrac{x^2-2x-15}{x^2-9}$

$$\frac{x+3}{x+1}-\frac{x+4}{x+2}+\frac{x-6}{x-4}=\frac{(x+3)(x-5)}{x^2-9}=\frac{x-5}{x-3}$$

$$\frac{x+3}{x+1}-\frac{x+4}{x+2}=\frac{x-5}{x-3}-\frac{x-6}{x-4},$$

by division $1+\dfrac{2}{x+1}-1-\dfrac{2}{x+2}=1-\dfrac{2}{x-3}-1+\dfrac{2}{x-4},$

$$\frac{1}{x+1}-\frac{1}{x+2}=\frac{1}{x-4}-\frac{1}{x-3},$$

or
$$\frac{1}{(x+1)(x+2)}=\frac{1}{(x-4)(x-3)},$$

whence
$$(x+1)(x+2)=(x-4)(x-3),$$

or
$$x=1$$

Ex 4 Solve $\dfrac{3x-2}{4}+\dfrac{x}{2}-11\tfrac{5}{6}=\dfrac{x-\dfrac{4x-9}{3}}{6}-5$

Multiply by 12, the L C D , thus

$$9x-6+6x-142=2x-\frac{8x-18}{3}-60,$$

or
$$13x=88-\frac{8x-18}{3},$$

multiply by 3, thus $39x=264-8x+18,$

whence $x=6$

Ex 5 Solve $\frac{bc(ax-1)}{b+c}+\frac{ca(bx-1)}{c+a}+\frac{ab(cx-1)}{a+b}=a+b+c$ [Cal, 1902]

By trans, we have

$$\frac{bc(ax-1)}{b+c}-a+\frac{ca(bx-1)}{c+a}-b+\frac{ab(cx-1)}{a+b}-c=0,$$

$$\frac{abcx-bc-ca-ab}{b+c}+\frac{abcx-bc-ca-ab}{c+a}+\frac{abcx-bc-ca-ab}{a+b}=0,$$

$$\text{or } (abcx-bc-ca-ab)\left(\frac{1}{b+c}+\frac{1}{c+a}+\frac{1}{a+b}\right)=0$$

Thus either the first factor$=0$, or the second factor$=0$, but the second factor cannot obviously be equal to 0,

$$abcx-bc-ca-ab=0,$$

whence $$x=\frac{bc+ca+ab}{abc}=\frac{1}{a}+\frac{1}{b}+\frac{1}{c}$$

Examples CXXXIX. *(Continued)*

Solve the following equations

28 $\frac{x}{x-1}-\left(1+\frac{2}{x}\right)=0$

29. $\frac{3}{x}-\frac{2}{x+1}=\frac{5}{4}\frac{1}{x+1}$

30 $\frac{30+6x}{x+1}+\frac{60+8x}{x+3}=14+\frac{48}{x+1}$

31 $5x+\frac{7x+9}{4x+3}=9+\frac{10x^2-18}{2x+3}$.

32 $\frac{3x+2}{x-3}-\frac{3x-2}{x+3}=\frac{4x+36}{x^2-9}$.

33 $\frac{2x}{3}-\frac{1-\frac{x}{2}}{4x}=\frac{x-1}{2}+\frac{x}{6}+\frac{7}{12}$

34 $\frac{25-\frac{1}{3}x}{x+1}+\frac{16x+4\frac{1}{5}}{3x+2}=5+\frac{23}{x+1}$

35 $\frac{6-5x}{15}-\frac{7-2x^2}{14(x-1)}=\frac{1+3x}{21}-\frac{2x-2^1}{6}+\frac{1}{105}$

36. $\frac{1}{2}\left(\frac{2x}{3}+4\right)-\frac{7\frac{1}{2}-x}{3}=\frac{x}{2}\left(\frac{6}{x}-1\right)$

37 $\frac{132x+1}{3x+1}+\frac{8x+5}{x-1}=52$

38 $\frac{4x-17}{9}-\frac{3\frac{2}{3}-22x}{33}=x-\frac{6}{x}\left(1-\frac{x^2}{54}\right)$

39 $\frac{\frac{3x}{2}-4}{6}-\frac{4x-7}{9}+x=\frac{8-\frac{x+4}{2}}{3}+2.$

Solve the following equations

40 $\frac{3x}{2}-\frac{81x^2-9}{(3x-1)(x+3)}=3x-\frac{3}{2}\frac{2x^2-1}{x+3}-\frac{57-3x}{2}$

41 $\frac{x+6}{x+2}+\frac{x+10}{x+3}=\frac{2x^2+10x+49}{x^2+5x+6}$.

42 $\frac{3+2x}{1+2x}-\frac{5+2x}{7+2x}=1-\frac{4x^2-2}{7+16x+4x^2}$

43 $\frac{1}{x-30}+\frac{2}{x+15}+\frac{3}{x-10}=\frac{6}{x-5}$

44. $\frac{(x+a)(x+b)}{x+a+b}=\frac{(x+c)(x+d)}{x+c+d}$

45 $\frac{1}{1+ax}-\frac{1}{1-bx}=\frac{a+b}{1+abx}$.

46 $\frac{x+a}{x+b}=\left(\frac{2x+a+c}{2x+b+c}\right)^2$

47 $\left(\frac{x-a}{x-b}\right)^3=\frac{x-2a+b}{x-2b+a}$

48 $\frac{x}{2}-\frac{\frac{2x-3}{3}-\frac{3x-1}{4}}{\frac{x-1}{2}}=\frac{3}{2}\frac{x^2+2}{3x-2}$

49 $\frac{x-b}{x-a}-\frac{x-a}{x-b}=\frac{2(a-b)}{x-(a+b)}$

50 $\frac{1}{x+6a}+\frac{2}{x-3a}+\frac{3}{x+2a}=\frac{6}{x+a}$.

51 $a-ax\left(1-\frac{1}{x}\right)=a(a+x)\left(1+\frac{1}{x}\right)+a^2\left(1-\frac{1}{x}\right)-a$

52 $\frac{x-a}{3b+5c}+\frac{x-3b}{5c+a}+\frac{x-5c}{a+3b}=3$

[Observe that $3=1+1+1$ See Ex. 5]

53 $\frac{x-a}{b+c+2a}+\frac{x-b}{c+a+2b}+\frac{x-c}{a+b+2c}+3=0$ [See Ex 5]

235 Irrational Equations AN IRRATIONAL EQUATION is that in which one or more terms are surds involving the variable

The principle to be followed in solving such equations, is to *reduce them to a rational form* This is done by leaving *one* of the surd terms on one side of the given equation and transposing *all the rest* to the other side, and then raising both sides to the power whose root the surd expresses ; this process being repeated till we finally get rid of all the radicals in the equation

Properly speaking most of these equations are not *simple equations*, for when reduced to a *rational form*, they will often be seen to be of a *higher degree* We have here given a few examples to shew that some of them can be solved as simple equations

Ex 1 Solve $\sqrt{x-2}=3$.

Here we get rid of the radical by simply squaring ,

$$x-2=9, \text{ or } x=11$$

Ex 2 Solve $\sqrt[m]{3x+a}=b$

Raise to the m^{th} power , thus

$$3x+a=b^m$$

$$x=\tfrac{1}{3}(b^m-a)$$

Ex 3 Solve $\sqrt{a^2-\sqrt{mx+b^4}}=c$

Square , thus $\quad a^2-\sqrt{mx+b^4}=c^2$

transpose , thus $\quad \sqrt{mx+b^4}=a^2-c^2,$

square again, $\quad mx+b^4=(a^2-c^2)^2,$

$$x=\frac{1}{m}\{(a^2-c^2)^2-b^4\}$$

Ex 4 Solve $\sqrt[n]{x+15}=\sqrt[2n]{x^2+75x-135}$.

We have $\quad (x+15)^{\frac{1}{n}}=(x^2+75x-135)^{\frac{1}{2n}}$,

raise to $2n^{th}$ power , thus

$$\{(x+15)^{\frac{1}{n}}\}^{2n}=\{(x^2+75x-135)^{\frac{1}{2n}}\}^{2n},$$

$$(x+15)^2=x^2+75x-135,$$

whence $\quad x=8$

Ex 5 Solve $\sqrt{7+x}-\sqrt{x}=1$

By transp , $\quad \sqrt{7+x}=1+\sqrt{x},$

squaring, $\quad 7+x=1+2\sqrt{x}+x,$

$$2\sqrt{x}=6, \text{ or } x=9$$

Ex 6 Solve $1+\sqrt{1+x}-\sqrt{1+x+\sqrt{1-x}}=0$

Transpose, thus $\sqrt{1+x+\sqrt{1-x}}=1+\sqrt{1+x},$

square, $\quad 1+x+\sqrt{1-x}=1+1+x+2\sqrt{1+x},$

or $\quad \sqrt{1-x}=1+2\sqrt{1+x},$

square again, $\quad 1-x=1+4(1+x)+4\sqrt{1+x},$

or $\quad -4\sqrt{1+x}=5x+4,$

$$16(1+x)=25x^2+40x+16,$$

$$25x^2=-24x,$$

divide by x, $\quad 25x=-24, \text{ or } x=-\tfrac{24}{25}$

Ex 7 Solve $\sqrt{x}+\sqrt{x-a}=\dfrac{a}{\sqrt{x-a}}$

Multiply by $\sqrt{x-a}$, thus

$$\sqrt{x^2-ax}+x-a=a,$$
$$\sqrt{x^2-ax}=2a-x,$$
$$x^2-ax=4a^2-4ax+x^2,$$
$$3ax=4a^2, \text{ or } x=\frac{4a}{3}.$$

Examples CXL

Solve the following equations

1 $\sqrt{5x-1}=7$

2 $\sqrt{3x-a}=\sqrt{2x}$

3 $\sqrt{30+2x}=5-\sqrt{2x}$

4 $\sqrt[3]{3x+7}-1=3$

5 $\sqrt[3]{3+\sqrt{x}}=2$

6 $\sqrt[3]{8x-1}=\sqrt[3]{3x+a}$

7 $\dfrac{\sqrt[4]{x+12}}{2}+\dfrac{3}{4}=1$

8 $\dfrac{2\sqrt[3]{7x-6}}{3}+\dfrac{3}{4}=\dfrac{25}{12}$

9 $\sqrt{5}\sqrt{x+2}=\sqrt{5x}+2$

10 $\sqrt{x+9}=1+\sqrt{x}$

11 $\sqrt{x}-\dfrac{\sqrt{x}}{5}=\sqrt{x-9}$

12 $\sqrt{x}-\sqrt{\dfrac{a}{x}}=\sqrt{a+x}$

13 $a\sqrt[4]{x+m}=\sqrt[3]{x+m}$

14 $\sqrt{x}+\sqrt{3+x}=\dfrac{12}{\sqrt{3+x}}$

15 $\sqrt[m]{a-x}=\sqrt[2m]{3a^2-6ax+x^2}$

16 $\sqrt{x}-\sqrt{x+\sqrt{1-x}}=1$

17 $\dfrac{\sqrt{x}+28}{\sqrt{x}+4}=\dfrac{\sqrt{x}+38}{\sqrt{x}+6}$

18 $\sqrt{1+x}+\sqrt{1-x}=4\sqrt[4]{1-x^2}$

19 $\sqrt{4a+x}+\sqrt{x}=2\sqrt{1+x}$

20 $\dfrac{5x-9}{\sqrt{5x}+3}=1+\dfrac{\sqrt{5x}-3}{2}$

21 $\dfrac{4x-9}{2\sqrt{x}-3}=3+\dfrac{2\sqrt{x}+20}{5}$

22 $\dfrac{6x-2}{\sqrt{3x}-1}=4+\dfrac{\sqrt{3x}+1}{2}$

23 $\dfrac{ax-b^2}{\sqrt{ax}+b}=\dfrac{\sqrt{ax}-b}{c}-c$

24 $\sqrt{x}-\sqrt{a-\sqrt{ax+x^2}}=\sqrt{a}$

25 $\sqrt{x+\sqrt{x}}-\sqrt{x-\sqrt{x}}=\dfrac{3}{2}\sqrt{\dfrac{x}{x+\sqrt{x}}}$

Solve the following equations

26 $\sqrt{\dfrac{a+x}{x}}+\sqrt{\dfrac{a-x}{x}}=\sqrt{\dfrac{x}{b}}$

27 $\dfrac{1}{a}\sqrt[n]{a+x}+\dfrac{1}{x}\sqrt[n]{a+x}=\dfrac{1}{c}\sqrt[n]{x}$ 28 $\dfrac{\sqrt{a}-\sqrt{a-x}}{\sqrt{a}+\sqrt{a-x}}=a$

29 $\dfrac{1}{x}+\dfrac{1}{a}=\sqrt{\dfrac{1}{a^2}+\sqrt{\dfrac{4}{a^2x^2}+\dfrac{9}{x^4}}}$

236 Examples of Irrational Equations in the form of the **sum or difference of two cube roots**

Ex Solve $\sqrt[3]{4+x}+\sqrt[3]{4-x}=2$

Cube both sides, thus

$$4+x+4-x+3\sqrt[3]{16-x^2}(\sqrt[3]{4+x}+\sqrt[3]{4-x})=8,$$

$$8+3\sqrt[3]{16-x^2}\times 2=8, \quad \because \sqrt[3]{4+x}+\sqrt[3]{4-x}=2,$$

$$\sqrt[3]{16-x^2}=0,$$

or $$16-x^2=0, \text{ whence } x=4$$

Examples CXL *(Continued)*

Solve the following equations

30 $\sqrt[3]{76+x}+\sqrt[3]{76-x}=8$

31 $\sqrt[3]{a+\sqrt{x}}+\sqrt[3]{a-\sqrt{x}}=\sqrt[3]{b}$

32 $\sqrt[3]{x+a}-\sqrt[3]{x-a}=c$

33 $\sqrt[3]{5x+158}-\sqrt[3]{5x-158}=4$

237 Exponential Equations Let $a^x=c$ be an equation in which the variable x occurs as an *exponent* The student at once sees that these equations are quite different in nature from those which we have treated in the previous articles Of the constants a and c, the former is called the *base* of x, and in solving these equations, the principle which we have to follow is to *reduce a given equation to such a form that the two sides may have the same base*

Ex 1 Solve $a^x=\dfrac{1}{a^{-3}}$ Here $\dfrac{1}{a^{-3}}=a^3$, $a^x=a^3$, whence $x=3$

Ex 2 Solve $2\times 4^x=8^{x-1}$

Here $2\times 4^x=2(2^2)^x=2\cdot 2^{2x}=2^{2x+1}$,

and $8^{x-1}=(2^3)^{x-1}=2^{3x-3}$,

$2^{2x+1}=2^{3x-3}$, or $2x+1=3x-3$, whence $x=4$

Examples CXLI

Solve the following equations

1 $m^{3x+1}=1.$ 2 $\frac{(27)^x}{3^{2a-1}}=9^x$ 3 $\left(\frac{a}{c}\right)^{x-m}=\left(\frac{c}{a}\right)^{3n}$

4 $2^{x-6}a^{x-8}=4$ 5 $a^{3x-2m}=b^{3x-2m}$

6 $5^{1-x}(25)^3=3^{1-x}9^3$ 7 $3^{a+2x}a^{2x-3}=3^{a+4}9a^{x}$

8 $e^x(e^{x+4}-e^{4-x})=\frac{e^{a-4}-1}{e^{-4}}$ 9 $1+4^x=2^{x+1}+9$

10 $a^{x^3}=\{(\sqrt{a})^{3x}\}^a$ 11 $a^{2x^m}=\{(a^x)^m\}^x$

*238 *Definitions* The word *expression* has been defined before [Art 9]

An **integral expression** is one in which no letter appears in the denominator of any term, or has a negative index Thus $x^2+\frac{1}{2}ax+\frac{2}{3}b$ is an integral expression, but not so is

$$x^2+\frac{1}{x}+a \text{ or } x^2+ax^{-1}+c$$

A **rational expression** is one in which the letters are free from radicals or fractional indices Thus ax^3-2abx^2+c is a rational expression, but not so is $ax^{\frac{1}{2}}-2a\sqrt{bx}+\sqrt[3]{c}$

An **integral function of** x is one in which x occurs in a form having only positive integral index, thus $ax^2+\frac{x}{b}+\frac{c}{d}$ is an integral function of x, but not of b or d

A **rational function of** x is one in which x occurs in a form free from radicals or fractional indices thus ax^3+bx^2+cx+d is a *rational* function of x, but not so is $ax+b\sqrt{x}+c$, where the second term is *irrational*

REMARK It is to be noted that these definitions *refer only to the symbol of reference* as x in the above examples, and therefore any one or more of the *coefficients* a, b, c, &c, may be *surd* or *fractional*, thus $px^2+\frac{x}{q}+\sqrt{r}\,x+s^{-1}$ is a rational and integral fraction of x, though some of the coefficients are surds and fractions

If two functions of a letter are equal to one another for some particular value or values of that letter, the equality is termed an **equation** [See Art 83.]

239. It is clear from the last article that we cannot ascertain the degree of an equation unless it is in a *rational and integral form*. For though the equations

$$\frac{1}{x}+c=0,\ 2\sqrt{x}+3=0,\ \frac{1}{x}+x=0,\ x+2\sqrt{x}+1=0,$$

appear to be of the first degree, some of them are in reality not of that degree. Hence to ascertain the degree of an equation, we must see whether it is in a rational and integral form as far as the variable is concerned. If any proposed equation is not in a rational and integral form, we must reduce it to that form and then ascertain its degree.

To reduce an equation to an integral form, we multiply by the denominator, thus $x+\frac{2}{x}=3$ becomes $x^2+2=3x$, or $x^2-3x+2=0$, when put in an integral form.

To reduce an equation to a rational form, we transpose and raise to the power denoted by the index of the radical, thus $ax+b\sqrt{x}+c=0$ becomes $(ax+c)^2=b^2x$, or $a^2x^2+(2ac-b^2)x+c^2=0$, when put in a rational form.

Hence we have the following

Definition. If after reducing an equation (if not already so reduced) to a rational and integral form with respect to the variable, it is found that the term of the highest degree in the variable is of *one*, *two*, *three*, *four* or n dimensions, it is said to be respectively of the *first*, *second*, *third*, *fourth* or n^{th} degree.

An equation of the *first* degree is commonly called a **simple** or **linear** equation, and an equation of the *second* degree is called a **quadratic** equation. Thus

$$ax+b=0,\ a\sqrt{x}+b=0,\ \frac{a}{x}+b=0,$$

are *simple* or *linear* equations, for when reduced to *rational and integral forms*, the last two become

$$a^2x-b^2=0, \text{ and } bx+a=0,$$

similarly

$$ax^2+bx+c=0,\ ax+\frac{b}{x}+c=0,\ ax+\sqrt{x}+b=0,$$

are *quadratic* equations, for the last two may be reduced to the forms

$$ax^2+cx+b=0, \text{ and } a^2x^2+(2ab-1)x+b^2=0,$$

and so on.

Examples CXLII

Determine the degree of the equations

1. $x-\frac{1}{x}=a$
2. $x^{-1}-x^2=c.$
3. $3x+\sqrt{x}=a\sqrt{x}$
4. $\frac{1}{x}-\frac{1}{y}=\frac{1}{a}$

***240 General Form.** *A simple equation in one variable can always be reduced to the general form* $ax+b=0$

For when the equation has been reduced to a rational and integral form [Art. 239], the terms involving x may be collected together, when we may bracket the coefficients of x, thus finding the a, and the constant terms being bracketed together, we find the b Thus $\frac{a}{x}+b-\frac{c}{x}=d$ becomes $a+bx-c=dx$ when we multiply by x, then by transposition, it assumes the form $(b-d)x+(a-c)=0$, hence here $b-d$ is the a, and $a-c$ is the b

Examples CXLIII

Reduce the following equations to the general form

1. $5x-\frac{3}{2}x+3+\frac{x}{2}=0$
2. $ax+b-cx=d$
3. $\frac{1}{x}+\frac{3}{2}+\frac{8}{x}=1$
4. $x=\frac{ax-b^2}{c}$
5. $\frac{a}{c}+\frac{b}{c}-\frac{d}{c}=0$
6. $1-\frac{x-2}{5}=\frac{x+2}{1}.$

***241 Solution of Simple Equations** We have seen [Art 240] that every simple equation in one variable can be reduced to the form $ax+b=0$ Thus the solution is $x=-\frac{b}{a}$

We shall examine this solution when a or b, or both a and b, have the special value 0

(i) When $a\neq 0$ but $b=0$, then $x=-\frac{b}{a}=\frac{0}{a}=0$

Thus *when* b=0, *the value of* x *is* 0

(ii) When $a=0$ and $b=0$, the equation takes the form $0x=0$, and any value of x will evidently satisfy it

Thus *when* a=0 *and* b=0, x *may have any value*

(iii) When $a=0$ and $b\neq 0$

Let a be very small, say, $\frac{1}{10^8}$, then $x=-\frac{b}{a}=-10^8b=-100000000b$, a very large quantity By taking a smaller than $\frac{1}{10^8}$, the value of x -

will be seen to be still greater than $-10^3 b$ Hence as a diminishes, x continually increases in absolute value, and at last when a is equal to 0, x has an infinitely great value

Thus *when* a=0, *and* b≠0, *the value of* x *is infinite*

242. Theorem *A simple equation has only one root and no more*

If possible, let the equation $ax+b=0$ have two *different* roots, *viz*, α and β Now since α and β are its roots, they will severally satisfy it [Art 85] therefore

$$a\alpha+b=0 \qquad \text{(i)},$$

$$a\beta+b=0 \qquad \text{(ii)}$$

Subtract (ii) from (i), thus $a(\alpha-\beta)=0$ Now either $a=0$, or $\alpha-\beta=0$ If $a=0$, the given equation reduces to $b=0$, which cannot be satisfied by any *finite value* of x [Art 241], therefore a is not $=0$

Therefore $\alpha-\beta=0$ or $\alpha=\beta$, *i e*, α and β are not two *different* quantities but *one* and the *same* quantity

Equations in two or more variables

243 Literal Equations The method of solution is the same as that for equations having numerical coefficients [see Art 96], and will be illustrated by the following examples

Ex 1. Solve $ax+y=a^2+b$ ...... (i),

$x-by=a-b^2$. ...... (ii)

Multiply (i) by b, thus

$$abx+by=a^2b+b^2$$

And from (ii) $x-by=a-b^2$

by addition, $(ab+1)x=a^2b+a=a(ab+1)$,

$$x=a \qquad \text{(iii)}$$

From (i) and (iii), $y=a^2+b-ax=a^2+b-a^2=b$

Ex 2 Solve $ax+by=c$ ...... (i),

$a'x+b'y=c'$ ...... (ii)

Multiply (i) by b', thus

$$ab'x+bb'y=b'c \qquad \text{(iii)}$$

Multiply (ii) by b, thus

$$a'bx+bb'y=bc'. \qquad \text{(iv)}$$

Subtract (iv) from (iii), thus

$$(ab'-a'b)x=b'c-bc',$$

whence

$$x=\frac{b'c-bc'}{ab'-a'b}$$

Substitute in (i), thus

$$by=c-ax=c-\frac{a(b'c-bc')}{ab'-a'b}=\frac{b(c'a-ca')}{ab'-a'b},$$

$$y=\frac{c'a-ca'}{ab'-a'b}$$

Examples CXLIV

Solve the equations

1 $x+y=a,$ $x-y=b$

2 $x+ay=b,$ $ax-by=c$

3 $bx+ay=b,$ $ax-by=a$

4 $ax+by=c,$ $mx-ny=d$

5 $ax+by+c=0,$ $a'x+b'y+c'=0$

6 $(a+b)x+(a-b)y=m$ $(a-b)x-(a+b)y=n$

7 $a(x+y)+b(x-y)=a^2-ab+b^2,$
$a(x+y)-b(x-y)=a^2+ab+b^2$
[Put $x-y=u$, $x+y=v$]

8 $a(x-y)-b(x+y)=c(x-y)-d(x+y)=2.$

9 $\frac{x-y}{a}+\frac{x+y}{b}=c,$ $\frac{x-y}{b}-\frac{x+y}{a}=c$

10 $ax+by=1=bx-\frac{b}{a}+ay-\frac{a}{b}.$

11 $2x-a-\frac{y-b}{a+b}=a,$ $2y-b+\frac{x-a}{b}=2b-y$

12 $\frac{x}{a}+\frac{y}{b}=1,$ $\frac{x}{b}+\frac{y}{a}=1$

13 $\frac{x}{a}+\frac{y}{b}=1,$ $\frac{x}{3a}+\frac{y}{6b}=\frac{2}{3}$

14 $\frac{x}{a}+\frac{y}{b}=1,$ $\frac{b}{x}+\frac{a}{y}=\frac{a^3}{bxy}$

15 $\frac{x}{a}+\frac{y}{b}=2,$ $ax-by=a^2-b^2$

16 $\frac{x}{a}-\frac{y}{b}=m,$ $\frac{x}{c}+\frac{y}{d}=n$

Solve the equations

17 $\frac{(m+n)x}{5}-\frac{(m-n)y}{3}=4mn,$ 18 $\frac{x}{a}+\frac{y}{b}=1-\frac{x}{c},$

$3x+\frac{5(m+n)y}{m-n}=30(m+n)$ $\frac{x}{b}+\frac{y}{a}=1+\frac{y}{c}$

19 $\frac{x}{a}+\frac{y}{b-a}=5m,$ 20 $\frac{ax}{b+c}+\frac{by}{a+c}=1,$ 21. $\frac{x}{b+c}+\frac{y}{a+c}=2,$

$\frac{x}{b}+\frac{y}{a-b}=7m$ $\frac{2x}{b+c}+\frac{2y}{a+c}=\frac{1}{a}+\frac{1}{b}$ $\frac{ax-by}{(a-b)c}=1$

22 $(a^2-b^2)x-(a^2-ab+c^2)y=a(a-2b)-\frac{bc^2}{a-b},$

$\frac{x}{a}+\frac{y}{b}=\frac{2a}{a^2-b^2}$

Ex 3 Solve $\frac{m}{x}+\frac{n}{y}=a$ (i),

$\frac{n}{x}+\frac{m}{y}=b$ (ii)

Here we proceed as in Art 101, Ex. 2, by finding the *reciprocals* of the variables.

Substitute u for $\frac{1}{x}$ and v for $\frac{1}{y}$, thus we have

$mu+nv=a$ (iii),

$nu+mv=b$ . . (iv)

Then proceed as in Ex. 2 to find u and v, and therefore x and y

Thus it will be found that $x=\frac{m^2-n^2}{ma-nb}$, $y=\frac{m^2-n^2}{mb-na}$.

Another method of solving this equation

Add together (iii) and (iv); thus

$(m+n)u+(m+n)v=a+b$

or $u+v=\frac{a+b}{m+n}$. (v)

Multiply (v) by n, and subtract from (iii); thus

$(m-n)u=a-\frac{n(a+b)}{m+n}=\frac{ma-nb}{m+n},$

$u=\frac{ma-nb}{m^2-n^2}.$

Similarly, or by substituting v in (iii) or (iv), we can find v.

A third method of solving this equation

Multiply (iii) by b and (iv) by a, and subtract, thus

$$(mb-na)u+(nb-ma)v=0,$$

$$i\ e,\quad (mb-na)u=(ma-nb)v,$$

or
$$\frac{u}{ma-nb}=\frac{v}{mb-na}=\lambda \text{ suppose}$$

Thus $u=\lambda(ma-nb),\ v=\lambda(mb-na)$ (vi)

Hence from (iii) and (vi), we have

$$\lambda m(ma-nb)+\lambda n(mb-na)=a,$$

whence
$$\lambda=\frac{1}{m^2-n^2}$$

from (vi),
$$u=\frac{ma-nb}{m^2-n^2},\ v=\frac{mb-na}{m^2-n^2}$$

Examples CXLIV (*Continued*)

Solve the equations

23 $\frac{1}{x}+\frac{1}{y}=\frac{1}{a}$, $\frac{1}{x}-\frac{1}{y}=\frac{1}{b}$

24. $\frac{a}{x}+\frac{1}{y}=m$, $\frac{1}{x}+\frac{b}{y}=n$

25 $\frac{a}{x}+\frac{b}{y}=m$, $\frac{c}{x}+\frac{d}{y}=n$

26 $\frac{b}{ax}+\frac{ay}{b}=a+b$, $\frac{a}{x}+by=a^2+b^2$

27 $\frac{a}{b+y}=\frac{b}{a-x}$, $\frac{c}{d-x}=\frac{d}{c+y}$

28 $\frac{m}{x}-\frac{n}{y}=a$, $px=qy$

29 $axy=c(bx+ay)$, $bxy=c(ax-by)$

30 $\frac{b}{x}+\frac{a+c}{y}=m$, $\frac{a-c}{x}+\frac{b}{y}=n$

31 $\frac{a-b}{x}+\frac{a+b}{y}=2\frac{a^2+b^2}{a^2-b^2}$, $\frac{a+b}{x}+\frac{a-b}{y}=2$

Ex. 4 Solve $\frac{x}{q}+\frac{y}{p}=p^2+q^2$ . . (i),

$$\frac{x-pq^2}{p-q}=\frac{y+p^2q}{p+q} \quad \text{(ii)}$$

When as in this example one of the equations [here (ii)] is such that x *alone* occurs on one side and y *alone* on the other, the equations are better solved as below

Let $$\frac{x-pq^2}{p-q}=\frac{y+p^2q}{p+q}=k,$$

thus $$x=k(p-q)+pq^2,\quad y=k(p+q)-p^2q \quad \text{... (iii)}$$

Substitute for x and y in (i), thus

$$\frac{k(p-q)+pq^2}{q}+\frac{k(p+q)-p^2q}{p}=p^2+q^2,$$

or $$kp(p-q)+p^2q^2+kq(p+q)-p^2q^2=pq(p^2+q^2),$$

whence $$k=pq$$

from (iii), $$x=pq(p-q)+pq^2=p^2q,$$
$$y=pq(p+q)-p^2q=pq^2$$

Examples CXLIV (*Continued*)

Solve the equations

32 $\dfrac{x}{a}=\dfrac{y}{b}$, $\dfrac{x}{b}+\dfrac{y}{a}=c$

33 $\dfrac{x+a}{b}=\dfrac{y+b}{a}$, $x+y=a+b$.

34 $\dfrac{x+a}{2a+b}=\dfrac{y+2b}{a+b}$, $bx+ay=a^2+b^2$

35 $\dfrac{x}{a+b}+\dfrac{y-1}{a-b}=0$, $\dfrac{x}{a-b}-\dfrac{y+1}{a+b}=0$

36 $\dfrac{x-mn}{m}=\dfrac{mn-y}{n}$, $mx+ny=m^2+n^2$

37 $\dfrac{x-a}{y-a}=\dfrac{a-b}{a+b}$, $\dfrac{x}{y}=\dfrac{a^3-b^3}{a^3+b^3}$

244 The following are examples of literal equations involving more than two variables.

Ex Solve
$$y+z=a \quad \text{. . (i)},$$
$$z+x=b \quad \text{(ii)},$$
$$x+y=c \quad \text{. (iii)}$$

Add the three equations together and from the sum subtract each in turn, thus

$$x=\tfrac{1}{2}(b+c-a),\quad y=\tfrac{1}{2}(c+a-b),\quad z=\tfrac{1}{2}(a+b-c)$$

Otherwise —Subtract (i) from (ii) and then add (iii), thus

$$x=\tfrac{1}{2}(b+c-a)$$

Similarly $$y=\tfrac{1}{2}(c+a-b),$$
$$z=\tfrac{1}{2}(a+b-c)$$

Examples CXLV

Solve the equations

1 $y+z=b+c-a,$
$z+x=c+a-b,$
$x+y=a+b-c$

2 $x+2y=a,$
$2x+4z=3a,$
$4y+5z=5a$

3 $x+y+2z=3a,$
$2x-y+z=6b,$
$3x+2y-z=6a$

4 $x+y+z=a+b+c,$
$x+a=y+b=z+c$

5 $y+z-x=a,$
$z+x-y=b,$
$x+y-z=c$

6 $bz+cy=a,$
$az+cx=b,$
$ay+bx=c$

7 $\dfrac{a}{x}+\dfrac{b}{y}=m,\ \dfrac{b}{y}+\dfrac{c}{z}=n,\ \dfrac{c}{z}+\dfrac{a}{x}=p$

8 $cx+ay=a-c,\ \dfrac{cx}{a-b}-\dfrac{ay}{b-c}=cx+ay+bz=0$

9 $a(y+z)=lyz,\quad b(z+x)=mzx,\quad c(x+y)=nxy$

10 $xyz=a(zx+xy-yz)=b(xy+yz-zx)=c(yz+zx-xy)$

Various Methods of Solution

245 Theorem *If*

$$ax+by+cz=0 \qquad (1),$$
$$a'x+b'y+c'z=0 \qquad (2),$$

then
$$\frac{x}{bc'-b'c}=\frac{y}{ca'-c'a}=\frac{z}{ab'-a'b} \qquad (3)$$

Multiply (1) by c' and (2) by c and subtract, thus

$$(ac'-a'c)x+(bc'-b'c)y=0,$$

transpose, thus $(bc'-b'c)y=-(ac'-a'c)x=(a'c-ac')x,$

i e, $(ca'-c\,a)x=(bc'-b'c)y,$

or
$$\frac{x}{bc'-b'c}=\frac{y}{ca'-c'a} \qquad (4)$$

Again, multiply (1) by a' and (2) by a, and subtract, thus

$$\frac{y}{ca'-c'a}=\frac{z}{ab'-a'b} \qquad (5)$$

Therefore from (4) and (5),

$$\frac{x}{bc'-b'c}=\frac{y}{ca'-c'a}=\frac{z}{ab'-a'b}$$

Another proof by the Method of Indeterminate Multipliers

Divide (1) and (2) by z, thus

$$a\frac{x}{z}+b\frac{y}{z}+c=0,\ a'\frac{x}{z}+b'\frac{y}{z}+c'=0\ ,$$

or putting $\qquad \frac{x}{z}=u$ and $\frac{y}{z}=v$,

we have
$$au+bv+c=0, \qquad \text{(i)},$$
$$a'u+b'v+c'=0, \qquad \text{(ii)}$$

Multiply (i) by λ, and add the product to (ii), thus

$$(\lambda a+a')u+(\lambda b+b')v+(\lambda c+c')=0 \qquad \text{(iii)}$$

Now since λ is an arbitrary multiplier, we may so choose its value that the co-efficient of v may vanish, that is, we put

$$\lambda b+b'=0, \text{ or } \lambda=-\frac{b'}{b} \qquad \text{(iv)},$$

and therefore (iii) reduces to

$$(\lambda a+a')u+(\lambda c+c')=0,$$

whence
$$u=-\frac{\lambda c+c'}{\lambda a+a'},$$

from (iv),
$$u=-\frac{-\frac{b'}{b}c+c'}{-\frac{b'}{b}a+a'}=\frac{bc'-b'c}{ab'-a'b}$$

And by putting $\lambda a+a'=0$, we get as above

$$v=-\frac{\lambda c+c'}{\lambda b+b'}=\frac{ca'-c'a}{ab'-a'b}$$

Now replacing u and v by their values, we have

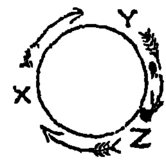
$$\frac{x}{z}=\frac{bc'-b'c}{ab'-a'b},\quad \frac{y}{z}=\frac{ca'-c'a}{ab'-a'b},$$

or
$$\frac{x}{bc'-b'c}=\frac{y}{ca'-c'a}=\frac{z}{ab'-a'b}$$

This Result is called the **Formula of Cross Multiplication**

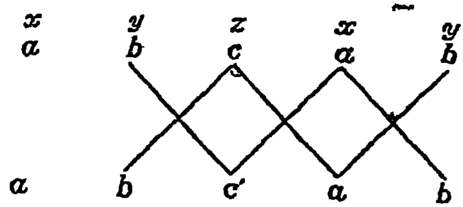
REMARK. The following device will help the student in remembering the relation (3) —It will be seen that

x has for denominator a combination of coefficients of y and z,



we have said in the second line z and x, and not of x and z, for in forming the denominators out of the co efficients of x, y and z, these *viz*, x, y and z are to be taken as following one another in *cyclic order*, *i e*, x being followed by y, y by z, z by x, etc, as shewn in the annexed diagram [See Art 146]

The combination is formed thus —



For x—Begin at b *under* y in the second column, proceed *diagonally from left to right to* c' *under* z, in the second row, and take the product bc', then go from c' to b', *i e*, go in a direction from *right to left*, and let this be a hint of affecting the next product with the sign $-$, to obtain this product, proceed from b' to c, again diagonally from left to right and take the product $b'c$ and put the negative sign, as we have said, before it Similarly for y and z

Ex 1. Solve
$$2x+3y+9=0,$$
$$4x+5y+13=0$$

We have
$$2x+3y+9\,1=0,$$
$$4x+5y+13\,1=0,$$

$$\frac{x}{3\times13-5\times9}=\frac{y}{9\times4-13\times2}=\frac{1}{2\times5-4\times3},$$

or
$$\frac{x}{-6}=\frac{y}{10}=\frac{1}{-2},$$

whence
$$x=\frac{-6}{-2}=3,\ y=\frac{10}{-2}=-5$$

Ex 2 Solve
$$ax+by+c=0,$$
$$a'x+b'y+c'=0 \quad \text{[Ex 5, p 334]}$$

Since $c=c\,1$ and $c'=c'\,1$, we have here $z=1$,

$$\frac{x}{bc'-b'c}=\frac{y}{ca'-c'a}=\frac{1}{ab'-a'b},$$

whence
$$x=\frac{bc'-b'c}{ab'-a'b},\ y=\frac{ca'-c'a}{ab'-a'b}$$

Note From Examples 1 and 2, we see that simple equations involving two variables can be easily solved by the Formula of Cross Multiplication Try by this method some of the equations given before.

Ex 3 Solve

$$3x+y-5z=0 \quad (a),$$
$$7x-3y-9z=0 \quad (b),$$
$$5x-3y+13z=12\tfrac{2}{3} \quad (c)$$

From (*a*) and (*b*), we get

$$\frac{x}{1(-9)-(-3)(-5)}=\frac{y}{-5\times 7-(-9)\times 3}=\frac{z}{3(-3)-7\times 1},$$

or
$$\frac{x}{-24}=\frac{y}{-8}=\frac{z}{-16},$$

$\therefore$
$$\frac{x}{3}=y=\frac{z}{2}=k \text{ say},$$

$$\therefore \quad x=3k,\ y=k,\ z=2k \quad (d),$$

Substitute these values in (*c*), thus

$$15k-3k+26k=12\tfrac{2}{3},$$

whence $k=\tfrac{1}{3}$,

from (*d*), $x=1,\ y=\tfrac{1}{3},\ z=\tfrac{2}{3}$

Ex 4 Solve

$$x+y+z=a+b+c \quad (a),$$
$$ax+by+cz=bc+ca+ab \quad (b),$$
$$(b-c)x+(c-a)y+(a-b)z=0 \quad (c)$$

We cannot apply the theorem here, unless two of the proposed equations are reduced to the forms (1) and (2)

From (*b*), $(ax-ca)+(by-ab)+(cz-bc)=0,$

or $a(x-c)+b(y-a)+c(z-b)=0 \quad (d),$

and from (*a*), $(x-c)+(y-a)+(z-b)=0 \quad (e),$

from (*d*) and (*e*), we have

$$\frac{x-c}{b\times 1-1\times c}=\frac{y-a}{c\times 1-1\times a}=\frac{z-b}{a\times 1-1\times b},$$

or
$$\frac{x-c}{b-c}=\frac{y-a}{c-a}=\frac{z-b}{a-b}=k \text{ suppose},$$

whence $x=c+k(b-c),\ y=a+k(c-a),\ z=b+k(a-b) \quad (f)$

Substitute these values in (*c*), thus

$$(b-c)\{c+k(b-c)\}+(c-a)\{a+k(c-a)\}+(a-b)\{b+k(a-b)\}=0,$$

or $k\{(b-c)^2+(c-a)^2+(a-b)^2\}=a^2+b^2+c^2-bc-ca-ab,$

whence $k=\tfrac{1}{2}$,

from (*f*), $x=\tfrac{1}{2}(b+c),\ y=\tfrac{1}{2}(c+a),\ z=\tfrac{1}{2}(a+b)$

Examples CXLVI

Solve the equations

1 $4x+3y+2z=0,$
$3x+5y+4z=0,$
$2x+\ y+3z=54$

2 $3x+4y-16z=0,$
$5x-8y+10z=0,$
$2x+6y+\ 7z=52$

3 $\frac{1}{3}x-\frac{2}{5}y+\frac{4}{3}z=0,$
$\frac{3}{5}y-\frac{5}{4}x-\frac{9}{5}z=0,$
$\frac{5}{11}x-\frac{3}{16}y+\frac{28}{9}z+1\frac{11}{15}=0$

4 $x+y+z=0,$
$(b+c)x+(c+a)y+(a+b)z=0,$
$bcx+cay+abz=1$

5 $x+y+z+(b-c)(c-a)(a-b)=0,$
$ax+by+cz=0,\ a^2x+b^2y+c^2z=0$

6 $x+y+z=0,\ ax+by+cz=0,$
$bcx+cay+abz+(b^2-c^2)(c^2-a^2)(a^2-b^2)=0$

7 $x+y+z=ax+by+cz=0,$
$$\frac{x}{b-c}+\frac{y}{a-c}+\frac{z}{a-b}=1$$

8 $x+y+z=a+b+c,\ bx+cy+az=cx+ay+bz=a^2+b^2+c^2$

9 $cx+ay+bz=bx+cy+az=0,$
$ax+by+cz=(b-c)^2+(c-a)^2+(a-b)^2$

10 $x+y+z=a+b+c,\ ax+by+cz=a^2+b^2+c^2,\ \frac{x}{a}+\frac{y}{b}+\frac{z}{c}=3.$

11 $ax+cy+bz=cx+by+az=bx+ay+cz=a^3+b^3+c^3-3abc$

12 $(b-c)x+(c-a)y+(a-b)z=0,$
$a(b^2-c^2)x+b(c^2-a^2)y+c(a^2-b^2)z=0,$
$a(b-c)x+b(c-a)y+c(a-b)z=(b-c)(c-a)(a-b)$

13 $ax+by+cz=0,\ \frac{x}{a}+\frac{y}{b}+\frac{z}{c}=0,\ \frac{x-a}{b-c}+\frac{y-b}{c-a}+\frac{z-c}{a-b}=0$

***246 Further application of Cross Multiplication** The method of solution will be illustrated by the following examples

Ex 1 Solve
$$x+2y+\ 3z=14 \qquad (1),$$
$$3x+4y+12z=47 \qquad (2),$$
$$2x+7y+\ 5z=31 \qquad (3)$$

Eliminate x between (1) and (2), *i e*, multiply (1) by 3 and subtract (2), thus
$$2y-3z=-5, \text{ or } 2y-3z+5=0 \qquad (4)$$

Again, eliminate x between (1) and (3), i e, multiply (1) by 2, and subtract from (3), thus

$$3y-z=3, \text{ or } 3y-z-3=0 \qquad (5)$$

Hence from (4) and (5) by Cross Multiplication

$$\frac{y}{(-3)(-3)-(-1)\times 5}=\frac{z}{5\times 3-(-3)\times 2}=\frac{1}{2(-1)-3(-3)},$$

or $$\frac{y}{14}=\frac{z}{21}=\frac{1}{7},$$

whence $$y=\tfrac{14}{7}=2, \text{ and } z=\tfrac{21}{7}=3$$

Substitute y and z in (1), thus

$$x+2\times 2+3\times 3=14, \text{ whence } x=1$$

Ex 2 Solve

$$2x+3y+6z=1 \qquad (1),$$
$$5x+2y-\ z=3 \qquad (2),$$
$$3x-\ y-5z=4 \qquad (3)$$

Multiply (1) by 3, and subtract (2), thus

$$x+7y+19z=0 \qquad (4)$$

Multiply (1) by 4, and subtract (3), thus

$$5x+13y+29z=0 \qquad (5)$$

Hence from (5) and (4) by Cross Multiplication

$$\frac{x}{247-203}=\frac{y}{29-95}=\frac{z}{35-13},$$

or $$\frac{x}{2}=\frac{y}{-3}=z=k \text{ suppose},$$

thus $$x=2k,\ y=-3k,\ z=k \qquad (6)$$

Substitute in (1), thus

$$4k-9k+6k=1,$$

whence $$k=1$$

$\therefore$ from (6), $x=2,\ y=-3,\ z=1$

These examples are sufficient to shew that we may apply the Formula of Cross Multiplication to solve three linear equations of the form

$$a_1x+b_1y+c_1z=d_1,\ a_2x+b_2y+c_2z=d_2,\ a_3x+b_3y+c_3z=d_3,$$

(1) by eliminating any one of the variables x, y and z as in Ex 1, or (2) by eliminating all the constants d_1, d_2 and d_3 as in Ex 2

*247 **Method of Indeterminate Multipliers** In Art 100, we have pointed out the advantage of this method We shall further illustrate it by solving equations involving three variables

Ex. 1 Solve
$$\left.\begin{aligned} x+2y+\ \ 3z&=14 \quad &(1),\\ 3x+4y+12z&=47 \quad &\dots(2),\\ 2x+7y+\ \ 5z&=31 \quad &(3)\end{aligned}\right\}\text{[Art 246, Ex. 1]}$$

Multiply (1) by λ and (2) by μ, and add to (3), thus
$$(\lambda+3\mu+2)x+(2\lambda+4\mu+7)y+(3\lambda+12\mu+5)$$
$$=14\lambda+47\mu+31 \quad \dots \quad (4)$$

To find x, let such values be given to λ and μ as will cause y and z to disappear from (4), which is done by putting

$$\left.\begin{aligned} 2\lambda+\ \ 4\mu+7&=0\\ \text{and} \qquad 3\lambda+12\mu+5&=0\end{aligned}\right\} \quad \dots \quad (5)$$

Thus (4) reduces to
$$(\lambda+3\mu+2)x=14\lambda+47\mu+31,$$
whence
$$x=\frac{14\lambda+47\mu+31}{\lambda+3\mu+2} \quad \dots \quad (6)$$

Thus x is found in terms of λ and μ whose values can be determined from (6) By Art 245, we have
$$\frac{\lambda}{4\times5-12\times7}=\frac{\mu}{7\times3-5\times2}=\frac{1}{2\times12-3\times4},$$
whence
$$\lambda=\frac{-64}{12}, \text{ and } \mu=\frac{11}{12}$$

Substitute λ and μ in (6), thus
$$x=\frac{14(-\frac{64}{12})+47\times\frac{11}{12}+31}{-\frac{64}{12}+3\times\frac{11}{12}+2}=1$$

Similarly by putting the coefficients of z and x equal to 0, we get y, and by putting the coefficients of x and y equal to 0, we get z.

Ex 2 Given
$$ax+by+cz=m \quad (1),$$
$$a^2x+b^2y+c^2z=m^2 \quad (2),$$
$$a^3x+a^3y+c^3z=m^3 \quad (3),$$
to find x

Multiply (1) by λ and (2) by μ, and add to (3), thus
$$ax(\lambda+\mu a+a^2)+by(\lambda+\mu b+b^2)+cz(\lambda+\mu c+c^2)$$
$$=m(\lambda+\mu m+m^2) \quad \dots \quad (4)$$

To find x, put
$$\left.\begin{aligned} \lambda+\mu b+b^2&=0\\ \text{and} \qquad \lambda+\mu c+c^2&=0\end{aligned}\right\} \quad \dots \quad (5)$$

Thus from (4). $x=\dfrac{m(\lambda+\mu m+m^2)}{a(\lambda+\mu a+a^2)}$ ... ...(6)

Now from (5) by Art 245,

$$\frac{\lambda}{bc^2-b^2c}=\frac{\mu}{b^2-c^2}=\frac{1}{c-b},$$

whence $\lambda=bc$ and $\mu=-(b+c)$

Therefore from (6),

$$x=\frac{m\{bc-(b+c)m+m^2\}}{a\{bc-(b+c)a+a^2\}}=\frac{m(m-b)(m-c)}{a(a-b)(a-c)}$$

REMARK. From the *symmetry* of the equations

$$y=\frac{m(m-c)(m-a)}{b(b-c)(b-a)},\quad z=\frac{m(m-a)(m-b)}{c(c-a)(c-b)}$$

The student will however do well to find y and z directly

Ex 3 Solve $a_1x+b_1y+c_1z=d_1$ (1),

$a_2x+b_2y+c_2z=d_2$ (2),

$a_3x+b_3y+c_3z=d_3$ (3)

Multiply (1) by λ and (2) by μ, and add to (3), thus

$$(\lambda a_1+\mu a_2+a_3)x+(\lambda b_1+\mu b_2+b_3)y+(\lambda c_1+\mu c_2+c_3)z$$
$$=\lambda d_1+\mu d_2+d_3$$

Now proceed as in Ex 1 To find x, put the coefficients of y and z equal to 0, to find y, put the coefficients of z and x equal to 0, to find z, put the coefficients of x and y equal to 0 The student is recommended to find the values of x, y and z

$$x=\frac{d_1(b_2c_3-b_3c_2)+d_2(b_3c_1-b_1c_3)+d_3(b_1c_2-b_2c_1)}{a_1(b_2c_3-b_3c_2)+a_2(b_3c_1-b_1c_3)+a_3(b_1c_2+b_2c_1)}$$

Similar values for y and z

Examples CXLVII

Solve the equations

1. $2-\dfrac{\dfrac{5x}{y}-3}{4}=7-\dfrac{4x+3}{2y},\quad 11y+\dfrac{6-\dfrac{y}{3}}{5}=35-\dfrac{\dfrac{x}{5}+4}{7}$

2 $x-\dfrac{2y-x}{23-x}=20-\dfrac{59-2x}{2},$

$y+\dfrac{y-3}{x-18}=30-\dfrac{73-3y}{3}$ [See Art 231]

Solve the equations

3. $$4x-\frac{11y-\frac{y}{2}}{17-3x}=20-\frac{103-8x}{2},$$
$$8y+\frac{3y-3}{5x-10}=50-\frac{147-24y}{3}$$ [See Art 231]

4. $$3y+11=\frac{4x^2-y(x+3y)}{x-y+4}+31-4x,$$
$$(x+7)(y-2)+3=2xy-(x+1)(y-1)$$

5. $$\frac{2}{3}\left(x-\frac{3}{5}y\right)+\frac{x+\frac{y}{5}}{6}=\frac{1}{3}-\frac{1}{2}\left\{\frac{\frac{4}{5}y-2}{6}-(x-y)\right\},$$
$$x-2y-\frac{3y-5x}{21}=\frac{11}{2}(x+y)+3(x-y).$$

6. $$xy+3y=20,$$
$$5y-4=2xy$$

7. $$\frac{x}{2+x}+\frac{y}{2+y}=\frac{x+y}{2y},$$
$$\frac{x}{y}-\frac{x-y}{x}=\frac{y}{x}$$

8. $$\frac{1}{x+\cfrac{1}{y-\cfrac{5}{x}}}=\frac{1}{x-\cfrac{1}{y-\cfrac{7}{x}}},\quad \frac{1}{y}\left(1-\frac{1}{x}\right)=1$$

9. $$3x+5y+7z+u=48,$$
$$7x+6y+5z+4u=53,$$
$$x+2y+3z+4u=27,$$
$$5x+8y+10z-2u=65$$

10. $$2u+5x=8,$$
$$x+2y=9,$$
$$7z+u=5,$$
$$3y+4z=14$$

11. $$19x-3u=4z-7y=7x-3y=11z-7u=1$$

12. $$3x-4y=7,$$
$$4y+u=12,$$
$$3z-5v=4,$$
$$5v-2x+5=0,$$
$$7v+3u-4z=7$$

13. $$9x-2z+u=41,$$
$$4y-3x+2u=5,$$
$$3y-4u+3v=7,$$
$$7y-5z-v=12,$$
$$7z-5u=11$$

14. $$x-ay+a^2z=a^3,$$
$$x-by+b^2z=b^3,$$
$$x-cy+c^2z=c^3$$

15. $$cx+(a+b)y+(a-b)z=a,$$
$$ax+(b+c)y+(b-c)z=b,$$
$$bx+(c+a)y+(c-a)z=c$$ [Ap]

Solve the equations

16. $(b+c)x+by+cz=(c+a)y+cz+ax=(a+b)z+ax+by,$

$$xyz-(c+a-b)yz-(a+b-c)zx-(b+c-a)xy$$
$$=2xyz\frac{a^2+b^2+c^2}{(a+b+c)^2} \quad [App.]$$

17. $ax+by+cz=0,$
$a^2x+b^2y+c^2z+(b-c)(c-a)(a-b)=0,$
$a^3x+b^3y+c^3z+(b-c)(c-a)(a-b)(a+b+c)=0$

18. $u+ax+a^2y+a^3z+a^4=0,$
$u+bx+b^2y+b^3z+b^4=0,$
$u+cx+c^2y+c^3z+c^4=0,$
$u+dx+d^2y+d^3z+d^4=0$

19. $(b+c)(y+z)=a(x+1),$
$(c+a)(z+x)=b(y+1),$
$(a+b)(x+y)=c(z+1)$ [App.]

20. $3^x9^y=27,$
$4^x8^y=32$

21. $2^x4^y=32,$
$\frac{3^x}{9^y}=3$

22. $x^y=y^x,$
$x^3=y^2.$

23. $a^x\times a^{y+1}=a^7,$
$a^{2y}\times a^{3x+5}=a^{20}$

24. $3^{x+1}+2^y=35,$
$3^x+2^{y+2}=41$

CHAPTER XXII

HARDER PROBLEMS

248. Problems Leading to Simple Equations in one Variable. In this Chapter we shall give a few examples of harder problems. The student is here recommended to revise the problems in Chapters IX and X.

249. Problems relating to Digits. As two or more variables are required to be found out in these problems, they will in general lead to Simultaneous Equations. [See Art. 105.] Sometimes however by virtue of the given relations between the digits, a problem may be solved by an equation in one variable. Of course it is to be carefully borne in mind that *every digit in such a problem has only an absolute value* and never a local value.

Ex. The digit in the tens' place of a number of two digits is $\frac{1}{4}$ of the digit in the units' place, and if the sum of the number and 5 be divided by the number formed by inverting the digits, the quotient will be $\frac{1}{2}$; find the number.

Let x = digit in the tens' place,

then $3x$ = " units' " ,

$10x+(3x)$ = required number,

$10x+(3x)+5$ = sum of the number and 5,

and $10(3x)+x$ = number formed by inverting the digits,

$$\frac{10x+(3x)+5}{10(3x)+x}=\frac{1}{2},$$

or $$\frac{13x+5}{31x}=\frac{1}{2}, \text{ whence } x=2,$$

∴ required number $=10x+(3x)=13x=26$

Note Here the relation that the tens' digit is $\frac{1}{3}$ of the units' digit enables us to solve the problem by means of an equation in one variable. It is however always *easy* to solve these problems by simultaneous equations, thus —

Let x = digit in the tens' place,

and y = " units' " ,

$10x+y$ = required number .... . (i),

$(10x+y)+5$ = sum of required number and 5 ... (ii),

and $10y+x$ = number formed by inverting the digits...(iii)

By the first condition of the problem, we have

$$x=\tfrac{1}{3}y \text{ or } y=3x \quad \ldots \quad \text{(iv)};$$

and by the second condition, we have from (ii) and (iii)

$$\frac{(10x+y)+5}{10y+x}=\frac{1}{2} \quad \ldots \quad \text{(v)}$$

Solving (iv) and (v), we get $x=2$, $y=6$,

∴ required number $=26$

Examples CXLVIII

1 A number consists of two digits, that in the tens' place being greater than the other by 3, if 13 be added to the number, the sum is 65 Find the number

2 The sum of the two digits of a number is 14, and the quotient of the one divided by the other is $\frac{4}{3}$, find the number

3 The sum of the two digits of a number is 10, and if 36 be subtracted from the number the digits are reversed, what is the number?

4 If from the sum of the digits of a number 10 be subtracted the remainder is 2, and the number formed by reversing the digits is equal to $\frac{1}{3}$ of the number, increased by 56, what is the number?

5 The difference between the digits of a number is 2, and if 3 times the units' digit, which is the greater of the two, be added to the number, the digits are inverted, find the number

6 The tens' digit of a number is twice the other, and if 3 times the greater digit together with 12 be taken from the number, the remainder is the number which is formed by interchanging the digits what is the number ?

7 A number consists of 3 digits of which the middle one is 0 and the sum 8, the number formed by interchanging the extreme digits is greater than the number itself by 198 what is the number ?

230 Problems relating to Work, Cisterns, &c If A performs a piece of work in 1 day, it is clear that in a days he will perform a times as much work, that is, *the capacity of an agent multiplied by the time he is employed on a work, is equal to the work.* Hence if

w=a *unit of work* done in a *unit of time*,

W=total work, done in t units of time,

then $$W=wt \qquad (1).$$

And from (1), $$w=\frac{W}{t} \qquad (2),$$

$$t=\frac{W}{w} \qquad (3),$$

that is, *capacity of agent=work ÷ time,*

and *time=work ÷ capacity of agent*

Now suppose A and B, whose capacities for work are w_1 and w_2 (*i e, the units of work which they respectively do in a unit of time*), can together perform the work W; required the time Here w is the sum of w_1 and w_2, and from (3)

$$\text{required time}=\frac{W}{w_1+w_2}$$

Similar reasoning will apply to Problems on Cisterns

Ex 1 A can do a piece of work in 5 hours and B in 6 hours, in what time can they together do it ?

Let W=whole work,

and x=required time, i e, number of hours, here *one hour being the unit of time*

Now A does the whole work W in 5 hours, in 1 hour he does

$\frac{1}{5}$th. of W, i e, $\frac{W}{5}$ Similarly in 1 hour, B does $\frac{W}{6}$ Hence in 1 hr, A and B *together* do $\frac{W}{5}+\frac{W}{6}$

$$\left(\frac{W}{5}+\frac{W}{6}\right)x=W,$$

or dividing by W, $$\frac{x}{5}+\frac{x}{6}=1,$$

whence $$x=\tfrac{30}{11}=2\tfrac{8}{11}$$

Ex 2 A cistern can be filled by 2 pipes in 5 and 6 hours respectively, and emptied by a third in 10 hours, if all the three be opened simultaneously, when will the cistern be filled?

Let x = time required, in hours

Now in 1 hour two of the pipes *fill* respectively $\frac{V}{5}$ and $\frac{V}{6}$, and the third *empties* $\frac{V}{10}$, if V represents the cubical content of the cistern It is evident therefore that in one hour, the *three* pipes *jointly fill* $\frac{V}{5}+\frac{V}{6}-\frac{V}{10}$

Hence $$\left(\frac{V}{5}+\frac{V}{6}-\frac{V}{10}\right)x=V,$$

solving which $x=3\tfrac{3}{4}$ hrs $=3$ hrs 45 min

Examples CXLVIII (*Continued*)

8 A man alone can reap a field in 10 days, and with the assistance of his son, in 6 days, how long will it take the son alone to reap it?

9 A vessel can be filled by means of one tap in 3 hours and by means of another tap in 5 hours In what time will it be filled, if both taps run together?

10 A can do a piece of work in 12 days, but when he has been at work for 4 days B is sent to help him, and they together finish it in 3 days, in how many days can B do the whole?

11 A can do as much work in 6 hours as B can do in 8 hours, or as C can do in 10 hours, in what time will A and B together complete a piece of work $\frac{1}{3}$ of which has been done by C in 25 hours?

12 A man can drink a cask of beer in 15 days, after he has been drinking for five days, he is joined by his wife and they together finish it in $6\tfrac{1}{4}$ days more, how long could the cask last the wife alone?

13 A rectangular bath can be filled by 3 spouts in 3, 4 and 5 hours respectively; if 65 cubic feet of water be first thrown in and the 3 spouts be then opened together, the rest can be filled in 1 hour; find the volume of the bath.

14 A leaky cistern is filled by 2 pipes in 15 and 20 hours respectively, if the leak be plugged; but if the leak as well as the pipes run together, the cistern can be filled in 12 hours; in what time can the leak empty the cistern when full, if the supply pipes be stopped?

15. After A has done $\frac{3}{5}$ths of a piece of work in 15 hours, B joins him and the two together finish it in 8 hours; when could they separately do it?

16 Two pipes A and B together fill a tank in 20 hours; A runs alone for 4 hours, when B is opened and in 15 hours more $\frac{7}{8}$ths of the tank is filled; in what time would each pipe have filled the tank separately?

17 A can do a piece of work in 15 days and B in 18 days; they work together for 3 days, when B leaves but A continues, and after 3 days is joined by C, and they together finish it in 4 days; how long would it take C to do the piece of work alone?

18 A, B and C together can fill a cistern in 4 minutes. A alone can fill it in 10 minutes, and B takes $1\frac{1}{2}$ times the time which C takes to fill it. In what time can B and C separately fill the cistern?

251 Problems relating to Motion It is easy to see [see p 106, *For* XIV] that A walking a miles per hour walks ab miles in b hours; that is, if d represent the distance passed over by a body in the time t at the rate of r per unit of time, then

$$d=rt \tag{1}$$

hence *distance = rate × time.*

From (1)
$$r=\frac{d}{t} \tag{2}$$

and
$$t=\frac{d}{r} \tag{3}$$

that is, *rate = distance ÷ time*; and *time = distance ÷ rate*

Again if A and B describe respectively the distances d, d' in the time t, t' at the rates r, r', then

$$d=rt \text{ and } d'=r't',$$

whence
$$\frac{d}{d'}=\frac{rt}{r't'} \tag{4}$$

that is, *the distances are proportional to the product of the rates into the times.*

If $d=d'$, from (4) $\qquad \dfrac{rt}{r't'}=1$ or $\dfrac{r}{r'}=\dfrac{t'}{t}$ (5),

that is, *the distance being the same, the rates are inversely proportional to the times*

If $r=r'$, from (4) $\qquad \dfrac{d}{d'}=\dfrac{rt}{r't'}=\dfrac{t}{t'}$ (6),

that is, *the rate being the same, the distances are proportional to the times*

If $t=t'$, from (4) $\qquad \dfrac{d}{d'}=\dfrac{rt}{r't'}=\dfrac{r}{r'}$ (7),

that is, *the time being the same, the distances are proportional to the rates*

REMARK In all algebraical problems relating to motion, *motion is always supposed to be uniform*

Ex 1 A messenger is sent to a town which is distant 80 miles after passing the midway station, he doubles his speed, if the whole time taken be 6 hours, what was his rate at first?

Let $\qquad x$ = required rate in miles per hour,

from (3), $\dfrac{40}{x}$ = time taken to travel the first half,

and $\qquad \dfrac{40}{2x}$ = time taken to travel the second half,

and these two times taken together = 6 hours, therefore

$$\frac{40}{x}+\frac{40}{2x}=6,$$

whence $\qquad x=10$

Thus the messenger at first travelled 10 miles per hour.

Ex 2 Two persons motor from A to B in 2 hours and 1 hour 40 min If one of them travels 3 miles an hour faster than the other, find the distance AB

Let $\qquad x$ = distance in miles from A to B

$\dfrac{x}{2}$ = rate of one man in miles per hr,

and $\qquad \dfrac{x}{1\frac{2}{3}}$ = " other " " "

Since the time taken by the second man is *less* than that taken by the first, his rate must be *greater*

$$\frac{x}{1\frac{2}{3}}-\frac{x}{2}=3, \text{ whence } x=30$$

Thus the distance $AB=30$ miles

Ex 3 A train, having to perform a journey of 300 miles, is obliged after going 60 miles to reduce its speed by one fifth, and is thus $1\frac{1}{2}$ hours late At what rate was it running at first?

Let x=the original rate of the train in miles per hr

Now the late arrival of the train is due to its performing the distance 300 − 60 or 240 miles at the reduced rate of $\frac{4}{5}x$ miles per hr; that is, $1\frac{1}{2}$ hrs is the difference between the times the train takes to run 240 miles at x and $\frac{4}{5}x$ miles per hour

$$\frac{240}{\frac{4}{5}x}-\frac{240}{x}=1\frac{1}{2},$$

whence $$x=40$$

Thus the original rate of the train was 40 miles per hour

Ex 4. A train, 96 yards long, travels at the rate of 45 miles per hour It is seen to pass completely a station platform in 9 seconds Find the length of the platform

Let x=length of the platform, in yds

The train takes 9 sec to pass a distance equal to its lown length + the length of the platform

Now the train passes $(96+x)$ yds at 45 miles, or 45×1760 yds per hour Thus the time taken

$$=\frac{96+x}{45\times1760}\text{hr}=\frac{96+x}{45\times1760}\times60\times60\text{ sec}=\frac{96+x}{22}\text{ sec.}$$

But by the question, this time=9 sec,

$$\frac{96+x}{22}=9$$

whence $$x=102$$

Thus the length of the platform is 102 yds

Ex 5 A man rides one-third of the distance from A to B at the rate of a miles per hour and the remainder at the rate of $2b$ miles per hour If he had travelled at a uniform rate of $3c$ miles per hour, he could have ridden from A to B and back again in the same time. Prove that

$$\frac{2}{c}=\frac{1}{a}+\frac{1}{b}\ [Cal,\ 1889]$$

Let $3x$=distance, in miles from A to B,

then since he rides x miles, at the rate of a miles, $2x$ miles at the

rate of $2b$ miles, and $6x$ miles at the rate of $3c$ miles per hour, we have

$$\frac{x}{a}=\text{time in hours, taken to ride } \frac{1}{3}AB,$$

$$\frac{2x}{2b} \text{ or } \frac{x}{b}= \quad \ldots\ldots \quad \ldots \quad \ldots \quad \frac{2}{3}AB,$$

$$\text{and } \frac{6x}{3c} \text{ or } \frac{2x}{c}= \ldots\ldots \quad \ldots \quad \ldots \quad 2AB,$$

and the first two times are together equal to the last, therefore

$$\frac{2x}{c}=\frac{x}{a}+\frac{x}{b},$$

divide by x, thus

$$\frac{2}{c}=\frac{1}{a}+\frac{1}{b}$$

Examples CXLVIII (*Continued.*)

19 Two cyclists start from the same place and reach a town in 3 hrs 12 min and 2 hrs 40 min If one of them goes $1\frac{3}{4}$ miles per hour quicker than the other, what is the distance of the town from the starting place ?

20 A person walked to the top of a mountain at the uniform rate of $2\frac{1}{3}$ miles an hour and down again by the same way at the rate of $3\frac{1}{2}$ miles an hour, and he was 5 hours in going and returning How many miles did he walk ?

21 Two persons started at the same time from A, one rode on horseback at the rate of $7\frac{1}{2}$ miles an hour and arrived at B 30 min later than the other, who travelled the same distance by train at the rate of 30 miles an hour Find the distance between A and B

22 Walking at a uniform rate a man can perform a journey in 10 hrs , but when he is half-way he increases his speed by 2 miles an hour, and thus reaches his destination an hour and a quarter earlier Find the distance travelled and his original rate

23 A and B start at the same time to go to a certain place , A, who walks $4\frac{1}{2}$ miles an hour, takes $\frac{5}{6}$ths of the time which B takes to reach the place , find the rate of B

24 A man can walk a certain distance in 4 hours , if he were to reduce his rate by one-sixteenth, he could walk one mile less in that time , what is his rate ?

25 A person after walking the first half of his journey, quickens his pace by one-fifth, and thus arrives at his destination 35 minutes earlier , how long does he take to walk the whole distance ?

26 A and B walk over the same ground, going out one way and

coming home the other, they start at the same time in opposite directions, A walking $3\frac{3}{4}$ miles and B 4 miles per hour A wants $\frac{1}{4}$ of a mile of being half-way when he meets B Required the length of the walk and the time each was out

27 A person walks to Baitsbite at the rate of 3 miles an hour, runs part of the way back at the rate of $8\frac{1}{2}$ miles an hour, and walks the remainder in 1 hour and 5 minutes, he is out 2 hours 44 minutes, find the distance he ran and that to Baitsbite

28 A student is allowed just 3 hours and 20 minutes for exercise, how far can he walk at the rate of $1\frac{3}{4}$ miles an hour so as to come home in time, riding back the distance at the rate of $10\frac{1}{2}$ miles an hour?

29. A railway train after running for sometime meets with an accident after which it proceeds with $\frac{4}{5}$ths of its former rate, had the accident occurred 30 miles further on, it would have arrived at the terminus 25 minutes sooner. Required the rate of the train

30 A and B start to run to a flagstaff 450 yards off, and back; A returning meets B 30 yards from the flagstaff, and arrives at the starting point half a minute before B How long did A take to run the whole distance?

252. Problems on Relative Motion. When two bodies are moving (for instance two persons are walking), it is clear that if their rates of moving be not the same, the interval between them will vary at the end of each unit of time. Thus in Ex 12 [p 114], the original distance between P and Q is diminished by $(4+5)$ miles every hour Hence if the original distance between two objects, which are moving towards each other (*i.e., in opposite directions*) at the rates of m and n miles an hour, be d miles, then

$$d-(m+n)h$$

will represent *the distance between them at the end of* h *hours* and if they meet after x hours, this distance vanishes, therefore

$$d-(m+n)x=0,$$

whence
$$x=\frac{d}{m+n} \tag{1}$$

Thus when the motions of two bodies are in *opposite directions, the time of meeting = distance ÷* sum *of the rates*

Again, if P and Q start at the same time from A and O respectively to proceed towards B, and if the distance $AO=d$, then evidently d is diminished every hour by $(m-n)$ miles, hence

$$d-(m-n)h$$

will represent *the distance between P and Q at the end of* h *hours*,

and at the instant when P overtakes Q, this distance will vanish Therefore if x represent the time of meeting of P and Q,

$$d-(m-n)x=0,$$

or $$x=\frac{d}{m-n}\text{ hours} \tag{2}$$

Thus, when the motions of two bodies are in the *same direction, the time of meeting = distance ÷* difference *of the rates*

Ex 1 A Goods train which runs 20 miles per hour, starts 6 hours before a Mail-train which runs 28 miles per hour, when and at what distance from the station will the Goods-train be overtaken?

Here $d=20\times 6=120$ miles, therefore from (2),

$$\text{time of meeting}=\frac{120}{28-20}\text{ hrs}=15\text{ hours},$$

and required distance $=(28\times 15)$ miles $=420$ miles

It is always well, however, to solve these problems from the principles explained above, thus

Let A represent the station, B the position of the Goods-train at the end of 6 hours, and C the place where the Goods-train is overtaken

$A \qquad\qquad B \qquad\qquad\qquad\qquad C$

Let x = time when the M will overtake the G,

then $28x$ = distance run by the M. in x hours $=AC$,

$20x$ = " " G. " " $=BC$,

and $AB=20\times 6=120$ miles,

hence, since $AC=AB+BC$, we have $28x=120+20x$,

whence $x=15$ hours,

and therefore required distance $=28\times 15=420$ miles

Otherwise —As before, let A represent the station and C the place where the Goods-train is overtaken

Thus $\frac{AC}{20}$ = time in hrs the G takes to run AC,

and $\frac{AC}{28}$ = " " M. " "

But the G has a start of 6 hrs., therefore the first time is greater than the second time by 6 hrs

$$\frac{AC}{20}=\frac{AC}{28}+6$$

Hence if x=distance in miles where the G. is overtaken, then $x=AC$

Thus $$\frac{x}{20}=\frac{x}{28}+6$$

Solving this we get $x=420$, i e, the G is overtaken at a distance of 420 miles from the station.

And time taken by the M to overtake G $=(420-28)$ hrs $=15$ hrs.

Ex 2 A railway train 84 yd long, going at the rate of 48 miles per hour, meets and passes completely in 4 seconds another train going at the rate of 42 miles per hour What is the length of the latter?

Let x=the required length in yds

In 1 hour the two trains together pass over $(48+42)$ miles $=90$ miles

To pass each other completely, they have to go a distance of $(84+x)$ yds Hence the time taken

$$=\frac{84+x}{90\times 1760}\text{ hr}=\frac{84+x}{90\times 1760}\times 60\times 60\text{ sec}=\frac{84+x}{44}\text{ sec.}$$

But by the question, they take 4 sec to pass each other

$$\therefore\ \frac{84+x}{44}=4\text{, whence }x=92.$$

Thus the length of the second train is 92 yds

Ex 3 AB is a railway 220 miles long, three trains (P, Q, R) travel upon it at the rates of 25, 20 and 30 miles per hour respectively, P and Q leave A at 7 A M and 8 15 A M respectively, and R leaves B at 10 30 A M When and where will P be equidistant from Q and R? [*Cal*, 1870]

Let x=time after R leaves B, when P is equidistant from Q and R.

Let the figure represent the positions of the trains at the required time

A B

Q P R

Now to come to their respective positions, P takes $(3\frac{1}{2}+x)$ hours, Q takes $(2\frac{1}{4}+x)$ hours, and R takes x hours, thus

$AQ=(2\frac{1}{4}+x)\times 20$ miles, $AP=(3\frac{1}{2}+x)\times 25$ miles, and $BR=30x$ miles,

and therefore $$AR=AB-BR=(220-30x)\text{ miles}$$

And from the geometry of the figure, we have $2AP=AQ+AR$, therefore $2(3\frac{1}{2}+x)25=(2\frac{1}{4}+x)20+(220-30x)$, whence $x=1\frac{1}{2}$ hours.

Hence P will be at its present position $1\frac{1}{2}$ hours after R starts, i e, at 12 o'clock, and at a place whose distance from A

$$=AP=(3\tfrac{1}{2}+1\tfrac{1}{2})\ 25 \text{ miles}=125 \text{ miles}$$

Examples CXLVIII (*Continued*)

31 B has $1\frac{3}{4}$ hrs, start of A, but only travels at the rate of 3 miles an hour, while A travels at the rate of $4\frac{1}{2}$ miles an hour, at what distance from the starting place does A overtake B?

32 A steamer running 14 miles an hour, descries another 21 miles off, going at the rate of 11 miles an hour, how many miles will the latter have run before she is overtaken?

33 A and B set out at the same time from P and Q, after 5 hours A meets B who walks 4 miles an hour, and reaches Q in 4 more hours, find the rate of A and the distance between P and Q

34 A person starts from Ely to walk to Cambridge, which is distant 16 miles, at the rate of $4\frac{4}{9}$ miles per hour, at the same time that another person leaves Cambridge for Ely, walking at the rate of a mile in 18 minutes, find where they meet

35 A is sent on an errand and travels at the rate of 5 miles an hour, 1 hour 24 minutes after, B is despatched to call him back, if after the first hour, B increases his pace by $\frac{1}{12}$ and overtakes A in 4 more hours, find the rate of B and the distance where he meets A.

36 A constable in pursuit of a thief at a uniform pace finds by inquiry that the thief is travelling $1\frac{1}{2}$ miles per hour quicker than himself, he therefore doubles his speed after the first 4 hours, and takes the thief at the end of 6 hours 20 minutes from the time of his starting Given that the thief had a start of 1 hour, and never varied his speed, find the rates of travelling of the parties, and the distance where the capture took place

37 A and B are two railway stations 60 miles apart, two minutes after a Passenger train has left A for B at 24 miles per hour, the station master of B sends a Pilot engine to repair a disabled bridge 18 miles off The Pilot takes 1 hour 4 minutes to repair the bridge, and then starts back at $\frac{3}{4}$ of its former rate and reaches B just in time to avert a collision What was the original rate of the Pilot, and how far was the Passenger train from the bridge at the time when the Pilot first reached the bridge?

38 A train leaves B at 9 A M and runs to C at the rate of 15 miles an hour, and another train leaves A at noon, and running through B to C at 25 miles an hour, arrives half an hour later than the train from B, if the distance AB be 15 miles, find the distance from A to C

39 A train starts from a station X at midday and arrives at Y

at 4 P. M. Another train starts from Y at the same time and arrives at X at 3 30 P. M They meet 28 miles from X Find the distance from X to Y.

40 The express leaves Bristol at 3 P. M and reaches London at 6 P M, the ordinary train leaves London at 1 30 P M. and arrives at Bristol at 6 P M If both trains travel uniformly, find the time when they meet

41 Two persons, who walk at the rates of 3 and 4 miles per hour respectively, started from two places 70 miles apart, after passing each other they continued their journey, reached the places, turned back immediately, and met again, when and where will the second meeting take place?

253 Problems relating to Motion on Streams Suppose a boat has an initial velocity of a miles per hour, *i e*, a velocity of a miles per hour in *still* water

It is clear that when the boat is on a stream which flows at the rate of b miles per hour, it has a velocity of $(a+b)$ miles per hour when it *goes down* the stream, for then the boats' initial velocity is increased by the velocity of the stream

Similarly when the boat *comes up* the stream, its velocity is $(a-b)$ miles per hour, for in this case the boat's initial velocity is diminished by the velocity of the stream

Thus *in going down a stream, the stream's velocity is* **added** *and in coming up, it is* **subtracted.**

Note Evidently when the boat comes up the stream, $a>b$

Ex 1 A boat goes 57 miles down a river in 6 hours, if the river flows at the rate of 4 miles an hour, find the rate of the boat in still water

Let x = the required rate of the boat in miles per hr,
then $(x+4)$ = the rate of the boat in miles per hour when it goes down stream.

by Art 251 (1), $6(x+4)=57$, whence $x=5\frac{1}{2}$

Thus the rate of the boat is $5\frac{1}{2}$ miles per hour

Ex 2 A boy, who can row 5 miles per hour in still water, takes 6 hours 40 min to come 10 miles up a river At what rate is the river flowing?

Let x = rate of the river in miles per hour,
then $5-x$ = rate of the boy when he comes up the river

Hence $(5-x)\times 6\frac{2}{3}=10$, whence $x=3\frac{1}{2}$

Thus the river is flowing at $3\frac{1}{2}$ miles per hr

Ex 3 If the number of the crew be doubled in coming up a river, a boat takes the same time to come up as to go down, the river flowing at the rate of $3\frac{1}{2}$ miles per hour Find the rate of the boat

In going down, the rate of the boat=its own rate+rate of river

Again, in coming up, the rate of the boat=its own rate−rate of river

Let x=required rate of the boat in miles per hour,

$x+3\frac{1}{2}$=the rate of the boat in going down,

and $2x-3\frac{1}{2}$= . .... coming up

Therefore, the distance and the time both being the *same*, we get from Art 251 (4),

$$\frac{x+3\frac{1}{2}}{2x-3\frac{1}{2}}=1,$$

whence $x=7$

the boat goes 7 miles per hour

Examples CXLVIII (*Continued*)

42 At what rate should a crew pull in still water in order to row with a stream flowing 5 miles an hour 3 times as fast as against it?

43 A crew, which can pull at the rate of 9 miles an hour, finds that it takes twice as long to come up a river as to go down, at what rate does the river flow?

44 A man rows to a place 48 miles distant and back in 14 hours. He finds that he can row 4 miles with the stream in the same time as 3 miles against it. Find the rate of the stream

45 Against a stream, which flows $3\frac{1}{2}$ miles an hour, a man rows 6 miles in an hour and a half How long would it take him to row $5\frac{1}{2}$ miles with the stream?

46 A steamer takes 4 hours less time to travel from A to B than from B to A If its rate in still water is 15 miles and that of the stream $4\frac{1}{2}$ miles an hour, find the distance from A to B, and the time taken in each journey

47 A cyclist, who rides at the rate of 7 miles per hour, takes 4 hours to go a certain distance with the wind and 7 hr 12 min to come back against the same wind At what rate was the wind blowing?

48 A and B can row respectively 6 and 8 miles per hour in still water P and Q are towns on a river which flows from P towards Q A starts from P and in 4 hours meets B, who has started

from Q 6 hr 40 min. before, at a place midway between P and Q Find the velocity of the river and the distance between P and Q

49 A and B are towns on a river 12 miles apart If it takes a man 1 hour longer to row from A to B and back than to row the same distance in still water at 8 miles an hour, what is the rate of the river ?

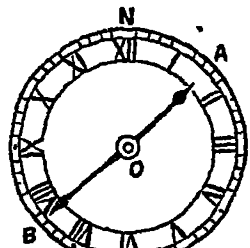
254. Problems relating to Watches and Clocks The chief peculiarity of these problems is that, unlike other problems, *one of their conditions is generally understood* In solving them therefore the student should carefully remember this condition, *viz.*, that *the minute-hand moves 12 times faster than the hour-hand*, and if the second-hand be also on the same axis as the other two, that *the second-hand moves 60 times faster than the minute-hand and 720 times faster than the hour-hand*

Two hands of a watch are said to be (1) *together* when there is *no interval* between them, (2) in the *same straight line*, when the interval between them is 30 *minute divisions*, and (3) *at right angles*, when the interval between them is 15 *minute-divisions*, and this last happens *twice* during the same hour

Ex At what time between 1 and 2 o'clock will the hour-hand and minute-hand of a watch be in the same straight line ?

Let x=number of minutes after One, when the two hands are in the same straight line

Then if AOB represent their position, it is easy to see that OB representing the minute-hand, has moved for x minutes, *i e*, the arc NB described by it represents x minute-divisions , and OA, which represents the hour-hand, and which has moved also for x minutes, has described an arc of $\frac{1}{12}$ of x, *i e*, the arc IA represents $\frac{x}{12}$ But



$$\text{arc NB} = \text{arc NI} + \text{arc IA} + \text{arc AB}$$

and arc NI $=5$ minute-divisions,

arc AB $=30$ . ..... ,

$$x = 5 + \frac{x}{12} + 30,$$

whence $x = 38\frac{2}{11}$,

i.e, two hands will be in the same line after $38\frac{2}{11}$ min. past One

Examples CXLVIII (*Continued*)

50 At what time between 1 and 2 o'clock will the two hands of a watch be, (1) together, and (2) at right angles ?

51 At what time between 6 and 7 o'clock will the hands of a clock be, (1) together, (2) in the same straight line, and (3) at right angles ?

52 At what time between 3 and 4 o'clock are the hour-hand and minute-hand of a watch (1) exactly in the same direction, (2) opposite each other, and (3) at right angles to each other ?

53 Find the times between 10 and 12 o'clock at which the two hands of a clock are exactly together

54 When between 2 and 3, will the minute-hand be exactly (1) one minute in advance of the hour-hand and (2) 7 minutes behind it ?

55 The hour-hand of a clock is between 6 and 7, the minute hand is 14 minute spaces apart What is the time ?

56 If the hands of a watch between 3 and 4 o'clock are 18 minute divisions apart, what is the time ?

57 It is between 8 and 9 o'clock, and the hands of a clock are 18 minute spaces apart Find the time

58 A watch has the second-hand on the same axis as the other two and the hands are all together at 12 o'clock, find when the minute-hand and the second-hand are next together

59 A man, who went out between 5 and 6 and returned between and 7, found that the hands of his watch had exactly changed places When did he go out ? [*Punj* 1894]

60 It is between 2 and 3 o'clock, but a person looking at the clock and mistaking the hour-hand for the minute-hand, fancies that the time of day is 55 minutes earlier than the reality, what is the true time ?

255 Problems relating to Mixtures If 5 mds of one kind of rice is mixed with 3 mds of another kind, then the mixture evidently contains (5+3) mds, or 8 mds and the quantity of the first kind of rice in the mixture is $\frac{5}{5+3}$ of (5+3) mds or $\frac{5}{8}$ of 8 mds and the quantity of the second kind is $\frac{3}{5+3}$ of (5+3) mds or $\frac{3}{8}$ of 8 mds

Thus generally if one ingredient in a mixture is x and the other ingredient is y, the quantity of the mixture is $x+y$, and the first ingredient $=\frac{x}{x+y}$ of the mixture and the second ingredient $=\frac{y}{x+y}$ of the mixture

Question A mixture contains 12 gallons of spirit and 4 gallons of water, another mixture contains 10 gallons of spirit and 2 gallons of water If 8 gallons are taken from the first mixture and 6 gallons from the second to form a new mixture, what are the quantities of wine and water in this mixture?

Ex 1 In what proportion must tea which cost R1 8a per lb be mixed with tea which cost R1 per lb, so that the mixture may be worth R1 5a per lb?

Let x = quantity, in lb, of the first kind of tea,

then since the mixture is to be of 1 lb,

$1-x$ = quantity, in lb, of the second kind of tea

$$x \times \text{R1 } 8a + (1-x) \times \text{R1} = \text{R1 } 5a,$$

or
$$\tfrac{3}{2}x + (1-x) = 1\tfrac{5}{16},$$

whence
$$x = \tfrac{5}{8}$$

Thus $\frac{5}{8}$ lb of the first kind is to be mixed with $\frac{3}{8}$ lb of the second kind to form the mixture

Ex 2 A cask A contains a mixture of 12 gallons of wine and 18 gallons of water, another cask B contains a mixture of 9 gallons of wine and 3 gallons of water, what quantity must be taken from A and B respectively, so that their mixture may contain 7 gallons of wine and 7 gallons of water?

The quantity in $A = (12+18)$ gallons $= 30$ gallons; therefore, since A contains 12 gals of wine, $\frac{12}{30}$ or $\frac{2}{5}$ of the mixture in A is wine

Similarly $\frac{9}{12}$ or $\frac{3}{4}$ of the mixture in B is wine

Now the quantity of the new mixture is to be $(7+7)$ gallons $= 14$ gallons Therefore if

x = quantity of mixture, in gallons, to be taken from A,

then $14-x$ = " " " " " B,

therefore in the new mixture

$\frac{2}{5}x$ = quantity of *wine* in gallons, drawn from A,

and $\frac{3}{4}(14-x)$ = " *wine* " " " B,

$$\tfrac{2}{5}x + \tfrac{3}{4}(14-x) = 7, \text{ whence } x = 10$$

Thus 10 gallons are to be taken from A and $14-10$ or 4 gallons from B, to form the required mixture

Note The student will do well to solve this problem by equating the quantities of *water*

Examples CXLVIII. (*Continued*)

61 How much tea worth 2*s* 6*d* per lb must be mixed with 9 lb of tea worth 3*s* 10*d* per lb, that the mixture may by worth 3*s* per lb ?

62 A mixture of 35 seers, worth 4*a* 6*p* per seer, is formed of two kinds of sugar at 4*a* 4*p* and 5*a* 2*p* per seer respectively What quantity of each was taken ?

63 A dealer buys wine at 16*s* a gallon, and after adding water to it, sells the mixture at 18*s* a gallon If his profits are $\frac{2}{7}$ths of the cost price, how much water is there in a gallon of the mixture ?

64 In what proportion must two kinds of oil at 8*a* 6*p* and 10*a* per seer be mixed, so that by selling the mixture at 9*a* 9*p* per seer a dealer may gain $12\frac{1}{2}$ per cent on his outlay ?

65 A hundred gallons of liquid contains 70 per cent of wine and the rest water How much wine should be added to the mixture to raise the proportion of wine to 80 per cent ?

66 A vessel is filled with a mixture of spirit and water in which there is 70 per cent of spirit, 19 gallons are taken out, and the vessel is filled up again with water, the proportion of spirit is now found to be 56 7 per cent Find how much the vessel contains

67 A milkseller mixes 28 seers of milk with 7 seers of water, he also mixes 30 seers of milk with 6 seers of water What quantity must he take from the first vessel to mix with 12 seers from the second to make a mixture containing 18 seers of milk ?

68 A vessel contains 42 gallons of wine and 28 gallons of water, a second vessel contains 35 gallons of wine and 21 gallons of water If a mixture formed by drawing the same quantity of liquid from each vessel contains 31 gallons of water, how much is drawn from each vessel and what is the quantity of wine in the new mixture ?

69 One cask contains a mixture of 12 gallons of wine and 18 gallons of water, another cask contains a mixture of 9 gallons of wine and 6 gallons of water, how many gallons must be drawn from each cask so as to produce a mixture containing 7 gallons of wine and 8 gallons of water ?

70 Two vessels contain mixtures of wine and water, in one there is twice as much wine as water and in the other three times as much water as wine Find how much must be drawn off from each to fill a third vessel of 15 gallons, so that its contents may be half wine and half water

256 Problems relating to Squares Let there be a number of men arranged in a solid rectangle as in the scheme below, where a star stands for a man

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In the front row there are 7 men, and as there are 4 such rows, the total number of men $=7\times4$ And generally if there are a men in the front row and b rows, then the total number of men $=a\times b$

Thus to ascertain the number of men in a **solid rectangle,** we *multiply the number of men in the "front" by the number of men in the "depth"*

Similarly we see that the number of men in a **solid square** is *the square of the number of men forming a side*, for in a square the number in the front is the same as the number in the depth.

The phrase "*four deep*" means that there are *four* rows of men including the front row

We shall now explain how the number of men in a **hollow square** can be found. We proceed by taking a numerical example though the reasoning is perfectly general

Suppose there is a hollow square, 4 deep, having 11 men in the front The annexed scheme shews how the men are arranged, the stars representing the men actually forming the hollow square and the dots the men wanted to fill up the hollow at the centre Thus the stars and the dots together form a solid square equal in area to the hollow one It is clear that the number of men in the hollow square = the number of men in the solid square *minus* the number filling the square hollow at the centre Now this square, as the dots shew, has a side containing 3 men $=(11-2\times4)$ men Therefore the number of men in the hollow square $=(11)^2-(11-2\times4)^2=112$

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Reasoning similarly we see that *the total number of men in a hollow square*

$$=a^2-(a-2b)^2,$$

where a represents the number of men in the front, and b the number of men forming the depth

Ex. An officer can form his men into a hollow square 5 deep, and also into a hollow square 6 deep, but the front in the latter formation contains 4 men fewer than in the former find the number of men [*Cal*, 1887].

Let x=number of men in the front of the first square, then by the question

$x-4=$ ... . second .. ,

$$x^2-(x-2\times 5)^2=\text{total number of men required}$$

also $$(x-4)^2-(x-4-2\times 6)^2=\text{total number of men required},$$

$$x^2-(x-10)^2=(x-4)^2-(x-16)^2,$$

whence $$x=35,$$

$$\text{required number of men}=(35)^2-(25)^2=(35+25)(35-25)$$
$$=60\times 10=600$$

Examples CXLVIII (*Continued*)

71 An officer can form the men in his battalion, numbering 1152, into a hollow square 12 deep, of how many men does the front consist?

72 A number of men can be formed into a hollow square 8 deep and also into a hollow square 4 deep, the side of the latter square containing 16 men more than the side of the former Find the number of men

73 A detachment from an army was marching in a regular column, with 5 men more in depth than in front, but upon the enemy coming in sight, the front was increased by 845 men, and by this movement, the detachment was drawn up in 5 lines Find the number of men in the detachment

74 On drawing up a regiment into a solid square, it is found that 40 men are left over On drawing up the same regiment into a solid column, it is found that there are 6 men more in the front and 4 men less in the depth than the number of men in the side of the square Find the number of men in the regiment

75 A regiment can be drawn up into a hollow square 6 deep, and with the addition of 140 men to its ranks, it can be drawn up into the same square, the depth being now increased by 1, find the number of men in the regiment

76 A regiment can be formed into a hollow square 8 deep, and if 260 men were to go away, the rest can be formed into a hollow square 7 deep with 5 men fewer in the front How many men are there in the regiment?

77 A company of soldiers was drawn up into a hollow square having 30 men in the front, another company was drawn up into a solid square having twice as many men in the front as there are men in the depth of the hollow square If the total number of men in the two detachments is 1440, find the number in each

78 A general wishing to draw up his regiment in the form of a hollow square found that he had 50 men over when it was 4 deep, but that he wanted 50 men to complete it when it was 5 deep, the number

of men in the front being the same in both cases Find the number of men in the regiment. [*Cal*, 1903]

79 At a review of an army, the troops were drawn up in a solid mass 40 deep, when there were just one-fourth as many men in front as there were spectators, had the depth, however, been increased by 5 and the spectators were drawn up in a mass with the army, the number of men in front would have been 100 fewer than before Determine the number of men of which the army consisted

257 The following are additional problems

Ex 1 A wall costs Rs 300 when brick sells at Rs 8 a thousand, and it costs Rs 365 when brick sells at Rs 10 a thousand, find the cost of labour in building the wall

Let x=required cost in rupees,
then $300-x$=price of brick, in rupees, by the first supposition,
and $365-x$= . second supposition

Since Rs 8 is the cost of 1000 bricks,

$$\frac{300-x}{8}=\text{so many thousands of bricks required for the wall,}$$

similarly $\dfrac{365-x}{10}=$ . ... ,

$$\frac{300-x}{8}=\frac{365-x}{10};$$

whence $x=40$

Thus the cost of labour is Rs 40

Ex 2 The expenses of a family are Rs 72 a month when rice is at 10 seers a rupee, and Rs 75 a month when rice is at 8 seers a rupee, if other expenses remain constant, find these

Here the quantity of rice consumed every month is supposed to be constant

Let x=this quantity in seers

Then $\dfrac{x}{10}$=price in rupees of x seers at 10 sr.

Also $\dfrac{x}{8}$=price in rupees of x seers at 8 sr

Thus by the first condition of the problem,

$$\text{other expenses}=\text{Rs}\left(72-\frac{x}{10}\right) \quad \ldots\ldots\ldots \quad (\text{i})$$

and by the second condition,

$$\text{the same expenses}=\text{Rs.}\left(75-\frac{x}{8}\right) \quad \ldots\ldots\ldots \quad (\text{ii})$$

Hence $$72-\frac{x}{10}=75-\frac{x}{8},$$

whence $$x=120$$

Thus the quantity of rice consumed every month is 120 seers, and therefore from (i) or (ii), the other expenses can be found

Thus from (i), the required expenses

$$=\text{Rs}\left(72-\frac{x}{10}\right)=\text{Rs}\ (72-12)=\text{Rs}\ 60$$

Ex 3 The expenses of a family are Rs 60 a month when rice is 14 seers per rupee, and Rs 57 a month when rice is 21 seers per rupee, other expenses remaining the same, what will they be when rice is 18 seers per rupee?

The problem will be solved if we find (i) the quantity of rice consumed every month, and (ii) the constant expenses

As in the last example, we shall find

(i) the quantity of rice consumed every month $=126$ sr,

(ii) the constant expenses $=$ Rs 51.

Hence the expenses required $=$ Rs $\frac{126}{18}+$ Rs $51=$ Rs 58

The solution of this Example may be better put in the following form.

Let $x=$ quantity in seers, of rice consumed every month,

and $y=$ other expenses in rupees

Then by the first condition of the problem Rs $\frac{x}{14}$ is the price of the rice consumed every month

$$\frac{x}{14}+y=60 \quad \ldots\ldots\ldots \quad (\text{i})$$

Similarly $$\frac{x}{21}+y=57. \quad \ldots\ldots\ldots \quad (\text{ii}),$$

and $$\frac{x}{18}+y=\text{expenses reqd to be found}\ldots\ldots \quad (\text{iii}).$$

From (i) and (ii), $$\frac{x}{14}-\frac{x}{21}=60-57,$$

whence $$x=126$$

from (i), $$y=60-\frac{x}{14}=51.$$

Hence from (iii), expenses reqd $=\frac{x}{18}+y=$ Rs. $(7+51)=$ Rs. 58.

Try Example 2 in this way.

Ex 4 A hare is 80 of her own leaps before a greyhound, she takes 4 leaps for every 3 that he takes, but he covers as much ground in one leap as she does in two How many leaps will the hare have taken before she is caught?

Let x = required number of leaps,

and a = length of ground covered by one leap of the hare;

thus, since the hare takes 4 leaps for every 3 leaps that the greyhound takes,

$\frac{3}{4}x$ = number of leaps, the greyhound takes in the same time that the hare takes x leaps,

and $2a$ = length of ground covered by one leap of the greyhound,

$80a + xa$ = distance of the place of capture from the place where the dog first was,

and $\frac{3}{4}x \times 2a$ = the distance the dog had to run to overtake the hare; now these two distances are the same, [see Art 252, Ex 1],

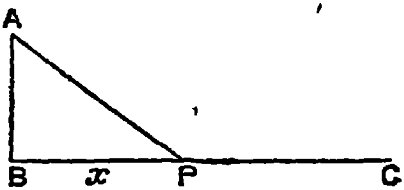
therefore $80a + xa = \frac{3}{4}x \times 2a$,

divide by a, thus $80 + x = \frac{3}{2}x$; whence $x = 160$

Thus the hare takes 160 leaps before she is caught

Ex 5 A peacock perched on the top of a tower 9 cubits high saw a snake approaching its hole at the foot of the tower from a distance equal to 3 times the height of the tower He immediately swooped down and caught it at a place equally distant from the top of the tower and the spot where the snake had been first seen Say quickly at what distance from its hole the snake was captured. *(Lilavati)*

Let A be the top of the tower, B the hole, C the spot where the snake was first seen and P the place where it was caught



Then $AB = 9$ cubits, $BC = 27$ cubits

Let $BP = x$ cubits

Thus $CP = (27 - x)$ cubits $= AP$, by the question

Hence from the right-angled $\triangle ABP$, we have

$$(27 - x)^2 = x^2 + 9^2,$$

whence $x = 12$

Thus the snake was captured at a place 12 cubits from its hole.

Examples CXLVIII (*Continued*)

80 There are 10 coins in a bag, crowns and half-crowns If the number of each kind were reversed, the value would be diminished by 5 shillings Find the number of each kind

81 A and B were employed together for 50 days, each at 5s a day, during which time A by spending 6d a day less than B, had saved 3 times as much as B and $2\frac{1}{2}$ days' pay besides What did each spend per day?

82 Find the number whose fourth root is equal to $\frac{1}{2}$ its cube root

83 A boy, selling oranges, sells half his stock and one more to A, half of what remains and two more to B, and three that still remains to C How many had he at first?

84 Find a number such that if you divide it by 13 or by 14, the remainder would be 1 and the difference between the quotients 1

85 A grocer buys a number of eggs at 6s 6d per hundred He sells all but 69 of them at the rate of 11 for a shilling, and then finds that he has received 30s more than he gave for the whole number How many eggs did he buy?

86 Wishing to buy a certain number of railway shares, I found that if I bought them in the E B Railway at Rs 400 a share, I should invest all my money, but if I bought them in the E I Railway at Rs 450 a share, I should not have money enough by Rs 2400 How much money had I to invest?

87 A tradesman buys an article and sells it at 12 per cent profit If it had cost him R1 more than it did and he had sold it for R1 more than he did, the rate of profit would have been 10 per cent What did it cost him?

88 A tradesman, after expending 100l a year, augments the remainder of his property by one third part of it, and at the end of 3 years his original property is doubled, what had he at first?

89. A and B began to pay off their debts with different sums, A's money at first was $\frac{2}{3}$rds of B's, but after A had paid £1 less than $\frac{3}{4}$ths of his money and B £1 more than $\frac{7}{8}$ths of his, it was found that B had only half as much as A What had each at first?

90 A gentleman, meeting with 3 beggars, gave to the first $\frac{1}{3}$ of what he had in his pocket and then 1 rupee more, to the second $\frac{1}{4}$ of what he had left and then one rupee more, to the third $\frac{1}{5}$ of what he had still left and then one rupee more, after which he had nothing left How much had he at first?

91 Find three numbers, differing in order from one another by 5, such that 12 times the product of the greatest and least may be equal to the square of the sum of the three numbers.

92 A garrison of 4000 men had provisions for 20 days, after 11 days it was reinforced and then the provisions were exhausted in 6 days, find the number of men in the reinforcement

93 A besieged garrison had provisions for 70 days, after 10 days a party of 1500 men made a sally and escaped, and the allowance per head being now reduced to $\frac{5}{6}$ths of what it was before, the garrison held out for 100 days more, what was the number of men in the garrison at first?

94 The expenses of a family when rice is 12 seers for a rupee are 50 rupees a month, when rice is 14 seers for a rupee the expenses are 48 rupees a month (other expenses remaining unaltered); what will they be when rice is at 16 seers per rupee?

95 A bag contains sixpences, shillings and half-crowns, the amount expressed by each kind is the same, if the total number of coins in the bag be 119, find the number of each

96 A man wished to enclose a piece of ground with palisades, and found that if he set them a foot asunder, he should have too few by 150, but if he set them a yard asunder, he should have too many by 70 How many had he?

97 A debt which might have been paid exactly with $5x$ half-sovereigns and x half-crowns, was paid out of a ten-pound note and the change was to be equal to $15x$ half-crowns and x half-sovereigns Find x and the amount of the debt

98 Her Majesty Empress Victoria was born May 24, A.D x, and Prince Albert was born August 26th in the same year, their united ages on the 26th August, 1819, amounted to 3 times the age of Prince Albert on the birthday next preceding his marriage, which took place February 10, 1840 In what year was each born?

99 A certain number of sovereigns, shillings, and sixpences together amount to £8 6s 6d, and the amount of the shillings is a guinea less than that of the sovereigns and a guinea and a half more than that of the sixpences Find the number of each coin

100 A farmer bought equal numbers of two kinds of sheep, one at £3 each the other at £4 each, if he had expended his money equally in the two kinds, he would have had 2 sheep more than he did Find how many of each kind he bought

101 A fruiterer sold a certain number of oranges for £6 10s If he had given 2 more oranges for a shilling, the same quantity would only have realised £5 17s How many oranges did he sell?

102 If 18 tolahs of gold weigh 17 tolahs in water, and 8 tolahs of silver weigh 7 tolahs in water, find the quantities of gold and silver in a compound of gold and silver, which weighs 100 tolahs in air and 90 tolahs in water

103 *A* and *B* agreed to reap a field for Rs 20 If they had worked together every day, the field would have been reaped in 15

days, but at the end of 7 days, A left off working for 4 days, and it consequently took $16\frac{1}{2}$ days to reap the field. In how many days could A alone, and in how many days could B alone have reaped the field? And what share of the Rs 20 ought each to receive for the work he actually did?

104. When flour costs 7*s* a bushel, the baker sold a loaf for 16*d*; when it rose to 10*s* 6*d*, he sold a loaf of the same weight for 21*d*; the price of baking being the same in each case, find that price.

105. The duty on salt being raised 8 annas per maund, it is found that the consumption has fallen one-eighth, but the revenue, instead of increasing, has remained stationary; what was the duty at first?

106. The annual rent of a paddy-field consists of Rs 55 and a corn-rent; when paddy sells at Re 1 12 as a maund, the landlord gets at the rate of Rs 3 per bigha; when it sells at Rs 2 per maund, he gets at the rate of Rs $3\frac{1}{6}$ per bigha. Required the amount of the corn rent and the area of the field.

107. The expenses of a tram-car company are fixed, and when it only sells three-penny tickets for the whole journey, it loses 10 per cent. It then divides the route into 2 parts and sells two-penny tickets for each part, thereby gaining 4 per cent and selling 3300 more tickets every week. How many persons used the cars weekly under the old system?

108. A carrier charges 3*d* each, for all parcels not exceeding a certain weight; and on heavier parcels he makes an additional charge for every 7 lb above that weight. The charge for half a cwt is 1*s* 3*d*, and the charge for 9 stones is 5 times that for 1 quarter. What is the scale of charges? (1 stone = 14 lb.)

109. A fraudulent trader has two weights, one as much over one seer as the other is under it; by using the heavier weight, he bought 6 maunds of an article at 8 annas a seer, which he sold at the same rate by using the lighter weight, thus gaining Rs 16. What were the weights?

110. In a certain tank, the tip of a bud of a lotus was seen a span above the surface of the water; forced by the wind it gradually advanced, and was submerged at a distance of two cubits; tell me quickly, O mathematician, what is the depth of the water. (*Lilavati.*)

111. Find a number such that if it be divided by a or by b, the remainder will be c, and the sum of the quotients will be c.

112. There are two kinds of coin, of which a and b pieces respectively are equivalent to £1; how many pieces of each kind must be taken so that c pieces together may be equivalent to £1?

113. A labourer is engaged for n days on condition that he receives a pence for every day he works and pays b pence for every

day he is idle; at the end of the time he receives m pence, how many days does he work and how many days is he idle?

114. A person has just a hours at his disposal, how far may he ride in a coach which travels b miles an hour, so as to return home in time, walking back at the rate of c miles an hour?

115. Two persons set out at the same time to meet each other, the one walking a miles per hour meets the other, who walks b miles per hour, at a place which is c miles from the mid-station of the road. Required the length of the road.

116. A merchant buys an article, subject to a duty of a per cent, and sells it at a loss of b per cent., but if he had sold it for Rs c more, he would have gained d per cent. on his bargain. What was the price of the article?

117. If a men or b boys can dig m acres in n days, shew that the number of boys whose assistance will be required to enable $(a-p)$ men to dig $(m+p)$ acres in $(n-p)$ days is

$$\frac{pb}{a}\left\{1+\frac{a}{m}\frac{m+n}{n-p}\right\}$$

238. Problems leading to Simple Equations in two or more variables

Ex 1. A number of 3 digits is such that if 198 be added to it, the digits are reversed, but if the tens' digit be subtracted from 5 times the hundreds' digit the difference is the units' digit. If the sum of all the digits is 6, find the number.

Let x, y and z represent the hundreds', tens' and units' digits respectively. Then the required number is $100x+10y+z$.

By the first condition,

$$100x+10y+z+198=100z+10y+x,$$

whence $z-x=2$ . . . (i),

By the second condition, $5x-y=z$. (ii),

and by the third condition, $x+y+z=6$ .. (iii)

Add (ii) to (iii), thus $6x=6$, or $x=1$

Hence from (i), $z=3$, and from (iii), $y=2$

Thus the required number $=123$

Ex. 2. A and B can finish a piece of work in 10 days, B and C in 15 days, and C and A in 18 days, how long would they take to finish the work together?

Let x, y, z be the number of days in which A, B and C respectively can finish the work.

Hence if W represent the whole work, then in 1 day A does

$\frac{1}{x}$ of W, B does $\frac{1}{y}$ of W and C does $\frac{1}{z}$ of W. But by the question, A and B together do $\frac{1}{10}$ W in one day,

$$\left(\frac{1}{x}+\frac{1}{y}\right)W=\frac{1}{10}W,$$

or $$\frac{1}{x}+\frac{1}{y}=\frac{1}{10} \quad \ldots \quad (i)$$

Similarly $$\frac{1}{y}+\frac{1}{z}=\frac{1}{15} \quad \ldots \quad (ii),$$

and $$\frac{1}{z}+\frac{1}{x}=\frac{1}{18} \quad \ldots \quad (iii)$$

From (i), (ii) and (iii) by addition we have

$$\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=\frac{1}{9} \quad \ldots \quad (iv),$$

i e, A, B and C do $\frac{1}{9}$ of the work in 1 day

∴ they do the whole work in 9 days

Ex. 3 In the last example, find how long would each take to do the work separately

Subtract (ii) from (iv), thus $\frac{1}{x}=\frac{2}{45}$, whence $x=22\frac{1}{2}$

Thus A takes $22\frac{1}{2}$ days to do the work

Similarly we get $y=18$ days by subtracting (iii) from (iv), and $z=90$ days by subtracting (i) from (iv)

Ex 4 Two passengers have together 7 maunds of luggage, and for the excess above the weight allowed free, one of them is charged Rs 3 and the other Rs 5 If all the luggage had belonged to one passenger, he would have been charged Rs 11 What amount of luggage is each passenger allowed free of charge and how much luggage had each passenger? [*Bom*, 1900]

Let x = weight in mds, allowed free of charge,

and y = . . . the first passenger had,

$7-y$ = . second

Since the passengers are charged Rs 3 + Rs 5 or Rs 8 for excess, the excess charge on $(7-2x)$ mds is Rs 8,

∴ the excess charge on every md = Rs $\frac{8}{7-2x}$. (i)

Similarly Rs. 11 is the excess charge on $(7-x)$ mds,

∴ the excess charge on every md. = Rs $\frac{11}{7-x}$. (ii)

Also the first passenger paid Rs 3 as excess charge on $(y-x)$ mds,

$$\therefore \text{ the excess charge on every md} = \text{Rs. } \frac{3}{y-x} \quad \ldots \text{ (iii)}$$

Thus from (i), (ii), and (iii), we get $\frac{8}{7-2x}=\frac{11}{7-x}=\frac{3}{y-x}$.

Hence from the first equation, we have $x=1\frac{1}{2}$, and from the other equation $y=3$

Thus $1\frac{1}{2}$ mds is allowed free of charge, the first passenger had 3 mds and the second had $(7-y)$ mds or 4 mds

Ex 5 A boat goes 52 miles down a river in 8 hours and 42 miles up the river in 12 hours Find the rates of the boat and the river

Let x = rate of the boat in miles per hour,
and y = .. river . . .

Thus in 1 hour the boat goes $(x+y)$ miles *down* the river and $(x-y)$ miles *up* the river

Hence $\frac{52}{x+y}$ = time in hours, the boat takes to go 52 miles *down*,

and $\frac{42}{x-y}$ = ... 42 .. *up*

$\therefore$ by the conditions of the problem, $\frac{52}{x+y}=8$ and $\frac{42}{x-y}=12$

Solving these we have $x=5$ and $y=1\frac{1}{2}$

Thus the required rates are 5 miles and $1\frac{1}{2}$ miles per hour

Ex 6 A boat goes up stream 30 miles and down stream 44 miles in 10 hours, it also goes up stream 40 miles and down stream 55 miles in 13 hours, find the rate of the stream and of the boat [*Cal*, 1880]

Let x = rate of boat in miles per hour,
and y = . . stream.. .. . . ,

then $\frac{30}{x-y}$ = time of coming up the stream,

$\frac{44}{x+y}$ = . going down . . . ,

and since these two times taken together = 10 hours, we get

$$\frac{30}{x-y}+\frac{44}{x+y}=10 \qquad (1).$$

Similarly $$\frac{40}{x-y}+\frac{55}{x+y}=13 \qquad (2)$$

Multiply (1) by 4 and (2) by 3, and subtract, thus

$$\frac{11}{x+y}=1, \text{ or } x+y=11 \qquad (3)$$

Again multiply (1) by 5 and (2) by 4, and subtract, thus

$$\frac{10}{x-y}=2, \text{ or } x-y=5 \qquad (4)$$

Add and subtract (3) and (4), thus $x=8$ and $y=3$

Ex 7 A challenged B to ride a bicycle race of 1040 yards He first gave B 120 yards start, but lost by 5 seconds, he then gave B 5 seconds start and won by 120 feet How long does each take to ride the distance? [*Cal*, 1881]

Let $x=A$'s rate and $y=B$'s rate, in yards per second,

$\therefore \frac{1040}{x}=$ time (in seconds) which A takes to run 1040 yards

$\frac{920}{y}=$ . . B 920 .

and $\frac{1000}{y}=$ . . .. B 1000

Now by the first condition of the problem the time which A takes to run 1040 yards is the same as B takes to run 1040 − 120 (or 920) yards + 5 seconds, therefore

$$\frac{1040}{x}=\frac{920}{y}+5 \qquad (1)$$

Again the time which A takes to run 1040 yards is, by the second condition, the same as B takes to run 1040 yards less 120 feet (*i.e.*, 1000 yards) − 5 seconds, therefore

$$\frac{1040}{x}=\frac{1000}{y}-5 \qquad (2)$$

Subtract (1) from (2), thus

$$\frac{1000}{y}-\frac{920}{y}-10=0, \text{ whence } y=8$$

Substitute y in (1), thus

$$\frac{1040}{x}=\frac{920}{8}+5, \text{ whence } x=\frac{26}{3}$$

Hence the time A takes to ride 1040 yards $=\frac{1040}{\frac{26}{3}}$ sec $=120$ sec

and. .... B. ... . $=\frac{1040}{8}$ sec $=130$ sec.

Ex 8 The price of a passenger's ticket on a French railway is proportional to the distance he travels, he is allowed 25 kilogrammes of luggage free, but on every kilogramme beyond this amount, he is charged a sum proportional to the distance he goes If a journey of 200 miles with 50 kilos of luggage cost 25 francs, and a journey of 150 miles with 35 kilos cost $16\frac{1}{2}$ francs, what will a journey of 100 miles with 100 kilos of luggage cost?

Let x francs = price of a ticket for each mile,

and y francs = fare charged per mile on every kilo of excess luggage,

thus $200x+200\times(50-25)y=25$, or $x+25y=\frac{1}{8}$ (1),

$150x+150\times(35-25)y=16\frac{1}{2}$, or $x+10y=\frac{11}{100}$ (2),

and $100x+100\times(100-25)y=$ required cost in francs (3)

Subtract (2) from (1), thus

$$25y-10y=\frac{1}{8}-\frac{11}{100}, \text{ or } y=\frac{1}{1000}.$$

Substitute y in (1), thus

$$x+25\times\frac{1}{1000}=\frac{1}{8}, \text{ or } x=\frac{1}{10}$$

Substitute x and y in (3), thus

$\therefore$ cost required $=(100\times\frac{1}{10}+100\times75\times\frac{1}{1000})$ francs $=17\frac{1}{2}$ francs

Examples CXLIX.

1. A and B play together, first A loses Rs 10 and then he has $1\frac{1}{2}$ times as much as B next B loses $\frac{1}{2}$ of what he had at first and one rupee more, and he has now half as much as A What had each at first?

2 A and B being at play severally, cut packs of cards so as to take off more than they left Now it so happened that A cut off twice as many as B left, and B cut off 7 times as many as A left. How many cards did each cut?

3 A certain number when divided by another gives a quotient 4 and a remainder 3 If twice the first number be divided by three times the second number, the quotient is 2 and the remainder 20 Find the numbers

4 A sum of money put out to simple interest amounts in 8 months to £297. 12s, and in 1 year 3 months to £306; find the sum and the rate of interest per cent per annum

5 A tradesman sells 2 articles together for Rs 46, making 10 per cent profit on one and 20 per cent on the other If he had sold each article at 15 per cent profit, the result would have been the same At what price does he sell each article?

6 A grocer gains 20 per cent by selling at 2s a lb a mixture formed by mixing with 7 lbs of a common tea, 2 lbs of a better

kind But if he had mixed 7 lbs of the latter with 2 lbs of the former kind, he would have lost 20 per cent by selling the mixture at that price What did each kind of tea cost him per lb ?

7 A, B and C are 3 villages connected by straight roads, from A to C through B, the distance is 64 miles, from B to A through C, the distance is 74 miles, and from C to B through A, the distance is 80 miles Find the length of each road

8 Divide Rs 908 among A, B and C, so that the shares of A and B together may exceed the share of C by Rs 300, and the shares of B and C together may exceed the share of A by Rs 278

9 A gentleman gave away in charity a certain sum of money, had there been 3 more beggars, each would have received 1s less than he did, but if there had been 2 fewer, each would have received 1s more Required the number of beggars and the amount each received

10 A boy at a fair spends his money in oranges, if he had received 5 more for his money, they would have averaged a half-penny each less, but if 3 less, a half-penny each more, how much did he spend and what was the number of oranges he bought ?

11 A number consists of two digits When the number is divided by the sum of the digits, the quotient is 7 The sum of the reciprocals of the digits is 9 times the reciprocal of the product of the digits Find the number

12 A certain number of 2 digits is equal to 7 times the sum of the digits If the digit in the units' place be decreased by 2 and that in the tens' place by 1, and the number thus formed be divided by the sum of its digits, the quotient is 10 Find the number

13 There is a number the 3 digits of which are in descending order of magnitude and differ from one another in succession by the same amount If the number be divided by the sum of its digits, the quotient will be 48, and if from the number 198 be subtracted, the digits of the difference will be the same as those of the original number but in reverse order Find the number

14 There is a number of 3 digits whereof the hundreds' digit is $\frac{1}{4}$ of the number formed by the other digits, if 45 be added to the number, the units' and tens' digits interchange places, and if the hundreds' digit be taken from the last digit, the difference is twice the tens' digit What is the number ?

15 The united ages of a man and his wife are six times the united ages of their children Two years ago their united ages were ten times the united ages of their children, and six years hence their united ages will be three times the united ages of their children How many children have they ?

16 A, B and C sit down to play, in the first game, A loses to

each of B and C as much as each of them has, in the second, B loses similarly to A and C, and in the third, C loses similarly to A and B, and now they have each 24s What had they each at first?

17 If A and B together can do a certain piece of work in 8 days, B and C together in 10 days and C and A together in 9 days, in how many days would each do the work if he worked by himself?

18 A cistern holding 1200 gallons is filled by 3 pipes A, B, C together in 24 min The pipe A requires 30 min more than C to fill the cistern, and 10 gallons less run through C per min than through A and B together Find the time in which each pipe would fill the cistern

19 A cistern is filled by 3 cocks, two of which are exactly of the same dimensions, when they are all open $\frac{5}{12}$ths of the cistern is filled in 4 hours, and if one of the equal cocks be stopped $\frac{7}{9}$ths of the cistern is filled in 10 hours and 40 minutes In how many hours would each cock fill the cistern?

20 A man rowing against a stream meets a log of wood which is being carried down by the current He continues rowing in the same direction for a quarter of an hour longer and then turns and rows down the stream, overtaking the log $1\frac{1}{2}$ miles lower down than the point where he first met it Find the rate at which the current flows

21 A person walks a certain distance in a certain time, had his rate been half a mile an hour faster, he would have performed his journey in $\frac{4}{5}$ths of the time, and had it been half a mile an hour slower, he would have been $2\frac{1}{2}$ hours longer on the road Required the distance and his rate of travelling

22 A and B start together from the same point on a walking match round a circular course After half an hour, A has walked three complete circuits, and B four and a half Assuming that each walks with uniform speed, find where B next overtakes A

23 At 7 40 A M an ordinary train starts from P, and reaches Q at 11 40 A M, an express train, which starts from Q at 9 A M, arrives at P at 11 40 A M, if both trains travel at a uniform speed without stopping, find the time when they meet

24 A Rhine steamer sails up the stream 40 miles and down the stream 48 miles in 8 hours on another occasion she sails up the stream 56 miles and down 96 miles in 13 hours Find the rate of the steamer and that of the river

25 A and B start to run a race to a certain post and back again, A returning meets B at 90 yards from the post and arrives at the starting place 3 minutes before him, if he had returned immediately to meet B, he would have met him at one-sixth of the distance between the post and the starting place Required the length of the course and the duration of the race

26 A railway train after running for half an hour meets with an accident, after which it proceeds with $\frac{3}{5}$ths of its former rate, and is thus 1 hour 10 minutes behind time, had the accident occurred 30 miles further on, it would have arrived 25 minutes earlier Required the rate of the train and the length of the line [See Ex 29, p 355]

27 The fore-wheel of a carriage makes 6 revolutions more than the hind-wheel in 120 yards, but only 4 revolutions more when the circumference of the fore-wheel is increased one-fourth and that of the hind-wheel one-fifth Find the circumference of each wheel

28 Two passengers have together 5 cwt of luggage, and are charged for the excess above the weight allowed 5*s* 2*d* and 9*s* 10*d* respectively Had the luggage all belonged to one of them, he would have been charged 19*s* 2*d* How much luggage is allowed free of charge and what amount of weight had each passenger?

29 In a quarter of a mile race, A gives B a start of 22 yards and beats him by 2 seconds In a 300 yards race, he gives B a start of 2 seconds and beats him by $10\frac{1}{2}$ yards Find the rate of each

30 A and B run a mile race In the first heat B receives 12 seconds start and is beaten by 44 yards In the second heat B receives 165 yards start, and arrives at the winning post 10 seconds before A Find the time in which each can run a mile

31 A mail coach runs between two places A and B, and back again A traveller, who starts walking from A 5 hours before the mail coach, is overtaken by it half way between A and B He then doubles his rate of walking and meets the mail coach on its return journey 3 miles from B The traveller then goes to B at the same rate and returns, and by the time he comes again midway between A and B, the mail coach reaches A Find the distance between A and B and the rate at which the mail coach runs

32 A railway train travels from A to C passing through B where it stops 7 minutes, 2 minutes after leaving B, it meets an express train which started from C when the former was 28 miles on the other side of B, the express travels at double the rate of the other, and performs the journey from C to B in $1\frac{1}{2}$ hours, if on reaching A, it returned at once to C, it would arrive 3 minutes after the first train Find the distances between A, B and C, and the speed of each train

33 If 22 oxen eat 33 acres of grass in 54 days, and 17 oxen eat 28 acres of grass in 84 days, how many oxen will eat 40 acres of the same grass in 24 days, the grass being supposed in all cases to grow uniformly? [See Ex 8]

34 If 22 oxen and 28 cows eat 24 acres of grass in 18 weeks, and 20 oxen and 38 cows eat 30 acres of grass in 27 weeks, and 41 oxen and 26 cows eat 50 acres of grass in 60 weeks, how long will 40 acres of the same grass last 35 oxen and 14 cows, the grass being supposed in all cases to grow uniformly? [See Ex 8]

35 A letter-carrier has a hours allowed to him for going from A to B and back again, including c hours for rest at B But he finds that he can get b hours for rest by going d miles an hour faster each way Find his ordinary speed, and the distance from A to B

36 Two trains, one a feet and the other b feet long, move with uniform speed on parallel rails, when they move in opposite directions they pass each other in m seconds, but when they move in the same direction, the faster train passes the other in n seconds Find the rates of the two trains

Ex. $a=125$, $b=115$, $m=3$, $n=30$.

37 A waterman rows a distance of a miles on a river and back again in t hours, and finds that he can row b miles with the stream in the same time that he can row c miles against it, determine the times of going and returning, and the velocity of the stream

Ex. $a=39$, $t=16$, $b=26$, $c=6$

38 To complete a certain work A requires m times as long a time as B and C together, B requires n times as long as A and C together, and C requires p times as long as A and B together Prove that

$$\frac{1}{m+1}+\frac{1}{n+1}+\frac{1}{p+1}=1.$$

CHAPTER XXIII

RATIO AND PROPORTION

Ratio

239 *Definitions* There are two ways of comparing quantities of the same kind with respect to magnitude (i) by finding by *how much* one is greater than the other, and (ii) by finding *how many times* one is contained in the other Thus 4 feet and 12 feet are compared by observing that one is greater than the other by 8 feet, or by ascertaining that one is 3 times as big as the other In the first mode of comparison, we take the difference between the quantities, and in the second, *the quotient of one divided by the other* It is with the latter mode that we are concerned in ratios Hence we have the following definition of Ratio

Ratio is the relation between two quantities of the same kind in respect of magnitude, the relation being ascertained by considering how many times one is contained in the other

Since a ratio expresses a certain number of times, it must necessarily be an **abstract number**, whether the quantities compared be themselves abstract or concrete

As we are unable to make a comparison between two quantities of different kinds, as for instance between a line and an area or between a horse and a cart, it is clear that *there can be no ratio between concrete quantities of different kinds*

The notation "$a : b$" expresses the ratio between a and b The quantities a and b are called the **terms** of the ratio, and the first, *viz*, a, is called the **antecedent** and the second, *viz*, b, the **consequent** of the ratio

If the antecedent of a ratio is *equal* to the consequent, the ratio is termed a ratio of **equality**, if *greater*, a ratio of **greater inequality** or of **majority**, and if *less*, a ratio of **less inequality** or of **minority**

Note In finding the ratio between two concrete quantities of the same kind, we must be careful to *reduce both of them to the* **same denomination** Thus the ratio between 5 rupees and 12 annas is not $5 : 12$ but $80 : 12$, for 5 rupees $= (5 \times 16)$ annas $= 80$ annas

260 A ratio is expressed as a fraction We have seen [Art 2] that every quantity whether abstract or concrete, is expressed by a number We have also seen [Art 180] that a fraction expresses the quotient when the numerator is divided by the denominator Thus the quotient when 4 bighas is divided by 5 bighas is $\frac{4}{5}$ Hence $4 : 5$ is the same as $\frac{4}{5}$

Thus the ratio between 4 bighas and 5 bighas is $4 : 5$ where $4 : 5$ expresses the number of times 5 is contained in 4 and is obtained by dividing 4 by 5

Hence *a ratio is expressed by a fraction, whereof the numerator is the antecedent and the denominator the consequent of the ratio*

Thus all the propositions regarding Fractions will apply to Ratios For example, since

$$\frac{a}{b} = \frac{am}{bm} = \frac{a-m}{b-m} \quad \text{[Art 181]},$$

we have the Theorem—*If the terms of a ratio be both multiplied, or both divided, by the same quantity, the ratio remains unaltered*

Since a ratio can be expressed as a fraction, we can compare ratios as we compare fractions

Thus the ratios $4 : 5$, $5 : 8$ and $7 : 9$ are compared by comparing the fractions $\frac{4}{5}$, $\frac{5}{8}$ and $\frac{7}{9}$

261. An important principle We see that

$$\frac{104}{117} = \frac{13 \times 8}{13 \times 9} = \frac{8}{9},$$

i e, 104 is the same multiple of 8 that 117 is of 9.

Reasoning similarly we may generally conclude that if $\frac{x}{y}=\frac{a}{b}$, then x is the same multiple of a that y is of b

Hence if $\frac{x}{y}=\frac{a}{b}$, we can legitimately assume that $x=ka$ and $y=kb$, where k is the constant factor common to x and y

Ex 1 If $5x+2y=8y-3x$, find the ratio $x : y$

By transp, we have $8x=6y$, or $\frac{x}{y}=\frac{6}{8}=\frac{3}{4}$

Otherwise —Divide by y, thus

$$5\left(\frac{x}{y}\right)+2=8-3\left(\frac{x}{y}\right),$$

whence $$8\left(\frac{x}{y}\right)=6, \text{ or } \frac{x}{y}=\frac{6}{8}=\frac{3}{4}$$

Ex 2 If $x : y$ be equal to $2 : 3$, find the ratio $6x-y : 3x+2y$

Since $x : y=2 : 3$, we have $x=2k$ and $y=3k$

Thus $$\frac{6x-y}{3x+2y}=\frac{12k-3k}{6k+6k}=\frac{9k}{12k}=\frac{3}{4}$$

Otherwise $$\frac{6x-y}{3x+2y}=\frac{6\left(\frac{x}{y}\right)-1}{3\left(\frac{x}{y}\right)+2}=\frac{6\times\frac{2}{3}-1}{3\times\frac{2}{3}+2}=\frac{3}{4}.$$

Ex 3 If $2x^2-11xy+15y^2=0$, find the ratio $x : y$.

Divide by y^2, thus we have

$$2\frac{x^2}{y^2}-11\frac{x}{y}+15=0,$$

or $$2k^2-11k+15=0, \text{ if } \frac{x}{y}=k,$$

i e, $$(k-3)(2k-5)=0,$$

$$\text{either } k=3, \text{ or } k=\tfrac{5}{2}$$

Thus the required ratio is either $3 : 1$ or $5 : 2$

Ex 4 What number must be added to the terms of the ratio $5 : 7$ to make it equal to the ratio $13 : 14$?

Let x=the number required

Thus $$\frac{5+x}{7+x}=\frac{13}{14}, \text{ whence } x=21$$

Ex 5 Find two numbers in the ratio $2 : 3$, such that the sum of their squares is 1053

Let x and y be the numbers

Then since they are in the ratio $2 : 3$, we have $x=2k$ and $y=3k$

By the question, $x^2+y^2=1053$,

i e, $(2k)^2+(3k)^2=1053$,

whence $13k^2=1053$, or $k=9$

Hence required numbers are $x=2k=18$, $y=3k=27$.

Examples CL

What is the ratio of

1 R 3 to $15a$? 2 1 md 10 sr to 2 mds ?

3 2 yd 8 in to 2 ft ? 4 £8 to 5 lb ?

5 Compare the ratios $3 : 4$, $5 : 6$ and $2 : 3$

6 If $17x-3y=24y-x$, find the ratio $x : y$

7 If $\dfrac{4a-3}{4(b-1)}=\dfrac{a+9}{b+12}$, find the ratio $a : b$

8 If $(x+y)^2=a(x-y)^2$, find the ratio $x : y$

9 $x : y=3 : 4$, find the ratio $3x+y : 5y-2x$

10 If $x : y=m : n$, find the ratio $3y-x : 2x+y$

11 If $4x-y : 2x+3y=1 : 4$, find $5x-2y : 2x+3y$

12 If $4x^2-3y^2 : 2x^2+5y^2=12 : 19$, find $x : y$

13. If $12x^2-7xy=12y^2$, find the ratio $x : y$

14 For what value of x will the ratio $x+2 : 5-x$ be equal to the ratio $5 : 6$?

15 What number must be subtracted from the terms of the ratio $53 : 67$ to make it equal to the ratio $3 : 4$?

16 The ratio $11 : 18$ becomes $2 : 3$ when a certain number is added to its terms, find the number

17 If the ratio $a : b$ remains unchanged when x and $2x$ are added to a and b respectively, find the value of $a : b$

18 Two numbers are in the ratio of $m : n$, and if c be subtracted from each, the ratio is $a : b$, find the numbers

19 Divide 135 into two parts which are the ratio $4 : 5$

20 Find two numbers in the ratio $5 : 7$, such that the sum of their squares is 1184

21 Find two numbers in the ratio $3 : 4$, such that the difference of their squares is 252

22. A number of two figures is altered in the ratio of $n : m$, if its digits be interchanged; shew that the digits are to each other in the ratio of $10m - n$ to $10n - m$.

262. **Composition of Ratios.** If the antecedents of any number of ratios be multiplied together, and also the consequents, the ratio of the two products is said to be a ratio **compounded** of the given ratios. Thus $ace : bdf$ is the ratio which is obtained by compounding the three ratios $a : b$, $c : d$, and $e : f$.

When *two equal* ratios are compounded, the resulting ratio is called the **duplicate ratio** of the given ratios. Thus if $a : b$ and $a : b$ are compounded, the ratio $a^2 : b^2$ is the duplicate ratio of $a : b$.

When *three equal* ratios are compounded, the new ratio is called the **triplicate ratio** of the given ratios. Thus if $a : b$, $a : b$ and $a : b$ are compounded, we obtain the ratio $a^3 : b^3$ which is called the triplicate ratio of $a : b$.

The ratio of the square roots of the terms of a ratio is called its **sub-duplicate ratio**. Thus $\sqrt{a} : \sqrt{b}$ is the sub-duplicate ratio of $a : b$. Similarly $\sqrt[3]{a} : \sqrt[3]{b}$ is the **sub-triplicate** ratio of $a : b$, and so on.

The ratio $a^{\frac{3}{2}} : b^{\frac{3}{2}}$ is called the sesquiplicate ratio of $a : b$.

Note. The duplicate, triplicate, sub-duplicate, &c., ratios are sometimes called respectively the *double*, *triple*, *half*, &c. ratios; hence $a^{\frac{1}{n}} : b^{\frac{1}{n}}$ is sometimes termed the $\frac{1}{n}$th of the ratio $a : b$.

Examples CLI.

Find the ratio compounded of

1. $2 : 3$, $5 : 8$ and $12 : 25$.
2. $3 : 4$, $2\frac{1}{4} : 3$ and $1\frac{2}{3} : 1-\frac{7}{8}$.
3. $3ax : 4by$, $a^2 - x^2 : c^2 - x^2$, $bc + bx : a^2 + ax$ and $c - x : a - x$.
4. $1 - x^2 : 1 + y$, $1 - y^2 : x + x^2$ and $1 + \dfrac{x}{1-x} : 1$.

5. Write down the duplicate ratio of $4 : 5$, the triplicate ratio of $1 : 2$, and the sub-duplicate ratio of $256 : 361$, and of $507 : 588$.

6. If $2x + 3 : 5x - 2$ in the duplicate ratio of $2 : 3$, find x.

7. If $x - 3 : 2 + 7x$ in the triplicate ratio of $1 : 2$, find x.

8. If $m : n$ in the duplicate ratio of $a + x : a - x$, shew that

$$\frac{1}{2}\left(\frac{a}{x} + \frac{x}{a}\right) = \frac{m+n}{m-n}.$$

9 If $a : b$ in the triplicate ratio of $x-a : x-b$, shew that

$$x^3-3abx+ab(a+b)=0$$

10 If $x : y$ in the duplicate ratio of m to n, and m to n in the sub-duplicate ratio of p^2+x^2 to p^2-y^2, shew that

$$p^2 : xy = x+y : x-y$$

11 If x has to y the triplicate ratio of a to b, and a to b, the sub-triplicate ratio of $c+x$ to $d+y$, prove that $x : y = c : d$

12 If $a+x : a-x$ equals the duplicate ratio of $a+b : a-b$, then

$$x-b : a-x = b(a+b) : a(a-b).$$

263 Change of Ratios Theorem—*A ratio of majority is diminished, and that of minority is increased by* **adding** *a positive quantity to both terms of the ratio*, and conversely, *a ratio of majority is increased and that of minority is diminished by* **subtracting** *a positive quantity from both terms of the ratio*

Let a positive quantity x be added to the terms of the ratio $\frac{a}{b}$, thus the new ratio is $\frac{a+x}{b+x}$

We have $$\frac{a}{b}-\frac{a+x}{b+x}=\frac{ax-bx}{b(b+x)}=\frac{x(a-b)}{b(b+x)},$$

now a, b, x being all supposed positive, $\frac{a}{b}-\frac{a+x}{b+x}$ is positive or negative, if $\frac{x(a-b)}{b(b+x)}$ is positive or negative, *i e*, if $a >$ or $< b$.

Therefore $\frac{a}{b} >$ or $< \frac{a+x}{b+x}$ according as $a >$ or $< b$ [Art 26]

The proof of the converse is exactly similar to the above proof

Cor From this theorem, it is easy to see that as x increases, the ratio $\frac{a+x}{b+x}$ continually approaches to unity, and by taking x sufficiently large, the difference between $\frac{a+x}{b+x}$ and unity may be made *smaller than any assignable magnitude* Therefore when x is *infinitely great, the ultimate value* of the ratio is said to be *unity* Hence we obtain the following important

Definition of "Equal to" *Two quantities are said to be equal when their ratio is unity* Thus if $\frac{a+x}{b+x}=1$ when x is infinitely large, $a+x=b+x$, where a and b may be *unequal*.

264 *Definitions* When the ratio of two quantities *is expressed by the ratio of two integers*, the quantities are termed **commensurable quantities** Thus $8\frac{1}{3}$ miles and 10 miles are commensurable quantities for their ratio is $5 : 6$

When the ratio of two quantities *cannot be expressed by the ratio of two integers*, the quantities are termed **incommensurable quantities** Thus the *diagonal* of a square and its *side* are incommensurable quantities, for their ratio is $\sqrt{2} : 1$, the *height* of an equilateral triangle and its *side* are incommensurable, for their ratio is $\sqrt{3} : 2$, &c But although in these cases we cannot find the *exact ratio*, yet we can find it to a *sufficient degree of approximation* that may be necessary.

Proportion

265 **Definitions** Four quantities are said to be **proportionals** when the first has the same ratio to the second, that the third has to the fourth, in other words, PROPORTION *is the equality of two ratios* Thus 4, 6, 10, 15 are proportionals for $4 : 6 = 10 : 15$, a, b, c, d are proportionals if $a : b = c : d$

It was usual formerly to use the sign $::$ for the sign of equality The proportion $a : b :: c : d$, or its equivalent $a : b = c : d$, is read thus "*a* is to *b* as *c* is to *d*."

The two terms a and d are called the **extremes**, and the two terms b and c the **means**, also the term d is called a **fourth proportional** to a, b and c

Four quantities are said to be **inversely proportional**, when the first and second are proportional to the *reciprocals* of the third and fourth Thus a, b, c, d are inversely proportional, if $a : b = \frac{1}{c} : \frac{1}{d}$ This is equivalent to $a : b = d : c$, hence four quantities are in inverse proportion, when the *first* has to the *second* the same ratio as the *fourth* has to the *third*.

266 **Proposition I** *If four quantities* a, b, c, d *be proportionals, the product of the extremes is equal to the product of the means.*

We have

$$a : b = c : d, \text{ that is, } \frac{a}{b} = \frac{c}{d}.$$

Multiply both sides by bd, thus $ad = bc$

REMARK If four concrete quantities are in proportion, they need not *all* be of the same kind It is sufficient that the *first two* be of one kind and the *other two* be of one kind, for in forming the first ratio we have to compare the first with the second, and in forming the second ratio, we have to compare the third with the fourth

Hence 2 ft. : 5 ft = Rs 4 : Rs. 10 is a correct proportion, but not so is 2 ft. : Rs 4 = 5 ft : Rs 10

267 Proposition II *If the product of two quantities be equal to the product of two others, the four are proportionals, the extremes being the factors of one of the products, and the means those of the other*

This proposition is the *converse* of the Proposition of Art 266

Let $ad=bc$, divide by bd, thus $\frac{a}{b}=\frac{c}{d}$, or $a : b=c : d$

Similarly by dividing by ac, cd and ab respectively, we shall obtain three other proportions. Thus if $ad=bc$, we get the following four proportions

$$a : b=c : d \qquad (1),$$

$$b : a=d : c \qquad (2),$$

$$a : c=b : d \qquad (3),$$

and

$$c : a=d : b \qquad (4)$$

Hence if any one of the proportions be true, all the others are true for any of them will give $ad=bc$

Cor It is evident that *any three* of the four quantities a, b, c and d being given, the *fourth* can be found

268 Proposition III *If four quantities* a, b, c, d, *be proportionals, they are proportionals when taken inversely* [Invertendo]

This is the same as (2) of Art 267

269 Proposition IV *If four quantities* a, b, c, d *be proportionals, they are proportionals when taken alternately* [Alternando]

This is the same as (3) of Art 267

REMARK From the definition of ratio [Art 259], it may seem that if the four quantities *a*, *b*, *c*, *d* be concrete, this alternation cannot take place *unless they be of the same kind* But when once the ratios are obtained, the numbers representing them become *abstract numbers*, and then alternation may take place

270 Proposition V *If four quantities* a, b, c, d *be proportionals, then* $a+b : b=c+d : d$ [Componendo.]

We have $\quad a : b=c : d$, that is, $\frac{a}{b}=\frac{c}{d}$,

therefore $\quad \frac{a}{b}+1=\frac{c}{d}+1$, or $\frac{a+b}{b}=\frac{c+d}{d}$,

that is, $\quad a+b : b=c+d : d$

271 Proposition VI. *If four quantities* a, b, c, d *be proportionals, then* $a-b : b = c-d : d$. [Dividendo]

We have $a : b = c : d$, that is, $\frac{a}{b}=\frac{c}{d}$,

therefore $\frac{a}{b}-1=\frac{c}{d}-1$, or $\frac{a-b}{b}=\frac{c-d}{d}$,

that is, $a-b : b = c-d : d$.

272 Proposition VII. *If four quantities* a, b, c, d *be proportionals, then* $a+b : a-b = c+d : c-d$. [Componendo and Dividendo.]

We have $\frac{a}{b}=\frac{c}{d}$.

$\therefore\ \frac{a}{b}+1=\frac{c}{d}+1$, or $\frac{a+b}{b}=\frac{c+d}{d}$. ....(i)

Again $\frac{a}{b}-1=\frac{c}{d}-1$, or $\frac{a-b}{b}=\frac{c-d}{d}$ .... (ii).

from (i) and (ii),

$$\frac{a+b}{b}\div\frac{a-b}{b}=\frac{c+d}{d}\div\frac{c-d}{d}.$$

Thus $\frac{a+b}{a-b}=\frac{c+d}{c-d}$ ....(iii),

i.e., $a+b : a-b = c+d : c-d$.

Cor. Hence $a-b : a+b = c-d : c+d$, [*Invertendo.*]

that is, $\frac{a-b}{a+b}=\frac{c-d}{c+d}$ .... (iv)

273 The following results are important. The student will have no difficulty in proving them.

(1) If $a : b = c : d$, then $a : a+b = c : c+d$.

(2) If $a : b = c : d$, then $a : a-b = c : c-d$. [*Convertendo.*]

(3) If $a : b = c : d$ and $m : n = p : q$, then $am : bn = cp : dq$.

Ex 1. The last 3 terms of a proportion are 4, 5 and 6; find the first term.

Denoting this term by x, we have

$x : 4 = 5 : 6$; whence $6x=4\times 5$, or $x=3\frac{1}{3}$.

Ex 2. Find the fourth proportional to 10, 12 and 15.

Let x = the required fourth proportional;

then $10 : 12 = 15 : x$; whence $10x=180$, or $x=18$.

Ex 3 Find the number that is to be added to each of the numbers 3, 4, 15 and 18 to give 4 numbers in proportion

Let x = the required number

Thus $\frac{3+x}{4+x}=\frac{15+x}{18+x}$, whence $x=3$

Ex 4 If $2x+3y : 2x-3y=23 : 5$, find (i) $x : y$ and (ii) $4x-5y : 4x+5y$.

We have $\frac{2x+3y}{2x-3y}=\frac{23}{5}$,

$$\frac{(2x+3y)+(2x-3y)}{(2x+3y)-(2x-3y)}=\frac{23+5}{23-5},$$

[Componendo and Dividendo, Art 272]

i e, $\frac{2x}{3y}=\frac{14}{9}$

or $\frac{x}{y}=\frac{7}{3}$, this proves (i)

Again $\frac{x}{y}=\frac{7}{3}$, $\frac{4}{5}\times\frac{x}{y}=\frac{4}{5}\times\frac{7}{3}$,

thus $\frac{4x}{5y}=\frac{28}{15}$,

$\frac{4x+5y}{4x-5y}=\frac{28+15}{28-15}=\frac{43}{13}$, [Componendo and Dividendo]

$\frac{4x-5y}{4x+5y}=\frac{13}{43}$ [Invertendo]

Ex 5. Solve the equation $\frac{x-3}{2} : \frac{2x+1}{3}=1 : 6$

The product of the extremes = the product of the means

$$\frac{x-3}{2}\times 6=\frac{2x+1}{3}\times 1,$$

thus $9x-27=2x+1$, whence $x=4$

Examples CLII

1 The first, second and fourth terms of a proportion are 2, 3 and 7, find the third term

2 The first, third and fourth terms of a proportion are 5, 8 and 12; find the second term

3 Find the fourth proportional to

(i) 8, 10 and 12, (ii) $\frac{1}{2}$, $\frac{2}{3}$, and $\frac{3}{4}$, (iii) $4a^2$, $3b$ and $4a^2b$;

(iv) $x+x^2$, $a-ax$ and x^2-1.

4 What number must be added to each of the numbers 3, 5, 7, 10 to give four numbers in proportion?

5 Find the number which must be subtracted from each of the numbers 8, 13, 18, 33 so as to give four numbers in proportion

6 If $7(x-y)=3(x+y)$, what is the ratio of x to y?

7 If $ax+by : ax-by=m : 1$, find the ratio $x : y$

8. Having given $b-a : b+a=4a-b : 6a-b$, find the ratio $a : b$.

9 If $5x+4y : 5x-4y=7 : 3$, find (i) $x : y$, and

(ii) $3x-5y : 3x+5y$

10 If $7x-3y : 7x+3y=13 : 43$, find (i) $x : y$, and

(ii) $8x+9y : 8x-9y$

11 If $(a+b+c+d)(a-b-c+d)=(a-b+c-d)(a+b-c-d)$, shew that a, b, c, d are proportionals

12 Solve $\dfrac{10+x}{} : \dfrac{4x-9}{7}=14 : 5$

13 Solve $\dfrac{17-4x}{4} : \dfrac{15+2x}{3}-2x=5 : 4$

274. We shall further illustrate Arts 266—273

Ex 1. If $a : b=c : d$, then $a+b : a-b=c+d : c-d$

Let $\dfrac{a}{b}=k$,* $\quad a=kb,\ c=kd$;

$$\therefore\ \frac{a+b}{a-b}=\frac{kb+b}{kb-b}=\frac{b(k+1)}{b(k-1)}=\frac{k+1}{k-1} \qquad (1),$$

and $$\frac{c+d}{c-d}=\frac{kd+d}{kd-d}=\frac{d(k+1)}{d(k-1)}=\frac{k+1}{k-1} \qquad (2);$$

$\therefore$ from (1) and (2), $\dfrac{a+b}{a-b}=\dfrac{c+d}{c-d}$

This is another proof of the Proposition of Art. 272

* This mode of representing each of the equal ratios by a constant (as k here) and then expressing the antecedent as the product of k and the consequent is extremely useful in Proportion

Ex 2. If $a : b = c : d$, prove that

$$ma+nc : pa+qc = mb+nd : pb+qd$$

Let $\qquad \dfrac{a}{b}=\dfrac{c}{d}=k$, thus $a=kb,\ c=kd \qquad\qquad (1).$

Now $\qquad \dfrac{ma+nc}{pa+qc}=\dfrac{mkb+nkd}{pkb+qkd}$ from $(1)=\dfrac{k(mb+nd)}{k(pb+qd)}=\dfrac{mb+nd}{pb+qd}$

Ex. 3 If $a : b = c : d$, prove that $a : c = \sqrt[3]{ma^3+nb^3} : \sqrt[3]{mc^3+nd^3}$.

Let $\qquad \dfrac{a}{b}=\dfrac{c}{d}=k$, then $a=kb,\ c=kd$,

$$\therefore \quad \frac{\sqrt[3]{ma^3+nb^3}}{\sqrt[3]{mc^3+nd^3}}=\frac{\sqrt[3]{mk^3b^3+nb^3}}{\sqrt[3]{mk^3d^3+nd^3}}=\frac{b\sqrt[3]{mk^3+n}}{d\sqrt[3]{mk^3+n}}=\frac{b}{d}=\frac{a}{c}$$

Ex 4. If $\dfrac{a}{b}=\dfrac{c}{d}$, prove that $\dfrac{(a+c)^3}{(b+d)^3}=\dfrac{a(a-c)^2}{b(b-d)^2}$ [*Cal*, 1888.]

Assume $\qquad \dfrac{a}{b}=\dfrac{c}{d}=k$, then $a=kb,\ c=kd$,

thus $\qquad \dfrac{(a+c)^3}{(b+d)^3}=\dfrac{(kb+kd)^3}{(b+d)^3}=\dfrac{k^3(b+d)^3}{(b+d)^3}=k^3,$

and $\qquad \dfrac{a(a-c)^2}{b(b-d)^2}=\dfrac{kb(kb-kd)^2}{b(b-d)^2}=\dfrac{k^3b(b-d)^2}{b(b-d)^2}=k^3;$

$$\therefore \quad \frac{(a+c)^3}{(b+d)^3}=\frac{a(a-c)^2}{b(b-d)^2}.$$

Ex. 5 If $\dfrac{a}{b}=\dfrac{c}{d}$, prove that $\dfrac{2a^2+ab+3b^2}{2a^2-ab+3b^2}=\dfrac{2c^2+cd+3d^2}{2c^2-cd+3d^2}$

Put $\qquad \dfrac{a}{b}=\dfrac{c}{d}=k$, then $a=kb,\ c=kd$,

$$\frac{2a^2+ab+3b^2}{2a^2-ab+3b^2}=\frac{2k^2b^2+kb^2+3b^2}{2k^2b^2-kb^2+3b^2}=\frac{2k^2+k+3}{2k^2-k+3} \quad \ldots\ldots(\text{i}),$$

and
$$\frac{2c^2+cd+3d^2}{2c^2-cd+3d^2}=\frac{2k^2d^2+kd^2+3d^2}{2k^2d^2-kd^2+3d^2}=\frac{2k^2+k+3}{2k^2-k+3} \quad \ldots\ldots(\text{ii}),$$

$\therefore$ from (i) and (ii), $\dfrac{2a^2+ab+3b^2}{2a^2-ab+3b^2}=\dfrac{2c^2+cd+3d^2}{2c^2-cd+3d^2}$.

Ex. 6. If $m : n = p : q$, prove that

$$\frac{(m-n)(m-p)}{m}=(m+q)-(n+p). \quad [Cal, 1859]$$

$$\frac{(m-n)(m-p)}{m}=\frac{m^2-m(n+p)+np}{m}=m-(n+p)+\frac{np}{m}$$

$$=m-(n+p)+\frac{mq}{m} \quad [\because np=mq, \text{ Art. } 266]$$

$$=m-(n+p)+q=(m+q)-(n+p)$$

Of course by putting $\frac{m}{n}=\frac{p}{q}=k$, we can prove the above relation

Examples CLIII

If $a : b = c : d$, shew that

1. $a : a-c = b : b-d$
2. $a : c = a+b : c+d$
3. $3a+4b : 3c+4d = 5a+6b : 5c+6d$
4. $14a+19b : 14c+19d = b^2c : ad^2$.
5. $7a-5b : 7a+5b = 7c-5d : 7c+5d$
6. $a^2+b^2 : a^2-b^2 = ac+ad : ac-bd$.
7. $a^2+b^2 : c^2+d^2 = (a-b)^2 : (c-d)^2$
8. $a^2+c^2 : b^2+d^2 = ac : bd$.
9. $a^2+ab : c^2+cd = ab-b^2 : cd-d^2$.
10. $ma-nb : ma+nb = mc-nd : mc+nd$
11. $ma+nb : mc+nd = pa+qb : pc+qd$.
12. $pa^2+qc^2 : pb^2+qd^2 = ma^2-nc^2 : mb^2-nd^2$
13. $a^2+ab+b^2 : a^2-ab+b^2 = c^2+cd+d^2 : c^2-cd+d^2$.
14. $(a^{\frac{1}{2}}+b^{\frac{1}{2}})^2 : (c^{\frac{1}{2}}+d^{\frac{1}{2}})^2 = a-b : c-d$
15. $ma+nc : mb+nd = \sqrt{a^2+c^2} : \sqrt{b^2+d^2} = a : b$
16. $a+\sqrt{a^2+b^2} : a-\sqrt{a^2+b^2} = c+\sqrt{c^2+d^2} : c-\sqrt{c^2+d^2}$.
17. $\sqrt{(3a^2+4c^2)} : \sqrt[3]{(5a^3-6c^3)} = \sqrt{(3b^2+4d^2)} : \sqrt[3]{(5b^3-6d^3)}$
18. $(a^3+a^2b+b^3)(a+b) : (c^3+c^2d+d^3)(c+d) = a^4+b^4 : c^4+d^4$

275 Examples converse of those given under Art 274, are also useful.

Ex. 1. If $3a+4b : 5a+6b = 3c+4d : 5c+6d$, then $a : b = c : d$.
[Cal., 1887.

From the given proportion, we have

$$(3a+4b)(5c+6d)=(5a+6b)(3c+4d) \quad [\text{Art } 266];$$

or $$15ac+20bc+18ad+24bd=15ac+18bc+20ad+24bd,$$

whence $$bc=ad,$$

i e, $$a : b=c : d$$

Ex 2 If $2a+3c+2b+3d : 3a-4c+3b-4d$
$=2a+3c-6b-9d : 3a-4c-9b+12d$, then $a : b=c : d$.

From the given proportion, we have

$$\frac{2a+2b+3c+3d}{3a+3b-4c-4d}=\frac{2a-6b+3c-9d}{3a-9b-4c+12d},$$

or $$\frac{2(a+b)+3(c+d)}{3(a+b)-4(c+d)}=\frac{2(a-3b)+3(c-3d)}{3(a-3b)-4(c-3d)}$$

Put $m=a+b$, $n=c+d$, $p=a-3b$, $q=c-3d$, thus

we have $$\frac{2m+3n}{3m-4n}=\frac{2p+3q}{3p-4q}.$$

Hence as in Ex. 1, we get $\frac{m}{n}=\frac{p}{q}$, i e, $\frac{a+b}{c+d}=\frac{a-3b}{c-3d}$,

whence as before, we have $\frac{a}{b}=\frac{c}{d}$.

Examples CLIII (*Continued*)

19 If $3a+4c : 5a+2c=3b+4d : 5b+2d$, then $a : b=c : d$

20 If $2a+3b : 2c+3d=4a+5b : 4c+5d$, then $a : b=c : d$

21 If $2a+c : 2b+d=3a-4c : 3b-4d$, then $a : b=c : d$

22. If $a^2+b^2 : a^2-b^2 =ac+bd : ac-bd$, then $a : b=c : d$.

23 If $a^2+c^2 : ab+cd=ab+cd : b^2+d^2$, then $a : b=c : d$

24 If $\dfrac{3a+4c+3b+4d}{5a+2c+5b+2d}=\dfrac{3a+4c+6b+8d}{5a+2c+10b+4d}$, then $a : b=c : d$

25 If $\dfrac{a+b+2c+2d}{2a-b+4c-2d}=\dfrac{3a+3b+4c+4d}{6a-3b+8c-4d}$, then $a : b=c : d$

276 Theorem If $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}=\dots$, then each of these ratios $=\left\{\dfrac{pa^n+qc^n+re^n+\dots}{pb^n+qd^n+rf^n+\dots}\right\}^{\frac{1}{n}}$, where p, q, r, n are any quantities whatever.

Let $$\frac{a}{b}=\frac{c}{d}=\frac{e}{f}=\dots\dots=k,$$

$$a=kb,\ c=kd,\ e=kf,\ \dots,$$

$$a^n=(kb)^n,\ c^n=(kd)^n,\ e^n=(kf)^n,\ \dots;$$

$$\therefore\ pa^n=p(kb)^n,\ qc^n=q(kd)^n,\ re^n=r(kf)^n,\ \dots$$

$$\therefore\ pa^n+qc^n+re^n+\dots=k^n(pb^n+qd^n+rf^n+\dots),$$

$$\therefore\ \left(\frac{pa^n+qc^n+re^n+\dots}{pb^n+qd^n+rf^n+\dots}\right)^{\frac{1}{n}}=k=\frac{a}{b}=\frac{c}{d}=\frac{e}{f}=\dots\dots$$

In the particular case when $n=1$, we have

$$\frac{pa+qc+re+\dots}{pb+qd+rf+\dots}=k=\frac{a}{b}=\frac{c}{d}=\frac{e}{f}=\dots\dots$$

The following three deductions from this theorem are very useful.

Cor 1 If $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}$, then each $=\frac{a+c+e}{b+d+f}$. Here $n=1, p=q=r$

Cor 2. If $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}$, then each $=\frac{a+c-e}{b+d-f}$. Here $n=1, p=q=-r$

Cor 3 If $n=1$, and $p=q=r=\dots$, it is easy to see that

$$\frac{a}{b}=\frac{c}{d}=\frac{e}{f}=\dots=\frac{a+c+e+\dots}{b+d+f+\dots}$$

That is, *if any number of quantities be proportionals, as any one antecedent is to its consequent so is the sum of all the antecedents to the sum of all the consequents.*

These results can of course be obtained immediately by the method employed to prove the theorem

Examples CLIV

If $a:b=c:d=e:f$, shew that

1 $a:b=mc-ne:md-nf$

2 $a+3c+2e:a-c=b+3d+2f:b-d$

3 $a:b=ma+nc+pe:mb+nd+pf$

4 $c:d=ab+cd+ef:b^2+d^2+f^2$

5 $e:f=\sqrt[4]{m^4a^4+n^4c^4+p^4e^4}:\sqrt[4]{m^4b^4+n^4d^4+p^4f^4}$

6 $(a^2+b^2)(ce+df)^2=(c^2+d^2)(ae+bf)^2=(e^2+f^2)(ac+bd)^2$

7 $a^2+c^2+e^2:ab+cd+ef=ab+cd+ef:b^2+d^2+f^2.$

8 $(a^2+ac+c\sqrt{ac})^2:(ac^2+ce\sqrt{ae}+e^3)^2$

$$=(b^2+bd+f\sqrt{bd})^2:(bd^2+df\sqrt{bf}+f^3)^2$$

9 $pa^2+qc^2+re^2 \quad pab+qcd+ref=lab+mcd+nef \quad lb^2+md^2+nf^2$.

If $x \quad a=y \quad b=z \quad c$, prove that

10. $\dfrac{la^3+mb^2y+ncz^2}{la^2x+mby^2+nz^3}=\dfrac{pa+qb+rc}{px+qy+rz}$

11. $\dfrac{ax+by}{a^2+b^2}=\dfrac{by+cz}{b^2+c^2}=\dfrac{cz+ax}{c^2+a^2}$

12 $(ax+by+cz)^2=(a^2+b^2+c^2)(x^2+y^2+z^2)$

13 If $x \quad a=y \;.\; b=z \quad c=$, shew that each of the ratios

$$=\left\{\frac{x^m+y^m+z^m+}{a^m+b^m+c^m+.}\right\}^{\frac{1}{m}}.$$

14 If $\dfrac{a-b}{d-e}=\dfrac{b-c}{e-f}$ then each of these ratios $=\dfrac{(a-b)f+(b-c)d}{(d-f)e}$.

277. The Proposition of Art 272 is very useful in solving equations We shall put it in the following form

If $\dfrac{a}{b}=\dfrac{c}{d}$, then $\dfrac{a+b}{a-b}=\dfrac{c+d}{c-d}$

[Componendo and Dividendo]

That is, *if two fractions be equal, the sum of the numerator and denominator of one divided by their difference, is equal to the corresponding expression for the other*

Cor Hence $\dfrac{a-b}{a+b}=\dfrac{c-d}{c+d}$.

REMARK. It is *generally* advantageous to use this formula, or the corollary when the variable occurs in *one side only*

1 Solve $\dfrac{2x+3a}{2x-3a}=\dfrac{5}{3}$

We have from the given equation

$$\frac{(2x+3a)+(2x-3a)}{(2x+3a)-(2x-3a)}=\frac{5+3}{5-3},$$

[Componendo and Dividendo]

or $\dfrac{4x}{6a}=\dfrac{8}{2}=4, \qquad x=6a$

2 Solve $\dfrac{3c+5ax}{3c-5ax}=\dfrac{4a+5c}{4a-5c}$

$$\frac{(3c+5ax)-(3c-5ax)}{(3c+5ax)+(3c-5ax)}=\frac{(4a+5c)-(4a-5c)}{(4a+5c)+(4a-5c)},$$

[Componendo and Dividendo]

$$\frac{5ax}{3c}=\frac{5c}{4a}, \quad . \; x=\frac{3c^2}{4a^2}$$

3 Solve $\dfrac{\sqrt{x}+\sqrt{a-x}}{\sqrt{x}-\sqrt{a-x}}=c$

$$\frac{\sqrt{x}+\sqrt{a-x}}{\sqrt{x}-\sqrt{a-x}}=c=\frac{c}{1},$$

$$\frac{\sqrt{a-x}}{\sqrt{x}}=\frac{c-1}{c+1}, \quad \text{[Comp and Divid.]}$$

$$\frac{a-x}{x}=\left(\frac{c-1}{c+1}\right)^2,$$

$$\frac{a}{x}-1=\left(\frac{c-1}{c+1}\right)^2,$$

$$\frac{a}{x}=1+\left(\frac{c-1}{c+1}\right)^2=\frac{2(c^2+1)}{(c+1)^2},$$

$$\frac{x}{a}=\frac{(c+1)^2}{2(c^2+1)}, \qquad x=\frac{a(c+1)^2}{2(c^2+1)}$$

Examples CLV.

Solve the equations

1 $\dfrac{\sqrt{4x+1}+\sqrt{4x}}{\sqrt{4x+1}-\sqrt{4x}}=9.$

2. $\dfrac{\sqrt{5x}-\sqrt{7-5x}}{\sqrt{5x}+\sqrt{7-5x}}=\dfrac{1}{4}.$

3 $\dfrac{\sqrt{x+1}-\sqrt{x-1}}{\sqrt{x+1}+\sqrt{x-1}}=\dfrac{1}{2}$

4. $\dfrac{\sqrt{a+c\sqrt{b-x}}}{\sqrt{a-c\sqrt{b-x}}}=c$

5. $\dfrac{\left(\dfrac{a+x}{a-x}\right)^{\frac{1}{2}}-b}{\left(\dfrac{a+x}{a-x}\right)^{\frac{1}{2}}+b}=\dfrac{2}{3}.$

6. $\dfrac{\sqrt{a+x}+\sqrt{a-x}}{\sqrt{a+x}-\sqrt{a-x}}=\dfrac{1}{b}.$

7 $\dfrac{\sqrt{x}-\sqrt{5x+2a}}{\sqrt{x}+\sqrt{5x+2a}}=\dfrac{\sqrt{x}+\sqrt{a+b}}{\sqrt{x}-\sqrt{a+b}}$

8 $\dfrac{\sqrt{a}-\sqrt{a-x}}{\sqrt{a}+\sqrt{a-x}}=a$

9 $\dfrac{\sqrt{a+bx^n}+\sqrt{a-bx^n}}{\sqrt{a+bx^n}-\sqrt{a-bx^n}}=c$

10 $\dfrac{\sqrt{a+x}+\sqrt{a-x}}{\sqrt{x+b}+\sqrt{x-b}}=\dfrac{\sqrt{a+x}-\sqrt{a-x}}{\sqrt{x+b}-\sqrt{x-b}}$

Solve the equations

11. $\dfrac{ax+1+\sqrt{a^2x^2-1}}{ax+1-\sqrt{a^2x^2-1}}=\dfrac{bx}{2}$ 12 $\dfrac{1-ax}{1+ax}\sqrt{\dfrac{1+bx}{1-bx}}=1$

The following equations demand greater skill.

13. $x-1=2\sqrt{x+2}$ 14 $\left(\dfrac{a+x}{a-x}\right)^3=1+\dfrac{cx}{ab}$

15 $\dfrac{a-\sqrt{2ax-x^2}}{a+\sqrt{2ax-x^2}}=b.$ 16 $\dfrac{a+x+\sqrt{2ax+x^2}}{a+x}=b$

17 $\dfrac{x}{\sqrt{1+x}+\sqrt{1-x}}=\dfrac{2a^3}{1-\sqrt{1-x^2}}$ [See *App*]

18 $1+\dfrac{x}{2}-\dfrac{x^2}{2(1+\sqrt{1+x})^2}=3$

19 $\sqrt{\dfrac{x+a}{x}}+2\sqrt{\dfrac{a}{x+a}}=b\sqrt{\dfrac{x}{x+a}}$

20 $\dfrac{1+\sqrt{x^2-1}}{1+2a\sqrt{x^2-1}}=\dfrac{\sqrt{x^2-1}-1}{x^2-2}$ 21 $\sqrt[4]{x^4-1}+x\sqrt{x^4-1}=x^3.$

22 $\sqrt{a^2-x^2}+x\sqrt{a^2-1}=a^2\sqrt{1-x^2}.$ [See *App*]

23 $\sqrt{\dfrac{x^3}{a^2}+1}-\sqrt[4]{\dfrac{x^2}{a^2}+1}=\dfrac{9a^2-1}{4}$

24 $(a+x)\sqrt{1+a}+(a-x)\sqrt{1-a}=2\sqrt{a^2+x^2}$ [See *App*]

25 $\dfrac{1+x-\sqrt{2x+x^2}}{1+x+\sqrt{2x+x^2}}=a^3\dfrac{\sqrt{2+x}+\sqrt{x}}{\sqrt{2+x}-\sqrt{x}}.$ [See *App*]

26 $\dfrac{(1+a^2)(1+x^2)}{(1+ax)^2}=\dfrac{1}{4}\left\{\dfrac{x}{a}+\dfrac{a}{x}+2\right\}$

278 The following is an important consequence of the Theorem of Art 276 which is sometimes useful in solving equations.

$$\text{If } \frac{a}{b}=\frac{c}{d}, \text{ then each fraction}=\frac{a+c}{b+d} \text{ or } =\frac{a-c}{b-d}$$

That is, *if two fractions be equal, each is equal to the sum or difference of the numerators divided respectively by the sum or difference of the denominators*

Cor Hence $\dfrac{a}{b}=\dfrac{a+mc}{b+md}=\dfrac{a-mc}{b-md}$ $\because$ $\dfrac{c}{d}=\dfrac{mc}{md}$, [Art. 181]

Ex 1 Solve $\dfrac{x-a+b-c}{s-p+q-r}=\dfrac{a-b+c}{p-q+r}$.

Each fraction = sum of numerators ÷ sum of denominators

$$\therefore \frac{a-b+c}{p-q+r}=\frac{x-a+b-c+a-b+c}{s-p+q-r+p-q+r}=\frac{x}{s},$$

$$x=s\,\frac{a-b+c}{p-q+r}$$

Ex 2 Solve $\dfrac{mx-(a+b)}{nx-(c+d)}=\dfrac{mx-(a+c)}{nx-(b+d)}$.

Each = difference of numerators ÷ difference of denominators

$$\frac{mx-(a+b)}{nx-(c+d)}=\frac{mx-(a+b)-mx+(a+c)}{nx-(c+d)-nx+(b+d)}=\frac{c-b}{b-c}=-1,$$

or $$mx-(a+b)=c+d-nx,$$

$$(m+n)x=a+b+c+d, \text{ or } x=\frac{a+b+c+d}{m+n}.$$

Ex 3 Solve $\dfrac{ax^2+bx+c}{px^2+qx+r}=\dfrac{ax+b}{px+q}$ [Ex 12, p 322]

Multiply numerator and denominator of second member by x, thus

$$\frac{ax^2+bx+c}{px^2+qx+r}=\frac{ax^2+bx}{px^2+qx},$$

$\therefore$ by Cor, $$\frac{ax+b}{px+q}=\frac{ax^2+bx+c-(ax^2+bx)}{px^2+qx+r-(px^2+qx)}=\frac{c}{r},$$

$$r(ax+b)=c(px+q),$$

whence $$x=\frac{cq-br}{ar-cp}$$

Examples CLVI

Solve the equations

1 $\dfrac{4x-9}{3x+8}=\dfrac{8x+9}{9x-8}$. 2 $\dfrac{2x-6}{3x-8}=\dfrac{2x-5}{3x-7}$ 3 $\dfrac{6x+a}{4x+b}-\dfrac{3x-b}{2x-a}=0$

4 $\dfrac{mx-a}{nx-b}=\dfrac{mx-c}{nx-d}$ 5 $\dfrac{x+a}{x+b}=\dfrac{x+3a}{x+a+b}$.

6 $\dfrac{\sqrt{ax}+b}{\sqrt{bx}-a}=\dfrac{\sqrt{ax}-c}{\sqrt{bx}+c}$. 7 $\dfrac{x^2-ax+b}{x^2-px+q}=\dfrac{x^2-ax-q}{x^2-px-b}$

8 $\dfrac{x^3+ax^2-bx+c}{x^3-ax^2+bx+c}=\dfrac{x^2+ax-b}{x^2-ax+b}$. 9 $\dfrac{(x+a)(x+b)}{x+a+b}=\dfrac{(x+c)(x+d)}{x+c+d}$.

10 $\dfrac{(x-a)(x-b)}{(x-c)(x-d)}=\dfrac{x-a-b}{x-c-d}$

279 Continued Proportion Quantities are said to be in **continued proportion** when the first : the second = the second : the third = the third : the fourth = &c Thus, if $a : b = b : c = c : d = d : e =$ &c, then a, b, c, &c are in continued proportion

If three quantities a, b, c be in continued proportion, b is said to be a **mean proportional** between a and c, and c is said to be a **third proportional** to a and b

The following properties of continued proportionals, should be carefully committed to memory

(i) If a, b, c are in continued proportion, then $b^2 = ac$

(ii) Conversely if $b^2 = ac$, then a, b, c are in continued proportion

(iii) If a, b, c are in continued proportion, then $a : c = a^2 : b^2 = b^2 : c^2$

Proof

(i) Since $a : b = b : c$, we have $b^2 = ac$ [Art 266],

(ii) $b^2 = ac$, $\therefore \frac{b^2}{ab} = \frac{ac}{ab}$, or $\frac{b}{a} = \frac{c}{b}$, $\therefore \frac{a}{b} = \frac{b}{c}$, i e, $a : b = b : c$

(iii) Take $\frac{a}{b} = \frac{b}{c} = k$, thus $k^2 = \frac{a}{b} \times \frac{b}{c} = \frac{a}{c}$, also $k^2 = \frac{a^2}{b^2} = \frac{b^2}{c^2}$ $\therefore \frac{a}{c} = \frac{a^2}{b^2} = \frac{b^2}{c^2}$.

Thus *if three quantities are in continued proportion, the first is to the third in the* **duplicate** *ratio of the first to the second, or of the second to the third*

The last result may evidently be generalised, for if a number of quantities $a, b, c, \ldots x$, be in continued proportion, giving n ratios, then

$$\frac{a}{x} = \left(\frac{a}{b}\right)^n = \left(\frac{b}{c}\right)^n = \left(\frac{c}{d}\right)^n = \&c$$

For let $$\frac{a}{b} = \frac{b}{c} = \frac{c}{d} = \ldots = k,$$

$$\therefore \frac{a}{b} \times \frac{b}{c} \times \frac{c}{d} \times \ldots \times \frac{v}{x} = kkk \ldots \text{ to } n \text{ factors} = k^n,$$

$$\therefore \frac{a}{x} = k^n = \left(\frac{a}{b}\right)^n = \left(\frac{b}{c}\right)^n = \left(\frac{c}{d}\right)^n = \&c$$

Hence in the particular case when four quantities a, b, c, d are in continued proportion, we have $\frac{a}{d} = \frac{a^3}{b^3} = \frac{b^3}{c^3} = \frac{c^3}{d^3}$

Thus *if four quantities are in continued proportion, the first is to the fourth in the* **triplicate** *ratio of the first to the second, or of the second to the third, or the third to the fourth.*

Of the four quantities $a, b, c, d,$ in continued proportion, b and c are said to be *two* mean proportionals between a and d.

Ex. 1 Find a mean proportional between 2 and 8

Required mean proportional $=\sqrt{(2\times 8)}=\sqrt{16}=4.$

Ex 2 Find a third proportinoal to 24 and 36

Let x = required third proportional

Then $\frac{24}{36}=\frac{36}{x}$, whence $x=54$

Ex 3 Find two mean proportionals between 2 and 16

Let x and y be the required mean proportionals.

Thus $\frac{2}{x}=\frac{x}{y}=\frac{y}{16}$

Hence $\frac{2}{16}=\frac{2^3}{x^3}$ and $\frac{2}{16}=\frac{y^3}{(16)^3}$,

thus $x=4$ and $y=8$

Ex 4 What quantity must be added to a, b and c to give three quantities in continued proportion?

Let x = required quantity.

Since $a+x$, $b+x$ and $c+x$ are to be in continued proportion, we have

$$(b+x)^2=(a+x)(c+x)$$

whence $$x=\frac{b^2-ac}{c+a-2b}$$

Ex If a, b, c, d are in continued proportion, shew that $a+2b$, $b+2c$ and $c+2d$ are in continued proportion.

Let $k=\frac{a}{b}=\frac{b}{c}=\frac{c}{d}$.

Thus by Art 276, $k=\frac{a}{b}=\frac{2b}{2c}=\frac{a+2b}{b+2c}$ .... (i),

and $k=\frac{b}{c}=\frac{2c}{2d}=\frac{b+2c}{c+2d}$ .... (ii)

Hence from (i) and (ii), $\frac{a+2b}{b+2c}=\frac{b+2c}{c+2d}$

or $a+2b$, $b+2c$ and $c+2d$ are in continued proportion

Ex. 6. If $\frac{a}{b}=\frac{b}{c}=\frac{c}{d}$, prove that

$$(ab+bc+cd)^2=(a^2+b^2+c^2)(b^2+c^2+d^2) \quad [Cal, 1887]$$

Let each of the given ratios $=k$, thus

$$k=\frac{ab}{b^2}=\frac{bc}{c^2}=\frac{cd}{d^2}=\frac{ab+bc+cd}{b^2+c^2+d^2} \text{ [Art 276, Cor 1] (i);}$$

also

$$k=\frac{a^2}{ab}=\frac{b^2}{bc}=\frac{c^2}{cd}=\frac{a^2+b^2+c^2}{ab+bc+cd} \text{ [Art 276, Cor 1] (ii)}$$

$\therefore$ from (i) and (ii), $\frac{ab+bc+cd}{b^2+c^2+d^2}=\frac{a^2+b^2+c^2}{ab+bc+cd}$,

or

$$(ab+bc+cd)^2=(a^2+b^2+c^2)(b^2+c^2+d^2)$$

Examples CLVII

1 Find a mean proportional between 12 and 48, also between x^3y and xy^3

2 Find a third proportional to 14 and 35

3 If $x-1$, x and $x+3$ be in continued proportion, find x

4 What number must be added to each of the numbers 1, 5 and 13, and what subtracted from each of the numbers 2, 7 and 17, to give three numbers in continued proportion ?

5. Find two mean proportionals between 3 and 24

6. Given that a, b, c, d are in continued proportion, prove that

$$\frac{a}{d}=\frac{b^3}{c^3} \text{ and } a^{\frac{2}{3}}=c^3d^{-\frac{4}{3}}$$

7 If a, b, c, d are in continued proportion, then also $a+b$, $b+c$ and $c+d$ are in continued proportion.

8 If a, b, c, d are in continued proportion, then also (i) $2a+b$, $2b+c$ and $2c+d$, and (ii) $a-2b$, $b-2c$ and $c-2d$, are in continued proportion

9 If a, b, c be in continued proportion, then a has to c the duplicate ratio of $a+b$ to $b+c$

10. If y be a mean proportional between x and z, then

$$x+y \quad x-y=y+z \quad y-z$$

11 If a be to b in the duplicate ratio of $a+x$ to $b+x$, shew that x is a mean proportional between a and b

12 If $a \quad b=c \quad d$ and x be a third proportional to a and b, and y to b and c, prove that the mean proportional between x and y is equal to that between c and d

13 If a, b, c are in continued proportion, then $ab+bc$ is a mean proportional between a^2+b^2 and b^2+c^2.

14 If a, b, c, d are in continued proportion, then b^2+c^2 is a mean proportional between a^2+b^2 and c^2+d^2.

15 $x : y = y : z$, find the simplest value of

$$\frac{xyz(x+y+z)^3}{(yz+zx+xy)^3}$$

16 If $a : b = b : c = c : d$, prove that

$$a : d = pa^3+qb^3+rc^3 : pb^3+qc^3+rd^3$$

280 We shall conclude this Chapter by working out a few examples illustrative of the various methods employed in the solution of questions on Ratio and Proportion

Ex. 1. If $\dfrac{x}{b+c-a}=\dfrac{y}{c+a-b}=\dfrac{z}{a+b-c}$,

shew that $(b-c)x+(c-a)y+(a-b)z=0$ [*Cal*, 1878]

Let each of the given ratios $=l$, thus we have

$$x=l(b+c-a) \qquad (1),$$
$$y=l(c+a-b) \qquad (2),$$
$$z=l(a+b-c) \qquad (3);$$

multiplying (1) by $b-c$, (2) by $c-a$ and (3) by $a-b$, and adding, we get $(b-c)x+(c-a)y+(a-b)z$

$$=l\{b^2-c^2-a(b-c)+c^2-a^2-b(c-a)+a^2-b^2-c(a-b)\}=l\times 0=0.$$

Ex. 2 Shew that if $\dfrac{ay-bx}{c}=\dfrac{cx-az}{b}=\dfrac{bz-cy}{a}$, then $\dfrac{x}{a}=\dfrac{y}{b}=\dfrac{z}{c}$.

Let $$\frac{ay-bx}{c}=\frac{cx-az}{b}=\frac{bz-cy}{a}=l,$$

$$\therefore \quad \frac{acy-bcx}{c^2}=\frac{bcx-abz}{b^2}=\frac{abz-acy}{a^2}=l \text{ [Art. 181]}$$

$$\text{by Cor. 1, Art. 276, } l=\frac{\text{sum of numerators}}{\text{sum of denominators}}=0,$$

hence each of the given ratios $=0$, therefore

$$ay-bx=0,\ cx-az=0,\ bz-cy=0,$$

whence $$\frac{x}{a}=\frac{y}{b}=\frac{z}{c}$$

Ex. 3 If $$a(y+z)=b(z+x)=c(x+y),$$

then $$\frac{y-z}{a(b-c)}=\frac{z-x}{b(c-a)}=\frac{x-y}{c(a-b)}.$$

We have $$a(y+z)=b(z+x)=c(x+y)$$

Divide by abc, thus

$$\frac{y+z}{bc}=\frac{z+x}{ca}=\frac{x+y}{ab}=k \text{ suppose}$$

Now by Art 276, $k=\dfrac{(x+y)-(z+x)}{ab-ca}=\dfrac{y-z}{a(b-c)}$

Similarly $\qquad k=\dfrac{(y+z)-(x+y)}{bc-ab}=\dfrac{z-x}{b(c-a)}$,

and also $\qquad k=\dfrac{(z+x)-(y+z)}{ca-bc}=\dfrac{x-y}{c(a-b)}$

Hence $\qquad \dfrac{y-z}{a(b-c)}=\dfrac{z-x}{b(c-a)}=\dfrac{x-y}{c(a-b)}$.

Ex 4 If $y^2-z^2 \quad yz=b-c \quad x$ and $z^2-x^2 \quad zx=c-a \quad y$, then prove that $\quad x^2-y^2 \quad xy=a-b \quad z$ [*Punj*, 1888]

From the given relations, we have [Art 266]

$$x(y^2-z^2)=yz(b-c) \qquad (1),$$

$$y(z^2-x^2)=zx(c-a) \qquad (2).$$

Multiply (1) by x and (2) by y, and add, thus

$$z^2(y^2-x^2)=xyz(b-a)$$

Divide by z and change signs, thus $z(x^2-y^2)=xy(a-b)$,

or $\qquad \dfrac{x^2-y^2}{xy}=\dfrac{a-b}{z}$,

i e, $\qquad x^2-y^2 \quad xy=a-b \quad z$

Ex 5 If $alx+bmy+cnz=apx+bqy+crz=ax^2+by^2+cz^2=0$, prove that $\quad x(mr-qn)+y(np-rl)+z(lq-pm)=0$ [*Punj*, 1887].

From the first two equations, we have

$$l\,ax+m\,by+n\,cz=0,$$

$$p\,ax+q\,by+r\,cz=0$$

Hence by Cross Multiplication,

$$\frac{ax}{mr-qn}=\frac{by}{np-r}=\frac{cz}{lq-pm},$$

or $\qquad \dfrac{ax^2}{x(mr-qn)}=\dfrac{by^2}{y(np-rl)}=\dfrac{cz^2}{z(lq-pm)}=k$ suppose,

$$\therefore\ k=\frac{ax^2+by^2+cz^2}{x(mr-qn)+y(np-rl)+z(lq-pm)},$$

or $\qquad k\{x(mr-qn)+y(np-rl)+z(lq-pm)\}=ax^2+by^2+cz^2=0$;

divide by k, thus $x(mr-qn)+y(np-rl)+z(lq-pm)=0$.

Ex 6. If $a : b = c : d$, shew that

$$4(a+b)(c+d)=bd\left(\frac{a+b}{b}+\frac{c+d}{d}\right)^2 \text{ [Cal, 1874.]}$$

Let each of the ratios $= k$, thus $a = kb$, $c = kd$,

$$\therefore \quad 4(a+b)(c+d)=4(kb+b)(kd+d)=4bd(k+1)^2$$
$$=bd\{2(k+1)\}^2=bd\{(k+1)+(k+1)\}^2$$
$$=bd\left\{\left(\frac{a}{b}+1\right)+\left(\frac{c}{d}+1\right)\right\}^2=bd\left(\frac{a+b}{b}+\frac{c+d}{d}\right)^2.$$

Ex. 7 Solve $\dfrac{ax+by}{cz}=\dfrac{cz+ax}{by}=\dfrac{by+cz}{ax}=x+y+z.$

By Art 276, Cor 1, $x+y+z=\dfrac{\text{sum of numerators}}{\text{sum of denominators}}=2$ (1).

And $$\frac{ax+by}{cz}+1=\frac{cz+ax}{by}+1=\frac{by+cz}{ax}+1,$$

$$\therefore \quad \frac{ax+by+cz}{cz}=\frac{cz+ax+by}{by}=\frac{by+cz+ax}{ax},$$

or $$ax=by=cz=k \text{ suppose} \qquad (2),$$

from (1), $k\left(\dfrac{1}{a}+\dfrac{1}{b}+\dfrac{1}{c}\right)=2$, whence $k=\dfrac{2abc}{bc+ca+ab}$;

from (2), $$ax=by=cz=\frac{2abc}{bc+ca+ab}.$$

Ex 8 A and B compared their incomes and found that A's income was to that of B as 7 : 9, and that the third of A's income was Rs 30 greater than the difference of their incomes. Find what each received [*Cal*, 1871]

Let x rupees $= A$'s income, and y rupees $= B$'s income,

thus $$\frac{x}{y}=\frac{7}{9} \qquad (1)$$

and $$\tfrac{1}{3}x=(y-x)+30, \quad y>x \text{ by hypothesis} \qquad (2)$$

From (1), $x=\frac{7}{9}y$, therefore from (2),

$$\tfrac{1}{3}\times\tfrac{7}{9}y=(y-\tfrac{7}{9}y)+30, \text{ whence } y=810$$

And from (1), $$x=\tfrac{7}{9}y=\tfrac{7}{9}\times 810=630$$

Ex 9 Two globes of gold whose radii are r, r', are melted and formed into a single globe, find its radius, having given that the volume of a sphere is proportional to the cube of its radius

Let v, v' denote the volumes of the given globes, and V and R, the volume and radius of the required globe

$$\frac{V}{R^3}=\frac{v}{r^3}=\frac{v'}{r'^3}=\frac{v+v'}{r^3+r'^3} \text{ [Art 276, Cor 1]},$$

and $$V=v+v', \text{ we get } R^3=r^3+r'^3, \text{ or } R=\sqrt[3]{r^3-r'^3}$$

Ex 10 If from a cask of wine containing a gallons, b gallons be drawn off and the vessel filled up with water, and this operation be repeated n times successively, find the quantity of wine then remaining

Let $a_1, a_2, a_3, \ldots, a_n$ denote the quantities of wine remaining after the *first*, *second*, *third*, ..., n^{th}, operation respectively, then it is easily seen that

$$a : a_1 = a : a-b \qquad (1),$$

but since the wine in the cask decreases every time in the ratio of a to $a-b$, we have

$$\left.\begin{array}{l} a_1 : a_2 = a : a-b, \\ a_2 : a_3 = a : a-b, \\ \ldots\ldots, \\ a_{n-1} : a_n = a : a-b, \end{array}\right\} \qquad (2)$$

Therefore compounding the n proportions in (1) and (2), we get

$$a : a_n = a^n : (a-b)^n,$$

whence $$a_n=\frac{(a-b)^n}{a^{n-1}}$$

Examples CLVIII

1. If $a : b = c : d$, prove that

 (1) $ac : bd = (a+c)(a-c) : (b+d)(b-d)$

 (2) $\dfrac{(a^2-c^2)b^2(c^2-d^2)}{(a^2-b^2)c^2(b^2-d^2)}=1.$

 (3) $\dfrac{1}{ma}+\dfrac{1}{nb}+\dfrac{1}{pc}+\dfrac{1}{qd}=\dfrac{1}{bc}\left\{\dfrac{a}{q}+\dfrac{b}{p}+\dfrac{c}{n}+\dfrac{d}{m}\right\}.$

2. If a, b, c, d be in continued proportion, shew that

 (1) $(b+c)(b+d)=(c+a)(c+d)$

 (2) $(a+d)(b+c)-(a+c)(b+d)=(b-c)^2$

3. If $a : b = c : d = e : f$, shew that

$$a^2 : b^2 = a(a+mc)+nec : b(b+md)+nfd$$

4. If $a_1 : b_1 = a_2 : b_2 = a_3 : b_3$, prove that

$$\sqrt{a_1b_1}+\sqrt{a_2b_2}+\sqrt{a_3b_3}=\sqrt{(a_1+a_2+a_3)(b_1+b_2+b_3)}$$

5 If $\frac{b+c}{a}=\frac{c+a}{b}=\frac{a+b}{c}$, shew that $a=b=c$

6 If
$$\frac{x}{b+c}=\frac{y}{c+a}=\frac{z}{a+b},$$
then
$$(b-c)x+(c-a)y+(a-b)z=0$$

7 If $\frac{x}{a(b-c)}=\frac{y}{b(c-a)}=\frac{z}{c(a-b)}$,

then
$$bc(b+c)x+ca(c+a)y+ab(a+b)z=0$$

8 If $\frac{x}{(b-c)(b+c-2a)}=\frac{y}{(c-a)(c+a-2b)}=\frac{z}{(a-b)(a+b-2c)}$

find the value of $x+y+z$

9 If $\frac{x}{a(b-c)(b+c-2a)}=\frac{y}{b(c-a)(c+a-2b)}=\frac{z}{c(a-b)(a+b-2c)}$,

find the value of $bcx+cay+abz$

10 If $\frac{x}{b+c}=\frac{y}{c+a}=\frac{z}{a+b}$, then $\frac{a}{y+z-x}=\frac{b}{z+x-y}=\frac{c}{x+y-z}$

11 If $x : ax+by+cz=y : bx+cy+az=z : cx+ay+bz$, shew that each of these ratios $=\frac{1}{a+b+c}$, supposing $x+y+z$ is not $=0$

12 If $a+b : b+c=c+d : d+a$, it is required to prove that
$$a=c \text{ or } a+b+c+d=0$$

13 If $(a+b+c)x=(b+c-a)y=(c+a-b)z=(a+b-c)w$, shew that
$$\frac{1}{x}=\frac{1}{y}+\frac{1}{z}+\frac{1}{w}$$

14 If $\frac{d-a}{x-u}=\frac{d-b}{y-u}=\frac{d-c}{z-u}=\frac{a+b+c+d}{3(u+x+y+z)}$,

prove that
$$\frac{a}{u+y+z}=\frac{b}{u+x+z}=\frac{c}{u+x+y}=\frac{d}{x+y+z}.$$

15 If $\frac{bx-ay}{cy-az}=\frac{cx-az}{by-ax}=\frac{z+y}{z+x}$, then each of these ratios $=\frac{x}{y}$, unless $b+c=0$ [*App*]

16 If $\frac{a}{y+z}=\frac{b}{z+x}=\frac{c}{x+y}$, then $\frac{a(b-c)}{y^2-z^2}=\frac{b(c-a)}{z^2-x^2}=\frac{c(a-b)}{x^2-y^2}$

17 If
$$\frac{bz+cy}{b-c}=\frac{cx+az}{c-a}=\frac{ay+bx}{a-b},$$
shew that
$$(a+b+c)(x+y+z)=ax+by+cz.$$

18 If $\dfrac{b+c-a}{y+z-x}=\dfrac{c+a-b}{z+x-y}=\dfrac{a+b-c}{x+y-z}$, then each of these ratios is equal to
$$\frac{a}{x}=\frac{b}{y}=\frac{c}{z};$$
also to
$$\frac{b+c-3a}{y+z-3x}=\frac{c+a-3b}{z+x-3y}=\frac{a+b-3c}{x+y-3z}$$

19 If $\dfrac{x'}{xy}=\dfrac{y'}{x^2}=\dfrac{z'}{yz}$, prove that $\dfrac{x}{x'y'}=\dfrac{y}{x'^2}=\dfrac{z}{y'z'}$.

20 If $\dfrac{c-a}{x-z}=\dfrac{c-b}{y-z}=\dfrac{a+b+c}{2(x+y+z)}$, prove that $\dfrac{a}{y+z}=\dfrac{b}{z+x}=\dfrac{c}{x+y}$.

21 If
$$\frac{la+mb}{l}=\frac{ma-lb}{m}=c,$$
shew that (1) $c^2=a^2+b^2$, (2) $ac=a^2+b^2$.

22 If
$$\frac{y+z}{3b-c}=\frac{z+x}{3c-a}=\frac{x+y}{3a-b},$$
prove that
$$\frac{x+y+z}{ax+by+cz}=\frac{a+b+c}{a^2+b^2+c^2}.$$

23 If $a+d=mc$ and $\dfrac{1}{a}+\dfrac{1}{d}=\dfrac{m}{b}$, shew that $a : b=c : d$, and find the value of m in terms of a, b and c

24 If $\dfrac{x}{a}=\dfrac{y}{b}=\dfrac{z}{c}$, $x^2+y^2+z^2=d^2$ and $ax+by+cz=f^2$,
prove that
$$(a^2+b^2+c^2)d^2=f^4.$$

25 If
$$\frac{\alpha}{x}=\frac{\beta}{y}=\frac{\gamma}{z} \text{ and } \frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1,$$
shew that
$$\frac{\alpha^2}{a^2}+\frac{\beta^2}{b^2}+\frac{\gamma^2}{c^2}=\frac{\alpha^2+\beta^2+\gamma^2}{x^2+y^2+z^2}$$

26 Solve $\dfrac{ax+by-cz}{b^2+c^2}=\dfrac{by+cz-ax}{c^2+a^2}=\dfrac{cz+ax-by}{a^2+b^2}=a+b+c$

27. Solve $\dfrac{1}{a+b+c}=\dfrac{\frac{b}{y}+\frac{c}{z}}{a}=\dfrac{\frac{c}{z}+\frac{a}{x}}{b}=\dfrac{\frac{a}{x}+\frac{b}{y}}{c}$.

28. Solve
$$\left.\begin{array}{l} y+z : z+x : x+y=a : b : c \\ (y+z)^2+(z+x)^2+(x+y)^2=1 \end{array}\right\}$$

29 If $x : a=y : b=z : c$, prove that $\dfrac{x^3}{a^2}+\dfrac{y^3}{b^2}+\dfrac{z^3}{c^2}=\dfrac{(x+y+z)^3}{(a+b+c)^2}$.

30 If $$\frac{bx+ay-cz}{a^2+b^2}=\frac{cy+bz-ax}{b^2+c^2}=\frac{az+cx-by}{c^2+a^2},$$

shew that $$\frac{x+y+z}{a+b+c}=\frac{ax+by+cz}{ab+bc+ca}$$

31. If $$\frac{x}{l(mb+nc-la)}=\frac{y}{m(nc+la-mb)}=\frac{z}{n(la+mb-nc)},$$

then $$\frac{l}{x(by+cz-ax)}=\frac{m}{y(cz+ax-by)}=\frac{n}{z(ax+by-cz)}$$ [App]

32 If $\frac{a+c}{b}=\frac{c}{a}=\frac{a}{c-b}$, determine the ratios $a : b : c$

33 If $\frac{b^2-a^2-c^2}{b-c}=\frac{b^2-a^2+c^2}{a+c}$, find the value of each ratio in terms of c

34 If $\frac{x}{y+z}=a$, $\frac{y}{z+x}=b$, $\frac{z}{x+y}=c$, shew that

$$x^2 : y^2 : z^2=a(1-bc) : b(1-ca) : c(1-ab)$$

35 Find two numbers such that their sum, difference and the sum of their squares, may be as the numbers 5, 3 and 51

36 What is the least integer that must be added to the terms of the ratio 9 : 23, so as to make it greater than the ratio 7 : 11 ?

37 *A*'s age is 25 years and *B*'s age is 6 years, find the least number of years after which the ratio of their ages will be less than the ratio 7 : 3

38 A person buys tea at 6*s* a lb, and also some at 4*s* a lb, in what proportion must he mix them so that by selling his tea at 5*s* 3*d*. a lb, he may gain 20 per cent on each lb sold ?

39 The length of a certain rectangular field is to its breadth as 6 : 5 One-sixth part of the area being planted, there remains for ploughing 625 square yards What are the dimensions of the field ?

40 A bill before Parliament was lost on a division, there being 600 votes recorded Afterwards, there being the same voters, it was carried by twice as many votes as it was before lost by, and the new majority was to the former as 5 : 4 How many members changed their minds ?

41. Two casks, *A* and *B*, contain mixtures of wine and water, *A* in the ratio of 8 : 3, and *B* in the ratio of 5 : 1 In what proportion must liquid be drawn from each cask to give a mixture containing wine and water in the ratio of 4 : 1 ?

42 A packet, sailing from Dover with a fair wind, arrives at Calais in two hours, on its return, the wind being contrary, it proceeds six miles an hour slower than it went When it is half-way

over, the wind changing, it sails two miles an hour faster, and reaches Dover sooner than it would have done, had not the wind changed, in the ratio of 6 : 7. Find the distance between Dover and Calais.

43. Having given that the illumination from a source of light is inversely proportional to the square of the distance, how much nearer to a candle must an object, which is now 10 inches off, be placed so as to receive just 9 times as much light?

44. Given that the areas of plane triangles are proportional to the product of their bases into heights, compare the areas of two triangles whose bases are as 3 and 4, and heights are as 8 and 7.

45. Three globes, whose radii are 3, 4 and 5 inches, are melted and formed into one; find its radius, having given that the volume of a sphere is proportional to $(\text{radius})^3$.

46. If m shillings in a row reach as far as n sovereigns, and a pile of p shillings be as high as a pile of q sovereigns, compare the values of equal bulks of gold and silver, having given that the volume of a coin is proportional to the product of its thickness into $(\text{radius})^2$.

CHAPTER XXIV

MISCELLANEOUS THEOREMS AND EXAMPLES

281. Theorem. *If the sum of any number of real positive quantities be zero, each of the quantities is severally zero.*

Let $A^2+B^2+C^2+\ldots=0$, where $A, B, C, \ldots$ are all *real* quantities.

Now whether the expression for which A stands be, after reduction, positive or negative, *its square must always be positive*; hence A^2 is essentially positive. Similarly $B^2, C^2, \ldots$ are all essentially positive. Now the *sum of positive quantities cannot be zero, unless each of them be severally zero*. Hence $A^2=0$, $B^2=0$, $C^2=0$, &c.; whence $A=0$, $B=0$, $C=0, \ldots$ which proves the Theorem.

Ex. If $a^2+b^2+c^2=2ab$, shew that $a=b$ and $c=0$.

By transposition, we have

$$(a-b)^2+c^2=0, \text{ whence } (a-b)^2=0 \text{ and } c^2=0;$$
$$\therefore a-b=0 \text{ or } a=b, \text{ and } c=0.$$

Examples CLIX.

1. If $x^2+y^2+z^2+a^2+b^2+c^2=2(ax+by+cz)$, shew that
$$x=a, y=b, z=c.$$

2 If $$a^2+b^2+c^2=ax+by+cz=x^2+y^2+z^2=1,$$
prove that $$\frac{x}{a}=\frac{y}{b}=\frac{z}{c}=1$$

3 If
$$(x-a)^2+(x-b)^2+(x-c)^2=(x-a)(x-b)+(x-b)(x-c)+(x-c)(x-a),$$
find the relations between a, b, c

4 If $\left\{x-\frac{m^2}{l-n}\right\}^2+\left\{x-\frac{n^2}{l-m}\right\}^2=0$, shew that $x=m+n$.

5 Solve $(1+a^2+b^2)(1+x^2+y^2)=(1+ax+by)^2$.

6 Solve $(x-a)^2+(y-b)^2+(a^2+b^2-1)(x^2+y^2-1)=0$

282 Theorem If $XY=0$, then (1) *either* $X=0$ or $Y=0$, or (2) $X=0$ and $Y=0$

For if $X=a,\quad Y=0$ \
or if $X=0,\quad Y=b$ } then $XY=0$,

and if $X=0,\quad Y=0$, then also $XY=0$

Hence *when nothing is known as to the values of X and Y,* (1) *is the only legitimate conclusion*

Conversely, the product XY cannot be 0 unless *one* of its factors is 0

Cor Hence if $XYZU\ldots=0$, *one at least of the factors must be zero*, and conversely, *the product $XYZU\ldots$ cannot be* 0 *unless one at least of its factors is* 0

Ex 1. If $x^3+y^3+z^3-3xyz=0$, then must $x+y+z=0$, supposing x, y and z all unequal

Now $x^3+y^3+z^3-3xyz$
$$=\frac{1}{2}(x+y+z)\{(y-z)^2+(z-x)^2+(x-y)^2\} \text{ [Art 155]},$$
$$\therefore \frac{1}{2}(x+y+z)\{(y-z)^2+(z-x)^2+(x-y)^2\}=0$$

Thus either the first factor or the second factor is zero, if the second factor be zero, we have
$$(y-z)^2+(z-x)^2+(x-y)^2=0,$$
whence $(y-z)^2=0$, $(z-x)^2=0$, $(x-y)^2=0$ [Art. 281],
that is $y-z=0$, $z-x=0$, $x-y=0$,
or $x=y=z$,
which is contrary to hypothesis, therefore $x+y+z=0$

Ex 2. Solve $\left(\dfrac{x-a}{x-b}\right)^3=\dfrac{x-2a+b}{x-2b+a}$ [Ex 47, p 326]

Add 1 to both sides, thus

$$\frac{(x-a)^3+(x-b)^3}{(x-b)^3}=\frac{2x-a-b}{x-2b+a},$$

whence $(2x-a-b)\left\{\dfrac{(x-a)^2-(x-a)(x-b)+(x-b)^2}{(x-b)^3}-\dfrac{1}{x-2b+a}\right\}=0$,

or after slight reduction,

$$(2x-a-b)\frac{(a-b)^2}{(x-b)^3(x-2b+a)}=0,$$

therefore either the first factor$=0$, or the second factor$=0$, but the second factor cannot be 0, for then $(a-b)^2$ would be equal to 0, *i e*, a would be equal to b, which is contrary to supposition, therefore $2x-a-b=0$, whence $x=\frac{1}{2}(a+b)$

Ex 3 Given $\dfrac{1}{x}+\dfrac{1}{y}+\dfrac{1}{z}=\dfrac{1}{x+y+z}$,

shew that $\left(\dfrac{1}{x}+\dfrac{1}{y}+\dfrac{1}{z}\right)^{2n+1}=\dfrac{1}{x^{2n+1}+y^{2n+1}+z^{2n+1}}$

Multiplying out, we have from the given relation

$$(x+y+z)(yz+zx+xy)-xyz=0,$$

or $\quad (y+z)(z+x)(x+y)=0$ [Art 160, Ex 2],

hence one at least of the factors must be 0 Let $y+z=0$, whence $y=-z$ and therefore $y^{2n+1}=(-z)^{2n+1}=-z^{2n+1}$, since $2n+1$ is always odd, [see Art. 196, *Remark*]

Now $\left(\dfrac{1}{x}+\dfrac{1}{y}+\dfrac{1}{z}\right)^{2n+1}=\left(\dfrac{1}{x}-\dfrac{1}{z}+\dfrac{1}{z}\right)^{2n+1}=\left(\dfrac{1}{x}\right)^{2n+1}=\dfrac{1}{x^{2n+1}}$

$$=\frac{1}{x^{2n+1}-z^{2n+1}+z^{2n+1}}=\frac{1}{x^{2n+1}+y^{2n+1}+z^{2n+1}}$$

283 Meanings of 0, $\dfrac{a}{0}$, $\dfrac{0}{0}$

(1) Ordinarily 0 stands for the difference of two equal positive quantities which is evidently *nothing* There is, however, another meanings of 0, not incompatible with this meaning. Suppose h represents the difference of a and x where $a>x$, then

$$a-x=h$$

Now it is clear that as x increases and approaches to a, h continually diminishes and by giving to x a value sufficiently near to

that of a, h may be made *smaller than any assignable quantity*. This is expressed by saying that *the limiting value of the difference* h, *when* a *and* x *are made to differ as little as we please, is* 0.

(ii) $\frac{a}{0} = \infty$, *i e*, an **infinitely large quantity**

In examining the solution of the equation $ax+b=0$, we have seen [Art. 241] that when the coefficient of x is 0, the value of x, represented by $\frac{-b}{0}$, is *infinite* Thus the result of dividing any quantity by 0 is an infinitely large quantity. Hence $\frac{a}{0} = \infty$, where ∞ is the symbol for infinity

(iii) $\frac{0}{0}$ is the symbol of an **indeterminate form**, *i e*, when an expression assumes this form for a particular value of any symbol, the value of the expression cannot be determined, as long as it has that form

For we have seen [Art 241], that when the value of x assumes the form $\frac{0}{0}$, the equation $ax+b=0$ is satisfied by *any finite value* of x

Ex Evaluate $\frac{x}{\sqrt{1+x} - \sqrt{1-x}}$, when $x=0$

If at the outset, we substitute 0 for x, in the given expression, it becomes

$$\frac{0}{\sqrt{1+0} - \sqrt{1-0}} = \frac{0}{0},$$

and we cannot find its value We must therefore *transform the proposed expression into another of equal value*, such that for a given value of x, it may not assume the form $\frac{0}{0}$

Rationalize the denominator, thus the expression becomes

$$\frac{x(\sqrt{1+x} + \sqrt{1-x})}{2x} = \frac{\sqrt{1+x} + \sqrt{1-x}}{2}.$$

Now put $x=0$, thus its value is 1

Examples CLIX (*Continued.*)

7 Evaluate $\dfrac{x^3-a^3}{x-a}$, when $x=a$

8 Evaluate $\dfrac{x^3-1}{x^3-2x^2+2x-1}$, when $x=1$

9 Evaluate $\dfrac{x^2(y+1)-xy-1}{x^2(y-1)-x(y-2)-1}$, when $x=1$.

284 The cases of the divisibility of $x^n \pm a^n$ by $x \pm a$ [Art 142] can be otherwise proved by the method of **Mathematical Induction**

(i) To investigate when x^n-a^n is divisible by $x-a$.

We have
$$x^n-a^n=x^n-ax^{n-1}+ax^{n-1}-a^n$$
$$=x^{n-1}(x-a)+a(x^{n-1}-a^{n-1});$$
$$\therefore \frac{x^n-a^n}{x-a}=x^{n-1}+a\frac{x^{n-1}-a^{n-1}}{x-a}.$$

Now since x^{n-1} is *integral*, x^n-a^n will be divisible by $x-a$, if $\dfrac{x^{n-1}-a^{n-1}}{x-a}$ be *integral*, or in other words, if $x^{n-1}-a^{n-1}$ be divisible by $x-a$, that is, if x−a *divide a quantity which is the difference between any the same power of* x *and of* a, *it will also divide a quantity which is the difference between the next higher power of* x *and of* a. But we know that $x-a$ divides x^2-a^2, therefore it will divide x^3-a^3, and since it divides x^3-a^3, it will divide x^4-a^4, and so on

Hence

$$x^n-a^n \text{ is always divisible by } x-a$$

(ii) To investigate when x^n+a^n is divisible by $x-a$

We have
$$x^n+a^n=x^n-a^n+2a^n,$$
$$\therefore \frac{x^n+a^n}{x-a}=\frac{x^n-a^n}{x-a}+\frac{2a^n}{x-a}$$

Now since x^n-a^n is divisible by $x-a$, x^n+a^n will be divisible by $x-a$, if $2a^n$ is so, but $2a^n$ is never divisible by $x-a$,

$$x^n+a^n \text{ is never divisible by } x-a$$

(iii) To investigate when x^n-a^n is divisible by $x+a$

From Art. 196, we know that $(-a)^n=+a^n$ only when n is even; therefore $x^n-a^n=x^n-(-a)^n$ only when n is *even* and $x+a=x-(-a)$. Hence from (i),

$$x^n-a^n \text{ is divisible by } x+a \text{ only when } n \text{ is even.}$$

(iv) To investigate when x^n+a^n is divisible by $x+a$

As proved above, $(-a)^n=-a^n$ only when n is *odd*, therefore $x^n+a^n=x^n-(-a)^n$ only when n is odd, and $x+a=x-(-a)$ Hence from (i)

x^n+a^n is divisible by $x+a$ only when n is odd

Note It is important to remember the form of the quotient in each of the above cases By actual division, we have

(1) $(x^n-a^n)\div(x-a)=x^{n-1}+ax^{n-2}+a^2x^{n-3}+\ldots\ldots+a^{n-1}$

(2) $(x^n-a^n)\div(x+a)=x^{n-1}-ax^{n-2}+a^2x^{n-3}-\ldots\ldots-a^{n-1}$

(3) $(x^n+a^n)\div(x+a)=x^{n-1}-ax^{n-2}+a^2x^{n-3}-\ldots\ldots+a^{n-1}$

Mark that in (1) *all* the signs in the quotient are $+$, and in (2) and (3) the signs are alternately $+$ and $-$.

Examples CLIX *(Continued)*

10 Shew that 8^n-1 is always divisible by 7

11. Shew that $(23)^n+1$ is divisible by 24, if n be odd

12 Shew that $(21)^n-1$ is divisible by 20 or 22, if n be even.

13. Shew that $(24)^n-(19)^n$ is divisible by 5 or 43, if n be even.

14. Shew that $7^{2n}-1$ is always divisible by 48

15. Shew that $8^{2n}-3^{2n}$ is always divisible by 55

285 Condition of Divisibility From the Corollary, Art 137, we have $D=dQ+R$, i e, $\frac{D}{d}=Q+\frac{R}{d}$. Now since Q is integral, D will be exactly divisible by d, when $\frac{R}{d}$ is either an integer or 0 But $\frac{R}{d}$ cannot be an integer, for the degree of R being lower than that of d, $\frac{R}{d}$ is *fractional* Therefore $\frac{R}{d}=0$, i e, $R=0$ Hence *the condition for exact divisibility of one expression by another is that the remainder shall vanish.*

If therefore the Divisor be of n dimensions and consequently the Remainder of $n-1$ dimensions, the Remainder will contain in general n terms [Art 133] and therefore n coefficients of the symbol of reference, *which must each vanish identically if the remainder vanishes*

Ex 1. For what value of q will x^2+5x+q have the factor $x+8$, and what will be the other factor?

$$\begin{array}{l} x+8\ )\ x^2+5x+q\ (x-3 \\ \quad\quad\quad \underline{x^2+8x} \\ \quad\quad\quad\quad -3x+q \\ \quad\quad\quad\quad \underline{-3x-24} \\ \quad\quad\quad\quad\quad q+24 \end{array}$$

Since $x+8$ is a factor, the remainder vanishes, thus $q+24=0$ or $q=-24$ Hence the other factor is $x-3$

Ex 2 Determine a and b in order that $3x^3+22x^2+ax+b$ may have a quadratic factor x^2+7x+8

By division, the remainder will be found to be $(a-31)x+(b-8)$

Now x^2+7x+8 being a factor of the proposed expression, this remainder vanishes, therefore each of the coefficients vanishes identically hence $a-31=0$ and $b-8=0$, thus $a=31$ and $b=8$.

Examples CLIX. (*Continued*)

16 Find for what value of x, the expression $x^3+4x^2+7x+21$ is divisible by x^2+3x+5

17 Find the value of x for which $21x^4-29x^3+16x^2+14x-236$ is divisible by $3x^2-5x+8$

18 Find the value of x for which $x^4+ax^3+bx^2+cx+d$ is divisible by x^2+px+q

19 Find the relation between the coefficients p and q in order that x^2+px+q may be a perfect square.

286 Identity is satisfied by all values We have seen [Art. 83], that an Identity holds whatever value be given to *any* of the symbols in it Thus the identity $x^2-(a+b)x+ab=(x-a)(x-b)$ is satisfied when we put x or a or b equal 0, 1, 2, 3, &c Hence as a particular case it is clear that an *Identity is satisfied by any value whatever of the symbol of reference*

287 Theorem *In an Identity the coefficients of the like powers of the symbol of reference are equal*

Suppose $A+Bx+Cx^2+Dx^3+\ldots=a+bx+cx^2+dx^3+\ldots$ to be an *identity* Since this is satisfied by *any value* of x [Art 286], put $x=0$, thus $A=a$, and

$$Bx+Cx^2+Dx^3+\ldots=bx+cx^2+dx^3+\ldots;$$

now divide by x, and we have

$$B+Cx+Dx^2+\ldots=b+cx+dx^2+\ldots,$$

hence, as before, $B=b$ Similarly it may be shewn that $C=c$, $D=d$, &c.

This theorem is known as the PRINCIPLE of **Indeterminate Coefficients**

REMARK 1. A and a may be considered as coefficients of x having the *zero* power, for they may be written respectively Ax^0 and ax^0.

REMARK 2 In *incomplete expressions*, the coefficient of the *absent power* is 0 Thus from the identity

$$Ax^3+Bx^2+Cx+D=ax^3+bx^2+d,$$

we get $\quad Ax^3+Bx^2+Cx+D=ax^3+bx^2+0x+d$ [see Art. 133];

therefore $\quad A=a,\ B=b,\ C=0,\ D=d.$

EX If $ax^3+bx^2+cx+d=(2x+1)(x^2-4)$ for all values of x, determine a, b, c and d,

We have

$$ax^3+bx^2+cx+d=(2x+1)(x^2-4) \text{ identically } =2x^3+x^2-8x-4;$$

$$a=2,\ b=1,\ c=-8 \text{ and } d=-4.$$

Examples CLIX (*Continued.*)

20 If $ax^4+bx^3+cx^2+dx+e=3x^3+2(x+1)(x-1)+7$ identically, find the values of a, b, c, d and e

21 If px^3+qx^2+rx+s be identically equal to $2(x-1)(x+2)(x+3)$, what are the values of p, q, r and s?

22 If $x^4+ax^3+bx^2+cx+d=(x^2+x-1)(x^2+2x+3)$, shew that

$$a+d=0 \text{ and } b-4c=0$$

23 Find the values of a, b, c, d, when

$$ax^3+bx^2+cx+d=(2x^2+4x+1)(x-2),$$

and shew that $a+d=0$

24 Determine l, m, n in order that x^3+lx^2+mx+n may be equal to the cube of $x-2$.

288 We shall further illustrate Arts 285 and 287.

EX 1. For what value of q will x^2+5x+q have the factor $x+8$ and what will be the other factor? [See Art 285, Ex. 1.]

Since the proposed expression is a quadratic, it can have but *two linear* factors therefore the other factor must be of the form $x+a$. Hence

$$x^2+5x+q=(x+8)(x+a)=x^2+(a+8)x+8a,$$

$$a+8=5 \text{ and } 8a=q$$

whence $a=-3$ and $q=-24$, and therefore the required factor is $x-3$.

Ex 2. If x^2+5x+a be the square of $x+b$, find a and b

We have $x^2+5x+a=(x+b)^2=x^2+2bx+b^2$,

$2b=5$ and $b^2=a$, whence $b=\frac{5}{2}$ and $a=\frac{25}{4}$.

[Work this example by the method of Art 285]

Ex 3 Determine a and b in order that $3x^3+22x^2+ax+b$ may have a quadratic factor x^2+7x+8 [See Art Ex 2]

Since the extreme terms of the given factor are x^2 and 8, the required factor, which is linear, must be $3x+\frac{b}{8}$, whence multiplying out and equating coefficients, we get $a=31$, $b=8$

Ex 4 If x^3+ax^2+bx+c be exactly divisible by x^2+px+q, shew that $p(p-a)=q-b$, $q^2=bq-cp$

The other factor must evidently be $x+\frac{c}{q}$, therefore multiplying out and equating coefficients, we have

$$p+\frac{c}{q}=a \ (1), \text{ and } q+p\frac{c}{q}=b \quad (2)$$

From (1), $\frac{c}{q}=a-p$, from (2), $q+p(a-p)=b$. The second relation at once follows from (2)

Examples CLIX. (*Continued*)

25 If $x-5$ be a factor of $2x^2-7x+a$, what is the other factor and what the value of a?

26 Find p and q in terms of a, when $x^2+px+q=(x+a)^2$, and hence shew that $p^2=4q$

27 If x^3+x^2+px+q be divisible by $(x-1)(x-2)$, determine p and q

28 If $4x^3-9x^2+ax-12$ be exactly divisible by $4x+3$, find a and the other factor

29 If $2x^4+x^3+lx^2+mx+n$ be divisible without remainder by x^3+2x^2+3x+4, find l, m and n

30 If $2x^4+3x^3+ax^2+bx+c$ be divisible by $(x-3)(x^2-1)$, find the values of a, b and c

31. For what values of a and b will the expression

$x^4+4x^3+ax^2+bx+25$ be a complete square?

32. If $x^4+ax^3+bx^2+cx+d$ be a perfect square, shew that

$$d=\frac{c^2}{a^2}, \quad \frac{a^2}{4}+2\sqrt{d}=b.$$

289. Symmetrical Expressions. An expression is said to be symmetrical with respect to a pair of symbols, when their interchange does not affect its value and with respect to all its symbols, when the interchange of *any pair* does not and affect its value

Thus $x+y+mz$ is symmetrical with respect to x and y, but not with respect to x and z, or y and z, for the interchange of x and y gives $y+x+mz$ which is equivalent to the given expression, whereas the interchange of y and z gives $x+z+my$ which differs from the proposed expression.

Again the expression $bc+ca+ab$ is symmetrical with respect to all its symbols, for the interchange of any pair b and c gives $cb+ba+ac$ which is the same in value as the given expression.

The following are other examples $x+y+z$, $a^3+b^3+c^3-3abc$, $(b+c-2a)(c+a-2b)(a+b-2c)$, &c

It is worthy of note that the expression $x+y+mz$ would be symmetrical with respect to x, y and z, if $m=1$, or if x and y have each the coefficient m Thus $mx+my+mz$ is the *only* general form of symmetrical expressions *of the first degree* in x, y, z, where m is independent of x, y and z

Similarly it is easy to see that the expression

$$ax^2+by^2+cz^2+dyz+ezx+fxy+gxyz$$

would be symmetrical if $a=b=c$, and $d=e=f$, or in other words, if the expression assumes the form

$$Ax^2+Ay^2+Az^2+Byz+Bzx+Bxy+gxyz$$

It is needless to multiply examples, the above being sufficient to shew that in symmetrical expressions all the terms of the *same type* must have the *same coefficient* Thus in the second example x^2, y^2, z^2, which are of the type of x^2, all have the same coefficient A, and yz, zx, xy which are of the type of yz, all have the same coefficient B

Again from the nature of symmetrical expressions, it follows that the sum, product, or quotient of two symmetrical expressions is symmetrical Thus the sum, product and quotient of $x^2+y^2+z^2$ and $a+b+c$ are respectively

$$x^2+y^2+z^2+a+b+c,$$
$$(x^2+y^2+z^2)(a+b+c),$$
$$(x^2+y^2+z^2)-(a+b+c),$$

which are all symmetrical, as the student can himself see We thus obtain the following **Laws of Symmetry** .—

(1) *In a symmetrical expression, all the terms of the* same type *must have the same coefficient.*

(ii) *The* **sum, product or quotient** *of two symmetrical expressions must also be symmetrical*

REMARK. The expression $(y-z)(z-x)(x-y)$ is also called *symmetrical* though by the interchange of a pair of symbols its *sign* is changed

290 Homogeneous Expressions In this article we shall give a more convenient definition than that of Art 49 A homogeneous expression of the r^{th} degree is such that when each of its symbols is multiplied by r, the expression itself is multiplied by r, *the degree of* r^n, *viz*, n, *denoting the degree of the expression* Thus x^2+xy+y^2, $(x^2+y^2)-(x+y)$, and $(x^2+y^2+z^2)-(yz+zx+xy)$ are respectively of the *second, first* and *zeroth* degrees, for when each of the symbols of these expressions is multiplied by r, we obtain respectively

$$r^2(x^2+xy+y^2),$$

$$\frac{r^2(x^2+y^2)}{r(x+y)}=r\frac{x^2+y^2}{x+y}, \text{ and } \frac{r^2(x^2+y^2+z^2)}{r^2(yz+zx+xy)}=r^0\frac{x^2+y^2+z^2}{yz+zx+xy}$$

Thus though the quotients are not obvious in the second and third examples, yet their degrees are apparent We have thus the following **Laws of Homogeneity** —

(i) *If a homogeneous expression of the* m^{th} *degree be multiplied by another of the* n^{th} *degree, the product will be a homogeneous expression of the* $(m+n)^{th}$ *degree*

(ii) *If a homogeneous expression of the* m^{th} *degree be divided by another of the* n^{th} *degree, the quotient will be a homogeneous expression of the* $(m-n)^{th}$ *degree*

291 Factor Theorem *If an integral function* $f(x)$ *vanish when* $x=a$, *then* $x-a$ *is a factor of* $f(x)$

Let Q be the quotient and R the remainder, when $f(x)$ is divided by $x-a$ Then as in Art 141, we have

$$f(x)=Q(x-a)+R$$

Now if $f(x)=0$ when $x=a$, then also $R=0$.

Hence $$f(x)=Q(x-a);$$

i e, $x-a$ is a factor of $f(x)$

This consequence of the Remainder Theorem [Art 141] is very useful in factorizing expressions

Ex. 1. Prove that $(b-c)^3+(c-a)^3+(a-b)^3=3(b-c)(c-a)(a-b)$

[Art. 155, Ex. 6]

Put $b=c$, *i e*, suppose $b-c=0$, then

$$(b-c)^3+(c-a)^3+(a-b)^3=0+(c-a)^3+(a-c)^3=0,$$

$b-c$ is a factor of $(b-c)^3+(c-a)^3+(a-b)^3$ Similarly it may be shewn that $c-a$ and $a-b$ are factors of the expression.

Therefore $(b-c)^3+(c-a)^3+(a-b)^3=\lambda(b-c)(c-a)(a-b)$ (A).

Now since the proposed expression is of the *third* degree in a, b and c, λ cannot contain a, b, c and therefore remains constant for *any values* that may be assigned to a, b and c And since (A) is an *identity*, we can give to a, b, c any value we please Suppose, then $a=0$, $b=1$, $c=2$, thus from (A)

$$-1+8-1=\lambda\times-1\times2\times-1\text{, whence }\lambda=3\text{,}$$

$$\therefore\quad (b-c)^3+(c-a)^3+(a-b)^3=3(b-c)(c-a)(a-b).$$

Ex 2 Factorize $a^2(b+c)+b^2(c+a)+c^2(a+b)+2abc$

Denoting for shortness the given expression by $f(a, b, c)$ [Art 140] we have, by putting $b=-c$,

$$f(a, b, c)=c^2(c+a)+c^2(a-c)-2ac^2=0\text{,}$$

thus $b+c$ is a factor $f(a, b, c)$ Therefore from the symmetry of the given expression, we see that $c+a$ and $a+b$ are likewise its factors

Now $f(a, b, c)$ is of the *third* degree, therefore it has *three linear* factors, and consequently if it has any other factor, that factor must not involve a, b and c Hence

$$f(a, b, c)=\lambda(b+c)(c+a)(a+b),$$

where λ is some constant To determine λ, proceed as in Ex. 1, or more simply, *compare the coefficients of some one term on both sides* [Art 289] Thus by comparing the coefficients a^2b, we have $\lambda=1$;

$$\therefore\quad \text{given expression}=(b+c)(c+a)(a+b)$$

Ex 3 Factorise $a^2(b-c)+b^2(c-a)+c^2(a-b)$

Denote the given expression by $f(a, b, c)$

Put $b=c$ in $f(a, b, c)$, thus

$$f(a, b, c)=0+c^2(c-a)+c^2(a-c)=0\text{,}$$

therefore $b-c$ is a factor of $f(a, b, c)$ Similarly it may be shewn that $c-a$ and $a-b$ are factors Now $f(a, b, c)$ is of the *fourth* degree, therefore besides the 3 factors $b-c$, $c-a$, and $a-b$, it has another *linear* factor, and $f(a, b, c)$ being symmetrical, this factor must also be symmetrical in a, b and c, and therefore it is $\lambda a+\lambda b+\lambda c$ [Art. 289] Hence

$$\begin{aligned}f(a, b, c)&=(b-c)(c-a)(a-b)(\lambda a+\lambda b+\lambda c)\\&=\lambda(b-c)(c-a)(a-b)(a+b+c)\end{aligned}$$

Compare the coefficients of a^3b, thus $\lambda=-1$ Therefore

given expression $=-(b-c)(c-a)(a-b)(a+b+c)$

Ex 4 Factorize $a(b-c)^3+b(c-a)^3+c(a-b)^3=f(a, b, c)$

As in Ex. 3, $b-c$, $c-a$, $a-b$ are factors of $f(a, b, c)$ But $f(a, b, c)$, being of the fourth degree has another linear factor, and as $f(a, b, c)$ is symmetrical in a, b and c, this factor must also be symmetrical, whence it is $\lambda a+\lambda b+\lambda c$ [Art 289] Therefore

$$f(a, b, c)=(b-c)(c-a)(a-b)(\lambda a+\lambda b+\lambda c)$$
$$=\lambda(b-c)(c-a)(a-b)(a+b+c),$$

whence comparing the coefficients of ab^3, we get $\lambda=1$,

$$\text{given expression}=(b-c)(c-a)(a-b)(a+b+c)$$

Ex 5 Factorize $(x+y+z)^3-(y+z-x)^3-(z+x-y)^3-(x+y-z)^3$
$=f(x, y, z)$

Put $x=0$, thus $f(x, y, z)=(y+z)^3-(y+z)^3-(z-y)^3-(y-z)^3=0$, and from the symmetry of $f(x, y, z)$, it is clear that $f(x, y, z)$ becomes zero, when $y=0$, $z=0$, thus x, y, z are factors of $f(x, y, z)$ Therefore since $f(x, y, z)$ is of the third degree, we have

$$f(x, y, z)=\lambda xyz,$$

where λ is some constant Comparing the coefficients of xyz, we obtain $\lambda=24$ We might obtain λ as in Ex. 1 above Put $x=y=z=1$, thus

$$(1+1+1)^3-(1+1-1)^3-(1+1-1)^3-(1+1-1)^3=\lambda\, 1\, 1\, 1,$$

whence $\lambda=24$

Examples CLIX. (*Continued*)

33 Prove that

$(bc+ca+ab)^3-(bc)^3-(ca)^3-(ab)^3=3abc(b+c)(c+a)(a+b)$

34 Prove the identity $x(x+y-z)(x+z-y)+y(y+z-x)(y+x-z)$
$+z(z+x-y)(z+y-x)+(y+z-x)(z+x-y)(x+y-z)=4xyz$

35 Shew that $x^n-nx+n-1$ is divisible by $(x-1)^2$, if n be a positive integer

36 Shew that $(1-x)^2$ is a factor of $1-x-x^n+x^{n+1}$, n being any positive integer

37 Shew that $(x-1)^2$ is a factor of $nx^{n+1}-(n+1)x^n+1$, where n is a positive integer

38 If n be a positive integer, shew that

$(ab)^n-(bc)^n+(cd)^n-(da)^n$ is divisible by $ab-bc+cd-da$

292 To investigate when $x-1$, $x+1$ and x^2-1 are factors Let

$$f(x)=p_0x^n+p_1x^{n-1}+p_2x^{n-2}+\dots+p_n$$

be an integral expression Divide $f(x)$ by $x-1$; thus by the Remainder Theorem,

$$R=p_0+p_1+p_2+\dots+p_n$$

Now if $x-1$ be a factor of $f(x)$, R must vanish, therefore

$$p_0+p_1+p_2+\dots+p_n=0 \qquad (1),$$

when $x-1$ is a factor of $f(x)$ Hence *if the algebraic sum of the coefficients of an integral expression vanish, the expression has a factor* $x-1$

Again divide $f(x)$ by $x+1$, thus by the Remainder Theorem

$$R=(-1)^n p_0+(-1)^{n-1}p_1+(-1)^{n-2}p_2+\dots+p_n$$

Now R must vanish if $x+1$ be a factor of $f(x)$; therefore

$$(-1)^n p_0+(-1)^{n-1}p_1+(-1)^{n-2}p_2+\dots+p_n=0 \qquad (2),$$

when $x+1$ is a factor of $f(x)$ Hence *if the algebraic sum of the coefficients of the odd powers of* $f(x)$ *be equal to that of the coefficients of the even powers, the expression has a factor* $x+1$

Lastly from (1) whether n be even or odd, we have

$$p_0+p_1+p_2+\dots+p_n=0 \qquad (\alpha);$$

and from (2), supposing n to be even; we have

$$p_0+p_2+p_4+\dots p_n=p_1+p_3+p_5+\dots+p_{n-1} \qquad (\beta)$$

Therefore from (α) and (β), we have $2(p_0+p_2+p_4+\dots+p_n)=0$; which shews that the left side cannot vanish [Art 282] unless

$$p_0+p_2+p_4+\dots+p_n=0,$$

thus if

$$\left.\begin{aligned} &p_0+p_2+p_4+\dots+p_n=0\\ \text{and from } (\beta),\ &p_1+p_3+p_5+\dots+p_{n-1}=0\end{aligned}\right\} \qquad (3),$$

$f(x)$ has a factor $(x-1)(x+1)$ or x^2-1

The same result will similarly follow by supposing n to be odd. Hence *if the algebraic sum of the coefficients of the odd powers and that of the even powers severally vanish, the expression has a factor* x^2-1

Ex 1 Factorize $x^4+6x^3-12x^2+2x+3$

Here the *algebraic sum* of the coefficients of the several powers of x is zero Therefore the expression has $x-1$ for a factor Hence

given expression $=x^3(x-1)+7x^2(x-1)-5x(x-1)-3(x-1)$

$$=(x-1)(x^3+7x^2-5x-3)$$

The algebraical sum of the coefficients of the second factor is also zero, therefore it has a factor $x-1$ Hence

$$x^3+7x^2-5x-3=x^2(x-1)+8x(x-1)+3(x-1)=(x-1)(x^2+8x+3),$$

$\therefore$ the required factors are $(x-1)^2$ and x^2+8x+3.

Ex 2. Factorize $2x^3+11x^2-26x-35$

The sum of the coefficients of odd powers is $2-26=-24$, and the sum of the coefficients of even powers is $11-35=-24$ Thus $x+1$ is a factor of the given expression, therefore

$$\text{given expression}=2x^2(x+1)+9x(x+1)-35(x+1)$$
$$=(x+1)(2x^2+9x-35)=(x+1)(2x-5)(x+7)$$

Ex 3 Factorize $15x^4-38x^3+9x^2+38x-24$

The sum of the coefficients of even powers is $15+9-24=0$, and the sum of the coefficients of odd powers is $38-38=0$ Thus x^2-1 is a factor of proposed expression, which is therefore

$$=15x^2(x^2-1)-38x(x^2-1)+24(x^2-1)$$
$$=(x^2-1)(15x^2-38x+24)=(x^2-1)(3x-4)(5x-6)$$

Ex. 4 Factorize $x^3-15x^2+71x-105$

Now 5 is a factor of 105, put $x=5$, thus given expression

$$=(5)^3-15(5)^2+71\times5-105=0 \text{ identically},$$

therefore $x-5$ is a factor of the given expression Hence the expression may be depressed to a *quadratic* and its factors may be found

Ex 5 Factorize $6x^4-41x^3+95x^2-86x+24$

Now 2 is one of the factors of 24, and by trial we see that when 2 is substituted for x in the proposed expression, it vanishes,

$x-2$ is a factor of the expression Thus

$$\text{given expression}=6x^3(x-2)-29x^2(x-2)+37x(x-2)-12(x-2)$$
$$=(x-2)(6x^3-29x^2+37x-12)$$

Let us resolve the second factor One of the factors of 12 is 3 and by trial we see that 3 being put for x, causes this factor to vanish, therefore $x-3$ is a factor of this expression Therefore

$$6x^3-29x^2+37x-12=6x^2(x-3)-11x(x-3)+4(x-3)$$
$$=(x-3)(6x^2-11x+4)=(x-3)(2x-1)(3x-4)$$

Therefore the proposed expression $=(x-2)(x-3)(2x-1)(3x-4)$

Examples CLIX. *(Continued)*

Factorize the expressions

39. $x^4+3x^3-7x^2+x+2$ 40 $12x^3+13x^2+4x+3$

41. $20x^4-29x^3-5x^2+17x-3$

42 $3x^5+34x^4-32x^3-94x^2+29x+60$ 43 $3x^5-10x^3+15x+8$.

44. $5x^6-27x^5+54x^3-15x^2-27x+10$ 45 $3a^3+4a^2b-5ab^2-2b^3$.

46 $8x^3+34x^2y+41xy^2+15y^3$ 47. $3a^4-2a^3x-11a^2x^2+2ax^3+8x^4$

Factorize the expressions

48. $x^4-6a^2x^2-8a^3x-3a^4$. 49. $2x^4-9x^3-4x^2+51x-36$

50. $6x^4+x^3y-x^2y^2-9xy^3+3y^4$

51. $a^4-10a^3x+35a^2x^2-50ax^3+24x^4$ 52. $x^4-8x^3+17x^2-8x+16$

293. Consistency of Equations We have seen that two equations are necessary to determine two variables But suppose there are two such equations as

$$2x+3y=14 \text{ and } 6x+9y=42$$

A glance at once shews that the second equation is derived from the first by multiplying the latter by 3 Hence we cannot solve these equations definitely, as there is in fact only one independent equation [Art. 95]. Hence these equations are not *sufficient* to determine x and y, for which purpose, we must have another equation *independent* of the first, *i e*, one which must not be deduced from it by multiplying it by a constant

Again let there be another pair of equations

$$2x+3y=14 \text{ and } 4x+6y=26$$

By dividing the second equation by 2, we get $2x+3y=13$; therefore from the first equation, we obtain $14=13$, an absurd result. Hence the proposed equations are not *consistent*, and the values of x and y cannot be determined from them

These simple cases present no difficulty; but the general ones, *viz*, those where the coeficients of the variables are letters, are not so simple. We therefore proceed to investigate them

Let there be two equations in their general forms

$$a_1x+b_1y+c_1=0 \qquad (1),$$

$$a_2x+b_2y+c_2=0 \qquad (2).$$

Solving these we get

$$x=\frac{b_1c_2-b_2c_1}{a_1b_2-a_2b_1},\ y=\frac{c_1a_2-c_2a_1}{a_1b_2-a_2b_1} \qquad (3)$$

Now if $a_1b_2-a_2b_1=0$, *i e*, if the denominators vanish, we get

$$\frac{a_2}{a_1}=\frac{b_2}{b_1}=\lambda \text{ suppose,} \qquad a_2=\lambda a_1,\ b_2=\lambda b_1$$

Substitute the values of a_2 and b_2 in (2), thus

$$\lambda a_1x+\lambda b_1y+c_2=0,$$

or

$$a_1x+b_1y+\frac{c_2}{\lambda}=0 \qquad (4).$$

Now (4) differs from (1) only in the *constant term*, therefore (1) and (2) are *inconsistent* equations. Hence *the condition of inconsistency of the proposed equations* is

$$a_1b_2-a_2b_1=0, \text{ or } \frac{a_1}{a_2}=\frac{b_1}{b_2} \qquad (5).$$

Again if $\frac{c_2}{\lambda}=c_1$ or $\frac{c_2}{c_1}=\lambda$, then (2) is *consistent* with (1), but now it follows from (1) and is therefore not *independent* of (1). Therefore, if this condition hold, the proposed equations are *insufficient* for finding the values of x and y. Hence *the conditions of insufficiency and consequently of consistency of the proposed equations are*

$$\frac{a_1}{a_2}=\frac{b_1}{b_2}=\frac{c_1}{c_2} \qquad (6)$$

The conditions (5) and (6) of inconsistency and insufficiency of the given equations may be expressed in another form, for if (5) is satisfied, the values of x and y assume the forms $x=\frac{P}{0}$, $y=\frac{Q}{0}$, and if also (6) is satisfied, they assume the forms $x=\frac{0}{0}$, $y=\frac{0}{0}$. [Art 283] The latter forms shew that the values of x and y are *indeterminate* as they should be, since now the equations are no longer independent of one another.

Let us now investigate the condition under which

$$a_3x+b_3y+c_3=0 \qquad (7),$$

is consistent with (1) and (2). It is clear that if (7) be consistent with (1) and (2), the values of x and y satisfying these, *i e*, the values (3), must also satisfy (7). Thus the required condition after reduction is

$$a_1(b_2c_3-b_3c_2)+b_1(c_2a_3-c_3a_2)+c_1(a_2b_3-a_3b_2)=0 \qquad (8)$$

The above investigation shews that unless this condition is satisfied, (7) will not be consistent with (1) and (2), and (1) and (2) being two independent and consistent equations to determine x and y, a third equation is not at all necessary for the purpose. Thus *to determine two variables two, and only two, independent and consistent equations are necessary and sufficient*.

Reasoning similarly we arrive at the general conclusion that *to determine* n *variables,* n *and only* n, *independent and consistent equations are necessary and sufficient*.

Ex 1 For what value of λ will the equations $5x+3y=21$ and $15x+\lambda y=57$ be inconsistent?

The of condition inconsistency is [see (5)]

$$5\lambda-45=0, \text{ whence } \lambda=9$$

Ex 2 Find for what values of a and b, the equations $x+6y=27$ and $2x+ay=b$ will be consistent

The conditions of consistency are [see (6)]

$$\frac{1}{2}=\frac{6}{a}=\frac{27}{b}, \text{ whence } a=12 \text{ and } b=54$$

Ex 3 If the equations

$$x-a=y-b, \quad \frac{x}{a+c}+\frac{y}{b+c}=1, \quad \frac{x}{a-c}+\frac{y}{b-c}=1$$

be consistent, shew that $c^2-ab=0$

We may apply the general formula (8), or we may proceed thus —Subtract the second equation from the third, thus

$$\left(\frac{1}{a-c}-\frac{1}{a+c}\right)x+\left(\frac{1}{b-c}-\frac{1}{b+c}\right)y=0, \text{ or } \frac{x}{a^2-c^2}+\frac{y}{b^2-c^2}=0;$$

whence $\qquad x=k(a^2-c^2), \; y=k(c^2-b^2)$ [Art 261]

where k is some constant. Therefore from the second equation

$$\frac{k(a^2-c^2)}{a+c}+\frac{k(c^2-b^2)}{b+c}=1 \text{ or } k=\frac{1}{a-b},$$

and therefore $\qquad x=\dfrac{a^2-c^2}{a-b}$ and $y=\dfrac{c^2-b^2}{a-b}$

Substitute these values in the first equation, therefore

$$\frac{a^2-c^2}{a-b}-a=\frac{c^2-b^2}{a-b}-b, \text{ whence } c^2-ab=0$$

294 Elimination It is the method of finding a relation among the coefficients of the variables We shall explain how this relation may be obtained when the variables occur in expressions of the first degree. We have seen that n independent equations in n variables are sufficient to determine the n variables [Art 293] Hence if there be *one more* consistent equation involving the same number of variables, the variables can be *eliminated* from the given equations *by simply substituting their values in the* $(n+1)^{\text{th}}$ *equation* This is the general method of elimination when the given equations are *non-homogeneous*, as for instance when they are of the form

$$ax+by+c=0, \; a'x+b'y+c'=0, \; a''x+b''y+c''=0.$$

But if they be *homogeneous, only* n *equations are sufficient*, for then by dividing each of the equations by one of the variables, we can reduce their number to *one* less, and thus have a sufficient number of equations to eliminate the variables Thus

$ax+by=0$, $a'x+b'y=0$ become $a\frac{x}{y}+b=0$ and $a'\frac{x}{y}+b'=0$, where the ratio $\frac{x}{y}$ ($=z$ say) may, for the purposes of elimination, be considered as one variable

Ex 1. Eliminate x and y from the equations

$$ax+by=c,\ a'x+b'y=c',\ a''x+b''y=c''.$$

From (1) and (2), we get [Art 245]

$$x=\frac{b'c-bc'}{ab'-a'b},\ y=\frac{c'a-ca'}{ab'-a'b},$$

substitute in (3), thus

$$a''\frac{b'c-bc'}{ab'-a'b}+b''\frac{c'a-ca'}{ab'-a'b}=c'',$$

$$a''(b'c-bc')+b''(c'a-ca')=c''(ab'-a'b)$$

Ex 2 Eliminate x, y and z from the equations

$$\frac{x}{y+z}=a,\ \frac{y}{z+x}=b,\ \frac{z}{x+y}=c$$

From the given equations, we have

$$x-ay-az=0,\ -bx+y-bz=0 \qquad (1),$$

$$-cx-cy+z=0 \qquad (2).$$

These are *homogeneous*, hence the three are sufficient to eliminate the variables x, y and z We have from (1) by Art 245,

$$\frac{x}{ab+a}=\frac{y}{ab+b}=\frac{z}{1-ab}=k \text{ suppose,}$$

$$x=k(ab+a),\ y=k(ab+b),\ z=k(1-ab)$$

Substitute in (2), thus

$$-ck(ab+a)-ck(ab+b)+k(1-ab)=0,$$

whence by dividing by k, and transposing,

$$ab+bc+ac+2abc=1.$$

Examples CLIX. (*Continued*)

53 Eliminate x and y from the equations

$$ax+by=c,\ mx-ny=d,\ nx+ay=m.$$

54 Eliminate x between the equations

$$(x-a)(x-b)=(x-c)(x-d)=(x-e)(x-f)$$

55 Eliminate x, y and z from the equations

$$ax+by+cz=0,\ bx+cy+az=0,\ cx+ay+bz=0$$

56 Eliminate a, b, c between the equations

$$bz+cy=a,\ cx+az=b,\ ay+bx=c.$$

57 If $y+z : z+x : x+y=a : b : c$, and $ax+by+cz=0$, shew that

$$2(a^2+b^2+c^2)=(a+b+c)^2$$

58 Given that

$$x=by+cz+du,\ y=ax+cz+du,\ z=ax+by+du,\ u=ax+by+cz,$$

prove that
$$1=\frac{a}{1+a}+\frac{b}{1+b}+\frac{c}{1+c}+\frac{d}{1+d}$$

295 We shall now give some examples of elimination where the variables occur in special expressions of higher degree.

Ex 1. Eliminate x, y and z from the equations

$$\frac{y}{z}+\frac{z}{y}=a,\ \frac{z}{x}+\frac{x}{z}=b,\ \frac{x}{y}+\frac{y}{x}=c.$$

We know [Art 193, Ex. 9] that

$$\left(\frac{y}{z}+\frac{z}{y}\right)^2+\left(\frac{z}{x}+\frac{x}{z}\right)^2+\left(\frac{x}{y}+\frac{y}{x}\right)^2=4+\left(\frac{y}{z}+\frac{z}{y}\right)\left(\frac{z}{x}+\frac{x}{z}\right)\left(\frac{x}{y}+\frac{y}{x}\right),$$

$$a^2+b^2+c^2=4+abc$$

Ex 2 Eliminate x between the equations

$$ax^2+bx+c=0,\ cx^2+ax+b=0$$

By Art 245, we have

$$\frac{x^2}{b^2-ca}=\frac{x}{c^2-ab}=\frac{1}{a^2-bc},$$

$$\therefore\quad x^2=\frac{b^2-ca}{a^2-bc},\ x=\frac{c^2-ab}{a^2-bc},$$

whence
$$\frac{b^2-ca}{a^2-bc}=x^2=\left(\frac{c^2-ab}{a^2-bc}\right)^2,\text{ or } a^3+b^3+c^3-3abc=0.$$

Ex. 3. Eliminate x from the equations

$$32\frac{c}{a}=\left(\frac{x}{a}\right)^5+10\frac{x}{a}+5\left(\frac{a}{x}\right)^3,\ 32\frac{a}{c}=\left(\frac{a}{x}\right)^5+10\frac{a}{x}+5\left(\frac{x}{a}\right)^3.$$

By addition we have

$$\left(\frac{x}{a}\right)^5+5\left(\frac{x}{a}\right)^3+10^x+10\frac{a}{x}+5\left(\frac{a}{x}\right)^3+\left(\frac{a}{x}\right)^5=32\left(\frac{c}{a}+\frac{a}{c}\right),$$

or $$\left(\frac{x}{a}+\frac{a}{x}\right)^5=32\left(\frac{c}{a}+\frac{a}{c}\right),\quad \frac{x}{a}+\frac{a}{x}=2\left(\frac{c}{a}+\frac{a}{c}\right)^{\frac{1}{5}} \tag{1}$$

Similarly by subtraction, we have $$\frac{x}{a}-\frac{a}{x}=2\left(\frac{c}{a}-\frac{a}{c}\right)^{\frac{1}{5}} \tag{2}$$

Squaring (1) and (2), we get

$$\frac{x^2}{a^2}+\frac{a^2}{x^2}+2=4\left(\frac{c}{a}+\frac{a}{c}\right)^{\frac{2}{5}},\ \frac{x^2}{a^2}+\frac{a^2}{x^2}-2=4\left(\frac{c}{a}-\frac{a}{c}\right)^{\frac{2}{5}},$$

$$4=4\left\{\left(\frac{c}{a}+\frac{a}{c}\right)^{\frac{2}{5}}-\left(\frac{c}{a}-\frac{a}{c}\right)^{\frac{2}{5}}\right\},\text{ or }\left(\frac{c}{a}+\frac{a}{c}\right)^{\frac{2}{5}}-\left(\frac{c}{a}-\frac{a}{c}\right)^{\frac{2}{5}}=1.$$

Examples CLIX. (*Continued*)

59 Eliminate x from $x+\frac{1}{x}=a,\ x^3+\frac{1}{x^3}=b^3-3b$

60 Eliminate x and y from $x-y=a,\ x^2+y^2=b^2,\ xy=c^2$.

61 Eliminate x and y between the equations

$$px-qy+q=py-qx+p,\ x^3+3xy^2=p^3,\ y^3+3x^2y=q^3$$

62 Eliminate x, y and z from

$$x^2(y+z)=a^3,\ y^2(z+x)=b^3,\ z^2(x+y)=c^3,\ xyz=abc.$$

63 Eliminate x between the equations

$$\frac{x^3}{a^3}+\frac{a^3}{x^3}+3\left(\frac{x}{a}+\frac{a}{x}\right)=m,\ \frac{x^3}{a^3}-\frac{a^3}{x^3}-3\left(\frac{x}{a}-\frac{a}{x}\right)=n$$

64 Eliminate x and y between the equations

$$ax+by=c\sqrt{(x^2+y^2)},\quad a'x+b'y=c'\sqrt{(x^2+y^2)}$$

65. Eliminate x and y from the equations

$$a=x^2y,\quad b=xy^3,\quad xy+1=c(x+y)$$

296 Miscellaneous Artifices We shall close this Chapter by giving a few examples of what is commonly called *Algebraical Artifices* More of these will be found in the Appendix

Ex 1 If $ax^2+\lambda x+c$ be a perfect square with respect to x, find the value of λ.

It is evident that the proposed expression must be the square of

$$\sqrt{ax}+\sqrt{c}, \text{ which } =ax^2+2\sqrt{(ac)}x+c,$$

whence by Art 287, $$\lambda=2\sqrt{ac}$$

Otherwise —Proceed as in the extraction of the square root

$$ax^2+\lambda x+c\left(\sqrt{a}x+\frac{\lambda}{2\sqrt{a}}\right.$$

$$\begin{array}{r|l} & ax^2 \\ \hline 2\sqrt{a}x+\frac{\lambda}{2\sqrt{a}} & \lambda x+c \\ & \lambda x+\frac{\lambda^2}{4a} \\ \hline & c-\frac{\lambda^2}{4a}. \end{array}$$

Now since the proposed expression is an exact square, the remainder must vanish, therefore

$$c-\frac{\lambda^2}{4a}=0, \text{ or } \lambda^2=4ac$$

Ex. 2. The expression $ax^2+2hxy+by^2+2gx+2fy+c$ will be an exact square, if $$abc=af^2=bg^2=ch^2$$

Let the proposed expression be the square of $\lambda x+\mu y+\sqrt{c}$ By expanding its square, we get

$$\lambda^2x^2+2\lambda\mu xy+\mu^2y^2+2\lambda\sqrt{c}x+2\mu\sqrt{c}y+c;$$

whence equating the coefficients [Art. 287], we have

$$\lambda^2=a,\ 2\lambda\mu=2h,\ \mu^2=b,\ 2\lambda\sqrt{c}=2g,\ 2\mu\sqrt{c}=2f.$$

Therefore from the first three, $h^2=ab$, from first and fourth, $g^2=ac$, and from the third and fifth, $f^2=bc$ Hence multiplying respectively by c, b and a, we have the required relations

Note We might have assumed the proposed expression to be the square of $\sqrt{a}x+\sqrt{b}y+\sqrt{c}$

Ex 3 Shew that $ax^3+3bx^2+3cx+d$ will be a perfect cube, if $ad=bc$

Let the proposed expression be the cube of $\lambda x+\mu$ By developing the cube of $\lambda x+\mu$, and equating the coefficients [Art 287], we get

$$\lambda^3=a,\ \lambda^2\mu=b,\ \lambda\mu^2=c,\ \mu^3=d,$$

whence $$ad=\lambda^3\mu^2=bc$$

Note Instead of assuming $\lambda x+\mu$ as the cube root of the given expression, we might have evidently assumed $\sqrt[3]{ax}+\sqrt[3]{d}$ as the cube root.

Ex 4 If $2a-3y=\frac{(z-x)^2}{y}$ and $2a-3z=\frac{(x-y)^2}{z}$, x, y, z being unequal, then will $2a-3x=\frac{(y-z)^2}{x}$, and $x+y+z=a$

Subtract the first equation from the second, thus

$$3y-3z=\frac{(x-y)^2}{z}-\frac{(z-x)^2}{y},$$

or $3(y-z)yz=y(x-y)^2-z(z-x)^2=x^2(y-z)-2x(y^2-z^2)+(y^3-z^3)$;
whence by dividing by $y-z$, which is not$=0$ by supposition, we get

$$x^2+y^2+z^2=2yz+2zx+2xy \qquad (a).$$

From the first equation

$$2a=\frac{(z-x)^2}{y}+3y=\frac{z^2+x^2-2zx+3y^2}{y}=\frac{2y^2+2yz+2xy}{y} \text{ using } (a)$$
$$=2(x+y+z),$$
$$x+y+z=a \qquad (\beta).$$

Again from (a), $y^2+z^2-2yz=2zx+2xy-x^2$,

or $$\frac{(y-z)^2}{x}=2z+2y-x=2a-3x, \text{ from } (\beta)$$

Ex 5 If $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=0$, and $(b-c)^2x+(c-a)^2y+(a-b)^2z=0$,

then will $(b-c)x=(c-a)y=(a-b)z$

Eliminating z between the given equations, we have

$$(b-c)^2x+(c-a)^2y-(a-b)^2\frac{xy}{x+y}=0,$$

whence after reduction $(b-c)^2x^2-2(b-c)(c-a)xy+(c-a)^2y^2=0$;

or $\{(b-c)x-(c-a)y\}^2=0$, $\therefore (b-c)x=(c-a)y$

And since the given expressions are symmetrical, we have

$$(c-a)y=(a-b)z;$$
$$(b-c)x=(c-a)y=(a-b)z.$$

Ex 6 If $a=bz+cy$, $b=cx+az$, $c=ay+bx$, shew that

$$\frac{a^2}{1-x^2}=\frac{b^2}{1-y^2}=\frac{c^2}{1-z^2}.$$

By transposition, we get

$$-a+bz+cy=0 \qquad (1),$$
$$az-b+cx=0 \qquad (2),$$
$$ay+bx-c=0 \qquad (3).$$

Thus from (1) and (2), and (2) and (3), we get, by Cross Multiplication [Art 245],

$$\frac{a}{zx+y}=\frac{c}{1-z^2}, \quad \frac{a}{1-x^2}=\frac{c}{zx+y};$$

whence $$\frac{a^2}{1-x^2}=\frac{c^2}{1-z^2} \qquad (4)$$

Similarly from (2) and (3), and (3) and (1), we have

$$\frac{a}{1-x^2}=\frac{b}{xy+z}, \quad \frac{a}{xy+z}=\frac{b}{1-y^2},$$

whence $$\frac{a^2}{1-x^2}=\frac{b^2}{1-y^2} \qquad (5)$$

Thus the required relations follow from (4) and (5)

Ex 7 If $$\frac{a}{x}(b-c)+\frac{b}{y}(c-a)+\frac{c}{z}(a-b)=0,$$

prove that (i) $$\frac{a(b-c)}{x(y-z)}=\frac{b(c-a)}{y(z-x)}=\frac{c(a-b)}{z(x-y)},$$

(ii) $$\frac{b-c}{x(bz-cy)}=\frac{c-a}{y(cx-az)}=\frac{a-b}{z(ay-bx)}$$

(i) Multiply the given relation by xyz, thus

$$ayz(b-c)+bzx(c-a)+cxy(a-b)=0 \quad \ldots\ldots \quad (A)$$

We have identically

$$a(b-c)+b(c-a)+c(a-b)=0 \text{ [Art. 149]}$$

Also from (A), $yz\,a(b-c)+zx\,b(c-a)+xy\,c(a-b)=0$

Hence by Cross Multiplication

$$\frac{a(b-c)}{xy-zx}=\frac{b(c-a)}{yz-xy}=\frac{c(a-b)}{zx-yz},$$

i e, $$\frac{a(b-c)}{x(y-z)}=\frac{b(c-a)}{y(z-x)}=\frac{c(a-b)}{z(x-y)}$$

(ii) From (A),

$$ayz(b-c)+bzx(c-a)+cxy(a-b)=0$$

Also identically, $(b-c)+(c-a)+(a-b)=0$ [Art. 149]

Hence by Cross Multiplication

$$\frac{b-c}{bzx-cxy}=\frac{c-a}{cxy-ayz}=\frac{a-b}{ayz-bzx},$$

$$\frac{b-c}{x(bz-cy)}=\frac{c-a}{y(cx-az)}=\frac{a-b}{z(ay-bx)}$$

Examples CLX

1 For what value of λ will $3x^2+\lambda x+1$ have two linear factors?

2 Find the value of h which will make $2x^2+hx+3$ a complete square

3 For what value of λ, μ and ν will $4x^2+\lambda xy+9y^2+\mu x+\nu y+16$ be an exact square?

4 If ax^3+bx^2+cx+d be a perfect cube, shew that $b^2=3ac$, $c^2=3bd$ and $c^3=27ad^2$

5 Extract the cube root of $x^6-6x^5+15x^4-20x^3+15x^2-6x+1$

6 Shew that $a(x+1)^2+bx^2+2cx(x+1)$ is a perfect square with respect to x, if $c^2-ab=0$

7 If $x^4+ax^3+bx^2+cx+d$ be an exact square, then the relations between the coefficients are

$$8c=a(4b-a^2),\ (4b-a^2)^2=64d$$

8 If $ax^4+bx^3+cx^2$ be subtracted from $(x^2+2x+4)^2$, the remainder is an exact square, find a, b, c

9 Shew that $4x(x+a)(x+b)(x+c)+a^2b^2$ is a perfect *square*, if $c=a+b$ and a perfect *fourth* power, if also $c^2=2ab$

10 Find the value of y which will make

$$2(y^2+y)x^2+(11y-2)x+4$$

and $$2(y^3+y^2)x^3+(11y^2-2y)x^2+(y^2+5y)x+5y-1$$

have a common factor, and find the factor

11. If $x+\frac{1}{x}=a+\frac{1}{a}$, shew that $x^n+\frac{1}{x^n}=a^n+\frac{1}{a^n}$

12 If $ax^3=by^3=cz^3$ and $x^{-1}+y^{-1}+z^{-1}=1$, shew that

$$ax^2+by^2+cz^2=(a^{\frac{1}{3}}+b^{\frac{1}{3}}+c^{\frac{1}{3}})^3$$

13 If $x^2-yz=a$, $y^2-zx=b$, $z^2-xy=c$, shew that

$$(a+b+c)(x+y+z)=\sqrt{a^3+b^3+c^3-3abc}$$

14 If $(a^2+bc)^2(b^2+ca)^2(c^2+ab)^2=(a^2-bc)^2(b^2-ca)^2(c^2-ab)^2$, then either $a^3+b^3+c^3+abc=0$, or $a^{-3}+b^{-3}+c^{-3}+a^{-1}b^{-1}c^{-1}=0$

15 If $$(a^2-bc)(b^2-ca)(c^2-ab)=0,$$

prove that $$\frac{1}{a^3}+\frac{1}{b^3}+\frac{1}{c^3}=\frac{a^3+b^3+c^3}{a^3b^3c^3}$$

16. If $a^2+b^2+c^2+\dots$ to n terms $=2x\left(a+b+c+\dots\dots-\frac{nx}{2}\right)$,

then $a=b=c=\dots\dots$, a, b, c, $\dots\dots x$ being all real.

17. If $(a^2+b^2+c^2)(x^2+y^2+z^2)=(ax+by+cz)^2$, shew that
$$x : y : z = a : b : c$$

18. If $\sqrt{ay^2-a^2}=yz$ and $\sqrt{ax^2-a^2}=xy$, then $\sqrt{az^2-a^2}=xz$.

19. If $x+\dfrac{1}{x}=2\sqrt{1+m^2}$, $y+\dfrac{1}{y}=2\sqrt{1+n^2}$ and $\dfrac{m}{a}=\dfrac{n}{b}$,

shew that
$$ay+\frac{b}{x}=bx+\frac{a}{y}$$

20. If $\dfrac{1}{x}+\dfrac{1}{y}+\dfrac{1}{z}=0$, then $\dfrac{1}{x^2-yz}+\dfrac{1}{y^2-zx}+\dfrac{1}{z^2-xy}=0$

21. If $x+y+z=0$, then $\dfrac{x^2}{2x^2+yz}+\dfrac{y^2}{2y^2+zx}+\dfrac{z^2}{2z^2+xy}=1$ [*App*]

22. If $xy+yz+zx=1$, then will
$$\frac{x}{1-x^2}+\frac{y}{1-y^2}+\frac{z}{1-z^2}=\frac{4xyz}{(1-x^2)(1-y^2)(1-z^2)}.$$

23. If $a=b\dfrac{y}{z}+c\dfrac{z}{y}$, $b=c\dfrac{z}{x}+a\dfrac{x}{z}$, $c=a\dfrac{x}{y}+b\dfrac{y}{x}$, prove that
$$\frac{1}{x^3}+\frac{1}{y^3}+\frac{1}{z^3}+\frac{1}{xyz}=0 \quad [App]$$

24. If $bc+ca+ab=1$, shew that
$$\left\{1-\frac{a^2}{1+a^2}-\frac{b^2}{1+b^2}-\frac{c^2}{1+c^2}\right\}^2=\frac{4a^2b^2c^2}{(1+a^2)(1+b^2)(1+c^2)}$$

25. If $(a+b)(x+y)=2(ab+xy)$ and $(c+d)(x+y)=2(cd+xy)$,

prove that
$$\left(\frac{x-y}{2}\right)^2=\frac{(a-c)(a-d)(b-c)(b-d)}{(a+b-c-d)^2}.$$

26. Given $U=\sqrt{1+x^2}-\sqrt{1+y^2}$ and $V=\dfrac{\sqrt{1+x^2}-1}{\sqrt{1+y^2}-1}\cdot\dfrac{y}{x}$,

shew that
$$\frac{1}{2}xy=\frac{U}{V-V^{-1}}$$

27. Prove that two of the quantities a, b, c, must be equal to one another, if
$$\frac{b-c}{1+bc}+\frac{c-a}{1+ca}+\frac{a-b}{1+ab}=0$$

28. If $x=\dfrac{a+1}{a-1}$, $y=\dfrac{b+1}{b-1}$, $z=\dfrac{c+1}{c-1}$, shew that
$$\frac{(1+x^2)(1+y^2)(1+z^2)}{(1+xy)(1+yz)(1+zx)}=\frac{(1+a^2)(1+b^2)(1+c^2)}{(1+ab)(1+bc)(1+ca)}$$

29 If $\dfrac{x^2-y^2}{a-b}=\dfrac{xy}{z}$ and $\dfrac{y^2-z^2}{b-c}=\dfrac{yz}{x}$, then $\dfrac{z^2-x^2}{c-a}=\dfrac{zx}{y}$

30 If x, y, z, be unequal and $\dfrac{x^2-yz}{x-xyz}=\dfrac{y^2-zx}{y-xyz}$, shew that each of these ratios is equal

$$\text{to } \frac{z^2-xy}{z-xyz}, \text{ to } x+y+z, \text{ and to } \frac{1}{x}+\frac{1}{y}+\frac{1}{z}$$

31 If x, y, z be unequal and $\dfrac{yz-x^2}{y+z}=\dfrac{zx-y^2}{z+x}$, prove that each of these $=\dfrac{xy-z^2}{x+y}$

32 If $ax(b-c)+by(c-a)+cz(a-b)=0$,

shew that (i) $\dfrac{b-c}{by-cz}=\dfrac{c-a}{cz-ax}=\dfrac{a-b}{ax-by}$,

(ii) $\dfrac{a(b-c)}{y-z}=\dfrac{b(c-a)}{z-x}=\dfrac{c(a-b)}{x-y}$

33 If $a+b+c=0$, shew that

$$\left(\frac{b-c}{a}+\frac{c-a}{b}+\frac{a-b}{c}\right)\left(\frac{a}{b-c}+\frac{b}{c-a}+\frac{c}{a-b}\right)=9$$

34 Shew that if

$$ax^2+by^2+cz^2=ax+by+cz=yz+zx+xy=0,$$

then $$abc=(b+c-a)(c+a-b)(a+b-c)$$

35. If $a+b+c=0$, and

$$a(by+cz-ax)=b(cz+ax-by)=c(ax+by-cz),$$

then will $$x+y+z=0 \quad [App]$$

CHAPTER XXV

QUADRATIC EQUATIONS

297 *Definition* A quadratic equation is one in which, when it is reduced to a rational and integral form, the highest term involving the variable is of the second degree. Thus $x^2+px+q=0$, $x+\dfrac{1}{x}=a$ and $x+\sqrt{x}+c=0$ are quadratic equations, for when the second is reduced to an *integral* form, and the third to a *rational* form, they become respectively $x^2-ax+1=0$ and $x^2+(2c-1)x+c^2=0$

Note It is well that the student should at the outset distinguish between a *quadratic expression* and a *quadratic equation* In the quadratic *expression* ax^2+bx+c, the variable x may have *any* value whatever, but *when that expression is equated to zero*, it becomes the quadratic *equation* $ax^2+bx+c=0$ and then the variable x must have a *definite number* of values

298 Different forms of Quadratics The **general** or **standard form** of a quadratic equation in one variable is

$$ax^2+bx+c=0 \quad \ldots \quad (1)$$

For when a quadratic equation is reduced to a rational and integral form and all its terms are transposed to one side, the coefficients of x^2 and x and the constant terms being bracketed together will give the quantities represented by a, b and c respectively

Hence a quadratic equation in the standard form consists of only **three terms**

For particular values of a, b and c, the equation (1) may assume **special forms** Thus by dividing this equation by a, we have

$$x^2+\frac{b}{a}x+\frac{c}{a}=0,$$

which, when p is put for $\frac{b}{a}$ and q for $\frac{c}{a}$, becomes

$$x^2+px+q=0$$

Thus when the coefficient of x^2 is made *unity* by division, (1) assumes the form

$$x^2+px+q=0 \quad \ldots \quad (2)$$

When $b=0$, (1) reduces to the form

$$ax^2+c=0 \quad \ldots \quad (3)$$

When $c=0$, (1) has the form

$$ax^2+bx=0 \quad \ldots \quad (4)$$

Lastly, when $b=0$, and $c=0$, the form of (1) is

$$ax^2=0 \quad \ldots \quad (5)$$

Thus *any equation which after reduction assumes any one of the above forms is a Quadratic Equation*

Equations of the form (3) are called **Pure Quadratics** and those of the form (1) or (2) are called **Adfected Quadratics** Thus *a* PURE *Quadratic is one in which only the second power of the variable occurs*, and *an* ADFECTED *Quadratic is one in which both the second and first powers occur*

239 Solution of Pure Quadratics Equations of the form (4) and (5) present no difficulty, so we shall solve equations of the form (3) which are called Pure Quadratics

To solve (3), we transpose c and divide by a, thus $x^2=-\frac{c}{a}$, or

$x=\pm\sqrt{-\frac{c}{a}}$, that is, the roots are $+\sqrt{-\frac{c}{a}}$ and $-\sqrt{-\frac{c}{a}}$ Hence the roots of a pure quadratic are *equal in magnitude but of opposite signs*

REMARK In extracting the square root, it is sufficient to affect with the double sign *only the right side* of an equation and not *both* of its sides Thus from $x^2=a^2$, we put $x=\pm a$, and not $\pm x=\pm a$, as the first virtually includes all the cases of the second For from $\pm x=\pm a$, we have

$$+x=+a \quad (\text{i}), \qquad -x=+a \quad (\text{iii}),$$
$$+x=-a \quad (\text{ii}), \qquad -x=-a \quad (\text{iv}),$$

and when the signs of (iii) and (iv) are changed, they are respectively the same as (ii) and (i)

Ex 1 Solve $x-\frac{27}{x}=\frac{4x-9}{6}+1\frac{1}{2}$

Reduce to an integral form by multiplying by $6x$,

thus $\qquad 6x^2-162=4x^2-9x+9x,$

or $\qquad x^2=81$, whence $x=\pm 9$.

Ex 2 Solve $x+\sqrt{9-2x-x^2}=1$

Transpose, thus $\qquad \sqrt{9-2x-x^2}=1-x$,

square, thus $\qquad 9-2x-x^2=1-2x+x^2$,

whence $\qquad 2x^2=8$, or $x=\pm 2$

Ex 3 Solve $\frac{a}{b+x}+\frac{a}{b-x}=c$

Divide by a, thus

$$\frac{1}{b+x}+\frac{1}{b-x}=\frac{c}{a},$$

or
$$\frac{2b}{b^2-x^2}=\frac{c}{a},$$

or
$$b^2-x^2=\frac{2ab}{c},$$

or
$$x^2=b^2-\frac{2ab}{c}=b^2\left(1-\frac{2a}{bc}\right),$$

$$x=\pm b\sqrt{1-\frac{2a}{bc}}$$

Ex. 4 Solve $\dfrac{\sqrt{a^2+x^2}+\sqrt{a^2-x^2}}{\sqrt{a^2+x^2}-\sqrt{a^2-x^2}}=b$

We have $\dfrac{\sqrt{a^2+x^2}}{\sqrt{a^2-x^2}}=\dfrac{b+1}{b-1}$, [compo and divido];

square thus $\dfrac{a^2+x^2}{a^2-x^2}=\left(\dfrac{b+1}{b-1}\right)^2$;

$$\therefore \quad \frac{x^2}{a^2}=\frac{(b+1)^2-(b-1)^2}{(b+1)^2+(b-1)^2}=\frac{2b}{b^2+1},$$

$$\text{i e, } x=\pm a\sqrt{\frac{2b}{b^2+1}}.$$

Ex 5 Solve $\dfrac{2a\sqrt{1+x^2}}{\sqrt{1-x^2}+\sqrt{1+x^2}}=a+b$

We have $2a\sqrt{1+x^2}=(a+b)\sqrt{1-x^2}+(a+b)\sqrt{1+x^2}$,

thus $(a-b)\sqrt{1+x^2}=(a+b)\sqrt{1-x^2}$,

or $\dfrac{\sqrt{1+x^2}}{\sqrt{1-x^2}}=\dfrac{a+b}{a-b}$

square, thus $\dfrac{1+x^2}{1-x^2}=\left(\dfrac{a+b}{a-b}\right)^2$;

$$\therefore \quad x^2=\frac{(a+b)^2-(a-b)^2}{(a+b)^2+(a-b)^2}=\frac{2ab}{a^2+b^2};$$

or $x=\pm\sqrt{\dfrac{2ab}{a^2+b^2}}$

Examples CLXI

Solve the equations

1. $3x^2-7=41$ 2 $6x^2+5=68-x^2$.

3 $5x^2-121=4(26-x^2)$. 4 $3(2x-3)^2=4x(2x-9)+43$

5 $2x+\dfrac{17}{x}=\dfrac{7x-10}{2\frac{1}{2}}+4$ 6. $(x-7)(x+7)=31-4x^2$.

7 $\dfrac{x-1}{x+2}=\dfrac{2x+1}{5x-2}$ 8. $\dfrac{2x-1}{x-2}=\dfrac{x-5}{3x-2}$

9 $\dfrac{x}{2}+\dfrac{2}{x}=\dfrac{x}{3}+\dfrac{3}{x}$. 10 $(x-a)^2+(x-b)^2=a^2+b^2$

11. $(x+4)(2x+9)=(2x+\frac{17}{2})2x-13$ 12 $\dfrac{a^2}{(x-a)^2}=\dfrac{b^2}{(x-b)^2}$

Solve the equations

13 $(2x-5)(3x-2)-(x-2)(2x-3)=4$

14 $\dfrac{2x}{x-2}+\dfrac{x-2}{x}=2$

15 $\dfrac{x-7}{x-4}+\dfrac{5}{2+x}=\dfrac{2}{x^2-2x-8}$

16 $\dfrac{x+7}{x(x-7)}-\dfrac{x-7}{x(x+7)}=\dfrac{7}{x^2-7^2}$.

17 $x+\dfrac{2}{1+\dfrac{1}{x}}=3$

18. $\sqrt{8x+1}-\sqrt{x+1}=\sqrt{3x}$

19. $\sqrt{2x+6}-\sqrt{x-1}=2$

20 $x\sqrt{x^2+12}+x\sqrt{x^2+6}=3$

21 $\sqrt{1+\dfrac{bx}{a^2}}+\sqrt{1-\dfrac{bx}{a^2}}=1\frac{3}{5}$

22. $\dfrac{2}{x+\sqrt{2-x^2}}+\dfrac{2}{x-\sqrt{2-x^2}}=x$.

23 $\dfrac{\sqrt{1-x}}{2-\sqrt{1+x}}=\dfrac{\sqrt{1+x}}{2+\sqrt{1-x}}$

24 $\dfrac{\sqrt{1+x}-1}{\sqrt{1-x}+1}+\dfrac{\sqrt{1-x}+1}{\sqrt{1+x}-1}=2a$.

25 $\dfrac{1+x^3}{(1+x)^3}+\dfrac{1-x^3}{(1-x)^3}=a$

26 $\sqrt{x^2+9}+\sqrt{x^2-9}=4+\sqrt{34}$

[*For other Examples, see pp 328—9 and pp 397—8*]

300 Methods of solving Quadratics There are five Methods of which we shall give only four The last of these four will however be found in a subsequent Chapter [See Art 337]

301 Method I—Solution by Factorization We have seen [Art 297, *Note*] that a quadratic expression, when equated to 0, gives a quadratic equation whose terms are all transposed to one side Hence a quadratic equation may be solved by factorizing the corresponding quadratic expression.

Ex. Solve the equation $8x^2+3=14x$

Transpose, thus $8x^2-14x+3=0$,

whence $(2x-3)(4x-1)=0$,

thus $2x-3=0$, or $4x-1=0$,

$x=\frac{3}{2}$ or $x=\frac{1}{4}$

Note The solution suggests that if an equation be given *as the product of factors equated to* 0, we can at once put down its roots

Example. Solve the equation $(x-1)(x-2)(x-3)=0$

Here either $x-1=0$, or $x-2=0$, or $x-3=0$ [Art. 282],

thus $x=1$, or 2, or 3

302 Method II—Solution by Completing the Square. In Art 153, we have seen that there are two methods of completing the square—(1) the *Common Method* and (2) *Sridhara's Method*,

commonly known as the *Hindu Method* The explanations given there suggest the following Rules.

Common Method

RULE —*Reduce the quadratic to the standard form, transpose the constant term, divide by the coefficient of* x^2, *and add to both sides the square of half the coefficient of* x.

Ex 1 Solve the equation $8x^2+3=14x$

Transpose $14x$, thus $\qquad 8x^2-14x+3=0$,

transpose 3, thus $\qquad 8x^2-14x=-3$,

divide by 8, thus $\qquad x^2-\frac{7}{4}x=-\frac{3}{8}$,

add $(\frac{1}{2}$ of $\frac{7}{4})^2$ or $(\frac{7}{8})^2$, thus $\qquad x^2-\frac{7}{4}x+(\frac{7}{8})^2=\frac{49}{64}-\frac{3}{8}$,

or $\qquad (x-\frac{7}{8})^2=\frac{25}{64}$,

whence $\qquad x-\frac{7}{8}=\pm\frac{5}{8}$,

$$x=\frac{7}{8}\pm\frac{5}{8}=\frac{3}{2} \text{ or } \frac{1}{4}$$

Thus $\qquad x=\frac{3}{2}$, or $x=\frac{1}{4}$.

Ex 2 Solve the equation $ax^2+bx+c=0$

Transpose c, thus $\qquad ax^2+bx=-c$,

divide by a, thus $\qquad x^2+\dfrac{b}{a}x=-\dfrac{c}{a}$,

add $\left(\dfrac{1}{2}\text{ of }\dfrac{b}{a}\right)^2$, i e, $\left(\dfrac{b}{2a}\right)^2$ to both sides, thus

$$x^2+\frac{b}{a}x+\left(\frac{b}{2a}\right)^2=\left(\frac{b}{2a}\right)^2-\frac{c}{a}$$

or

$$\left(x+\frac{b}{2a}\right)^2=\frac{b^2-4ac}{4a^2},$$

extract the square root, thus

$$x+\frac{b}{2a}=\pm\frac{\sqrt{b^2-4ac}}{2a},$$

$$x=-\frac{b}{2a}\pm\frac{\sqrt{b^2-4ac}}{2a}=\frac{-b\pm\sqrt{b^2-4ac}}{2a}.$$

Thus the two roots are $\dfrac{-b+\sqrt{b^2-4ac}}{2a}$ and $\dfrac{-b-\sqrt{b^2-4ac}}{2a}$

Hindu Method

RULE—*Reduce the quadratic to the general form, transpose the constant term, multiply by four times the coefficient of* x^2, *and add to both sides the square of the coefficient of* x.

Explanation The *coefficient* of x^2 may be *unity* or some other number

Ex 3 Solve the equation $x^2+7x+12=0$

Transpose, thus $\qquad x^2+7x=-12,$

multiply by 4×1 (see *Explanation*) or 4, thus

$$4x^2+28x=-48,$$

add 7^2, thus $\qquad 4x^2+28x+7^2=7^2-48,$

or $\qquad (2x+7)^2=49-48=1,$

extract the square root, thus $\quad 2x+7=\pm\sqrt{1}=\pm 1,$

transpose, thus $\qquad 2x=-7\pm 1=-6$ or $-8,$

divide by 2, thus $\qquad x=-3,$ or $x=-4$

Ex 4 Solve the equation $ax^2+bx+c=0$

Transpose c, thus $\qquad ax^2+bx=-c,$

multiply by $4a$, thus $\qquad 4a^2x^2+4abx=-4ac,$

add b^2 to both sides, thus

$$4a^2x^2+4abx+b^2=b^2-4ac, \text{ or } (2ax+b)^2=b^2-4ac,$$

extract the square root, thus

$$2ax+b=\pm\sqrt{b^2-4ac},$$

transpose b, and divide by $2a$,

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}.$$

Examples CLXII

Solve the equations by both Factorizing and Completing the Square

1 $x^2+11x+24=0$	2 $x^2-50x+429=0$
3 $x^2+11x-80=0$	4. $x^2+105x+2000=0$
5 $x^2+\frac{15}{2}x=81$	6 $x^2-\frac{35}{6}x+6=0$
7 $3x^2+10x=88$	8 $6x^2-11x=10$
9 $10x^2-31x=-15$	10 $24x^2-55x=-14$
11 $3x^2-26x=169$	12 $42x^2-41x=20$
13 $\frac{2}{3}x^2+\frac{3}{4}x=105$	14 $\frac{6}{5}x^2-\frac{31}{10}x=-2$

Ex 5 Solve the equation $\dfrac{3x-7}{x}=3\tfrac{1}{2}-\dfrac{4(x-2\frac{1}{2})}{x+5}$

Reduce the given equation to the standard form by multiplying by $2x(x+5)$, the L C M of denominators thus

$$2(x+5)(3x-7)=7x(x+5)-8x(x-2\tfrac{1}{2}),$$

or $$2(3x^2+8x-35)=7x^2+35x-(8x^2-20x),$$

whence $$7x^2-39x-70=0$$

The equation is now reduced to the required form and can be solved as before The solutions are $x=7$, or $x=-1\frac{3}{7}$

Ex 6 Solve the equation $x^2-6x+7=0$

Transpose, thus $$x^2-6x=-7,$$

add 3^2, thus $$x^2-6x+3^2=3^2-7, \quad \text{or } (x-3)^2=2,$$

whence $$x-3=\pm\sqrt{2},$$
$$x=3\pm\sqrt{2}$$

REMARK Here the two roots are *irrational* but *real* as $\sqrt{2}$ can be found though approximately

Ex 7 Solve the equation $9x^2-12x+8=0$

Transpose, thus $$9x^2-12x=-8,$$

divide by 9, thus $$x^2-\tfrac{4}{3}x=-\tfrac{8}{9},$$

add $(\frac{1}{2}\times\frac{4}{3})^2$, thus $$x^2-\tfrac{4}{3}x+(\tfrac{2}{3})^2=(\tfrac{2}{3})^2-\tfrac{8}{9},$$

or $$(x-\tfrac{2}{3})^2=\tfrac{4}{9}-\tfrac{8}{9}=\tfrac{-4}{9},$$

extract the square root, thus

$$x-\frac{2}{3}=\pm\sqrt{\frac{-4}{9}}=\pm\frac{\sqrt{-4}}{3},$$

$$x=\frac{2}{3}\pm\frac{\sqrt{-4}}{3}=\frac{2\pm\sqrt{-4}}{3}$$

REMARK Here the two roots are *not real* but *imaginary*, as -4 has no square root.

Examples CLXII (*Continued*)

Solve the equations

15 $2x^2+3x+5=3x^2+4x-1$ 16. $5x(x+1)-2=x(x-2)$

17 $16x^2+3p^2=16px$ 18 $12x^2+5ax=3a^2$

19 $5x^2+7ax=20a^2-x^2$ 20 $12(x^2+ax-2a^2)=11a(a+x)$

21 $\dfrac{x^2}{3}+\dfrac{5x}{2}=27.$ 22 $\dfrac{1}{x}-7x=6$ 23. $\dfrac{3}{x}+2x=7.$

Solve the equations

24 $15x+\frac{96}{x}=76$ 25 $x+\frac{28}{x}+4=2x-7$ 26 $1-\frac{3}{x}=\frac{2}{x}+\frac{6}{x^2}$.

27 $\frac{x}{4}+\frac{3}{x}=2$ 28 $\frac{x}{5}+\frac{5}{x}=\frac{3}{5}+\frac{5}{3}$ 29 $x-\frac{1}{x}=\frac{3}{2}$ 30 $x+\frac{1}{x}=4$

31 $x^2-2ax+a^2-b^2=0$ 32 $ax^2+2bx+c=0$

33 $(a^2-b^2)(x^2+1)=2(a^2+b^2)x$ 34 $(a+b+x)^2=2bx+a^2+2b^2$

35 $(x-7)(x-16)=0$ 36 $(2x+3)(3x-4)=0$

37 $(a-x)(2m-x)=(a-x)(x-2n)$ 38 $\frac{x}{a}\left(1+\frac{x}{b}\right)+1+\frac{x}{b}=0$

39 $(x-1)(x-2)=12$ 40 $(x-8)(x-10)=5\times 3$

41 $(x-a)(x-2a)=12a^2$ 42 $(x-1)(x-2)=2(x-3)(x-4)$

43 $(x-1)(x+3)=(2x-5)(3x-5)$ 44. $x^2-(a-b)x=(c-a)(c-b)$

45 $\frac{x+11}{x}=7-\frac{9+4x}{x^2}$ 46 $\frac{10}{x}-\frac{14-2x}{x^2}=\frac{22}{9}$

47 $\frac{x}{6}+\frac{6}{x}=\frac{5(x-1)}{4}$. 48 $\frac{2x-1}{x-3}=\frac{5x+2}{x+1}$

49 $\frac{x-7}{2(x+3)}=\frac{x-6}{x+24}$. 50 $\frac{x+1}{x-1}=\frac{4x-3}{x+9}$.

Ex 8 Solve the equation $\frac{2x+3}{x}+\frac{4}{2x+3}=4\frac{1}{3}$.

Here the second term on the left side is 4 times the *reciprocal* of the first term

Put $$y=\frac{2x+3}{x} \quad \ldots \quad (\text{i}),$$

thus we have $$y+\frac{4}{y}=4\tfrac{1}{3},$$

whence $3y^2-13y+12=0$, or $(y-3)(3y-4)=0$,

thus $$y-3=0, \text{ i e}, y=3 \quad \ldots \quad (\text{ii}),$$

or $$3y-4=0, \text{ i e}, y=\tfrac{4}{3} \quad \ldots; \quad (\text{iii})$$

From (i) and (ii), $\frac{2x+3}{x}=3$, whence $x=3$,

and from (i) and (iii), $\frac{2x+3}{x}=\frac{4}{3}$, whence $x=-\frac{9}{2}=-4\frac{1}{2}$.

Ex. 9 Solve the equation $\frac{x+5}{x-5}+\frac{x-5}{x+5}=\frac{37}{6}$.

Put $$y=\frac{x+5}{x-5} \quad \ldots \ldots \ldots (i),$$

thus $$y+\frac{1}{y}=\frac{37}{6}, \text{ or } 6y^2-37y+6=0,$$

whence $$(y-6)(6y-1)=0$$

Thus $$y-6=0, \text{ i e}, y=6 \quad \ldots \ldots (ii),$$

or $$6y-1=0, \text{ i e}, y=\tfrac{1}{6} \quad \ldots \ldots (iii)$$

From (i) and (ii), $\frac{x+5}{x-5}=6$, or from (i) and (iii), $\frac{x+5}{x-5}=\frac{1}{6}$

Hence compo and divido, $x=7$, or $x=-7$

[For *another* method, see Ex 10 below]

Ex. 10. Solve the equation $\frac{x-2}{x+2}+\frac{x+2}{x-2}=\frac{2(x+3)}{x-3}$

By actual division of numerator by denominator, we have

$$\left(1-\frac{4}{x+2}\right)+\left(1+\frac{4}{x-2}\right)=2\left(1+\frac{6}{x-3}\right),$$

whence cancelling and dividing by 4, we get

$$-\frac{1}{x+2}+\frac{1}{x-2}=\frac{3}{x-3},$$

or $$-(x-2)(x-3)+(x+2)(x-3)=3(x^2-4),$$

or after reduction, $3x^2-4x=0$, or $x(3x-4)=0$,

hence $x=0$, or $3x-4=0$, i e, $x=\frac{4}{3}$

Examples CLXII (*Continued*)

Solve the equations

51 $2x-5+\frac{1}{2x-5}=3\frac{1}{3}$

52 $\frac{x}{x+1}+\frac{x+1}{x}=2\frac{1}{30}$

53 $\frac{x-2}{x}+\frac{x}{x-2}=\frac{5}{2}$

54 $\frac{x-6}{x-12}+\frac{x-12}{x-6}=\frac{13}{6}$.

55 $\frac{x+2}{x-2}-\frac{x-2}{x+2}=\frac{5}{6}$.

56 $\frac{2x-3}{3x-5}+\frac{3x-5}{2x-3}=\frac{5}{2}$

57 $\frac{3x-2}{2x-5}-\frac{2x-5}{3x-2}=\frac{8}{3}$

58 $\frac{1}{x-2}+\frac{2}{x-1}=\frac{6}{x}$

Solve the equations

59 $\frac{x+2}{x-1}-\frac{4-x}{2x}=2\frac{1}{3}$

60 $\frac{1}{x}+\frac{3}{x+1}=\frac{5}{x+2}$

61 $\frac{2}{x+1}+\frac{3}{x+2}=\frac{8}{x+3}$

62 $\frac{6}{x+2}+\frac{7}{x+3}=\frac{16}{x+4}$

63 $\frac{2}{x-3}+\frac{1}{x-4}=\frac{7}{6}$

64 $\frac{4}{x-1}=\frac{1}{18}+\frac{3}{x+7}$

65 $\frac{x+1}{x-1}+\frac{x+2}{x-2}=7$

66 $\frac{x+1}{x+2}+\frac{x+2}{x+3}=1\frac{5}{12}$

67 $\frac{2x+3}{3x+2}-\frac{2x-3}{3x-2}=1$

68 $\frac{x+2}{x-2}-\frac{2x-3}{2(x-1)}=\frac{23}{6}$

69 $\frac{2x-7}{2x-8}+\frac{5}{2}=\frac{3x+1}{3(x-5)}$

70 $\frac{2x+7}{3(x-3)}-3=\frac{x+6}{2x-3}$

71 $\frac{x+3}{x+2}+\frac{x-3}{x-2}=\frac{2x-3}{x-1}$

72 $\frac{x+1}{x-1}+\frac{x+2}{x-2}=\frac{2(x+3)}{x-3}$

73 $\frac{4x^2(2x-3)+3}{2(x-1)-x}=4x(x-1)+15$

74 $(a+x)(x-b)=2(x-b)^2+ab$

75 $(a-b)x^2-(a+b)x+2b=0$

76 $ax\left(\frac{ax}{b^2}-\frac{1}{c}\right)+\frac{1}{c}\left(\frac{b^2}{c}-ax\right)=0$

77 $\frac{5}{2x-a}+\frac{1}{2x-5a}=\frac{2}{a}$

78 $\frac{a^2(x-b)}{a-b}+\frac{b^2(x-a)}{b-a}=x^2$

79 $\frac{b}{x-a}+\frac{a}{x-b}=2$

80 $x+\frac{1}{x}=a+\frac{1}{a}$

81 $(b+c)x^2+ax-a-b-c=0$

82 $(b-c)x^2+(c-a)x+(a-b)=0$

83 $a(b-c)x^2+b(c-a)x+c(a-b)=0$

84 $(a+b)x^2+(a-b)x=\frac{ab}{a+b}$

85 $\frac{1+a}{1-ax}+\frac{1-a}{1+ax}=1$

86 $\frac{1}{a}+\frac{b}{x+ab}+\frac{b}{2x+ab}=0$

87 $\left(\frac{a}{b}-\frac{b}{a}\right)\left(\frac{1}{x-1}-\frac{1}{x+1}\right)=\frac{2}{x}$.

88 $\frac{1}{a}+\frac{1}{b}+\frac{1}{x}=\frac{1}{a+b+x}$

89 $\frac{x+2}{x-2}+\frac{x-2}{x+2}=\frac{6x+16}{3x}$

90 $\frac{2x-1}{x+1}-\frac{x-7}{x-1}=4-\frac{3x-1}{x+2}$

91 $\frac{2x}{x-1}+\frac{3x-1}{x+2}-\frac{5x-11}{x-2}=0$

92. $\frac{7x-11}{4x-7}+\frac{3x-2}{12x-1}=\frac{2x+5}{x+2}$

93 $(c+a-2b)x^2+(a+b-2c)x+(b+c-2a)=0$

94 $(x+a)(x+b)(x+c)=abc$

95 $x^3=(x-a)(x-b)(x-c)$

96 $\frac{1}{x-a}+\frac{1}{x-b}+\frac{1}{x-c}=0$

97 $\frac{x+a}{x-a}+\frac{x+b}{x-b}+\frac{x+c}{x-c}=3$

Solve the equations

98 $\left(\frac{2a}{x}-\frac{x}{a}-1\right)\left(1-\frac{a}{x}+\frac{2x}{a}\right)=0$ 99 $\frac{a}{x-a}+\frac{b}{x-b}=\frac{a}{b}+\frac{b}{a}$.

100. $\frac{(x+a)(x+a+b)}{(x+c)(x+c+b)}=\frac{(x-a)(x-a-b)}{(x-c)(x-c-b)}$ 101 $\frac{a+b}{x+b}+\frac{a+c}{x+c}=\frac{2(a+b+c)}{x+b+c}$

102 $\frac{ax^2-b}{ax+b}+\frac{a+bx^2}{a-bx}=\frac{2(a^2+b^2)}{a^2-b^2}$. 103 $x=\cfrac{3}{4-\cfrac{3}{4-\cfrac{3}{4-x}}}$.

104 $\sqrt{16-x}+\sqrt{2x-5}=6$ 105. $\sqrt{x+3}+\sqrt{3x-3}=10$

106 $\sqrt{3x+1}-\sqrt{4x+5}+\sqrt{x-4}=0$

107 $\sqrt{2x+7}+\sqrt{3x-18}=\sqrt{7x+1}$

108 $\sqrt{3x-3}+\sqrt{5x-19}=\sqrt{3x+4}$

109 $\sqrt{a-x}+\sqrt{b-x}=\sqrt{a+b-2x}$

110 $\sqrt{a+x}-\sqrt{2a-x}=\sqrt{4a+x}$

303 Method III—Solution by Formula Since the equation $ax^2+bx+c=0$ is the most general form of a quadratic equation, its solution [Art. 302, Ex 2], viz,

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

is used as a **Formula** to find the roots of any particular quadratic.

Ex. 1 Solve $x^2+7x+12=0$ [Art 302, Ex. 3]

Here $a=1$, $b=7$ and $c=12$, therefore

$$x=\frac{-7\pm\sqrt{49-4\cdot1\cdot12}}{2\cdot1}=\frac{-7\pm\sqrt{1}}{2}=\frac{-7\pm1}{2}=-3 \text{ or} -4$$

Ex 2 Solve $3x^2+10x=88$

By transposition, we have $3x^2+10x-88=0$, which is of the form $ax^2+bx+c=0$

Hence here $a=3$, $b=10$, $c=-88$, therefore

$$x=\frac{-10\pm\sqrt{(10)^2-4\cdot3(-88)}}{2\cdot3}=\frac{-10\pm\sqrt{100+1056}}{6}$$

$$=\frac{-10\pm\sqrt{1156}}{6}=\frac{-10\pm34}{6}=\frac{24}{6} \text{ or } -\frac{44}{6}=4 \text{ or} -\frac{22}{3}$$

Ex 3 Solve $\frac{7-x}{2x+1}+\frac{3x+2}{2x-1}-3=\frac{7x-1}{4x^2-1}+2$

Reduce this equation to the general form

Multiply by $4x^2-1$, thus

$$(7-x)(2x-1)+(3x+2)(2x+1)-3(4x^2-1)=7x-1+2(4x^2-1),$$

clear and transpose all the terms to one side, thus $16x^2-15x-1=0$ Now proceed as before

304 Theorem *The quadratic expression ax^2+bx+c has two linear factors* and no more*

Let $x-\alpha$ be a factor of the given expression, thus it is divisible by $x-\alpha$, therefore

$$ax^2+bx+c\equiv Q(x-\alpha),$$

where Q is of $(2-1)^{th}$ degree in x, i e, *linear* Now if $x-\beta$ be another factor of the given expression, then Q is divisible by $x-\beta$, therefore

$$Q\equiv q(x-\beta),$$

thus

$$ax^2+ax+c\equiv q(x-\alpha)(x-\beta),$$

where q is of $(2-2)^{th}$ degree or *zeroth* degree in x, that is, q is a *constant* Hence if the proposed expression has any other factor, that factor must be independent of x, and by comparing the co efficients of x^2 [Art 287], we find $q=a$ Thus

$$ax^2+bx+c\equiv a(x-\alpha)(x-\beta)$$

305 Theorem *A quadratic equation has only* TWO ROOTS *and no more*

First Proof Let the quadratic equation be $ax^2+bx+c=0$

Then since $ax^2+bx+c\equiv a(x-\alpha)(x-\beta)$ [Art 304], we have

$$a(x-\alpha)(x-\beta)=0$$

Now by supposition $\quad a$ is not $=0$,

$$(x-\alpha)(x-\beta)=0,$$

i e, either $\quad x-\alpha=0$, or $x-\beta=0$,

$$x=\alpha, \text{ or } x=\beta$$

Second Proof If possible let the quadratic equation $ax^2+bx+c=0$ have three *different* roots α, β and γ

Since each of these roots satisfies the equation, we have

$$a\alpha^2+b\alpha+c=0 \qquad (1),$$
$$a\beta^2+b\beta+c=0 \qquad (2),$$
$$a\gamma^2+b\gamma+c=0 \qquad (3).$$

* By *factor* is here meant a *factor in x*

Subtract (2) from (1), thus

$$a(\alpha^2-\beta^2)+b(\alpha-\beta)=0,$$

or dividing by $\alpha-\beta$, which is by supposition not 0,

$$a(\alpha+\beta)+b=0 \qquad (4).$$

Similarly from (2) and (3), we get

$$a(\beta+\gamma)+b=0 \qquad (5).$$

Subtract (5) from (4), thus

$$a(\alpha-\gamma)=0 \qquad (6)$$

Hence either $a=0$, or $\alpha-\gamma=0$ but this is impossible since by supposition neither $a=0$, nor $\alpha-\gamma=0$, α and γ being two different quantities

Thus the quadratic cannot have more than two roots

306 If three distinct values α, β and γ be found to satisfy the quadratic $ax^2+bx+c=0$, it is no longer an equation but an identity

For then from (6) [Art 305], we have $a=0$, and therefore from (4), $b=0$ and from (1), $c=0$ Thus the coefficients being all 0, the quadratic is clearly satisfied by *all values of* x

Hence *when a quadratic is satisfied by* **more than two values** *of the variable, it is not an equation but an* **identity**

Ex Prove the identity

$$\frac{(x-b)(x-c)}{(a-b)(a-c)}+\frac{(x-c)(x-a)}{(b-c)(b-a)}+\frac{(x-a)(x-b)}{(c-a)(c-b)}=1$$

We see that the quadratic is satisfied by $x=a$, $x=b$ and $x=c$, that is, by more than two values of x, it is therefore an identity, and not an equation

307 Relations between the roots and coefficients. Theorem. In the quadratic equation $ax^2+bx+c=0$,

(i) **the sum of the roots** $=-\dfrac{b}{a}$,

(ii) **the product of the roots** $=\dfrac{c}{a}$

Let α and β be the roots of the equation, thus

$$\alpha=\frac{-b+\sqrt{b^2-4ac}}{2a} \text{ and } \beta=\frac{-b-\sqrt{b^2-4ac}}{2a} \quad \text{[Art 302, Ex. 2].}$$

Now $\alpha+\beta=\dfrac{-2b}{2a}=-\dfrac{b}{a}$ (i)

and $\alpha\beta=\dfrac{(-b)^2-(\sqrt{b^2-4ac})^2}{4a^2}=\dfrac{4ac}{4a^2}=\dfrac{c}{a}$ (ii)

By dividing by a, we can put the equation in the form

$$x^2+\frac{b}{a}x+\frac{c}{a}=0$$

Thus the above relations can be expressed in words thus —*If in a quadratic equation the coefficient of* x^2 *is unity, the sum of the roots is equal to the coefficient of* x *with its sign changed and the product of the roots is equal to the constant term*

The relations (i) and (ii) given above should be carefully remembered. One of their uses may be seen from the following examples

Ex 1 Solve $(a-b)x^2-(a+b)x+2b=0$ [Ex. 75, p 446]

Evidently $x=1$ satisfies the given equation, therefore 1 is a root of the equation The product of the roots is $\dfrac{2b}{a-b}$, therefore the other root is $\dfrac{2b}{a-b}$

Ex 2 Solve $(a+b)x^2+(a-b)x=\dfrac{ab}{a+b}$ [Ex 84, p 446]

Here the sum of the roots is $-\dfrac{a-b}{a+b}$, which are therefore $\dfrac{-a}{a+b}$ and $\dfrac{b}{a+b}$, for their product is $-\dfrac{ab}{(a+b)^2}$, as it should be

Ex 3 Solve $\dfrac{ax^2-b}{ax+b}+\dfrac{a+bx^2}{a-bx}=\dfrac{2(a^2+b^2)}{a^2-b^2}$ [Ex. 102, p 447]

Here $x=1$ obviously satisfies the given equation, therefore 1 is a root. Again after reduction to the integral form the equation becomes

$$(a^2+2ab-b^2)x^2-(a^2-b^2)x-2ab=0$$

Hence the product of the roots is $\dfrac{-2ab}{a^2+2ab-b^2}$, and as one of them is 1, the other must evidently be $\dfrac{-2ab}{a^2+2ab-b^2}$

[*Solve equations Nos* 81, 82, 83, 93, 99 *and* 101 *of* CLXII]

308 Theorem *If* α *and* β *be the roots of the quadratic equation* $ax^2+bx+c=0$, *then the quadratic* expression

$$ax^2+bx+c\equiv a(x-\alpha)(x-\beta)$$

For $ax^2+bx+c=a\left(x^2+\frac{b}{a}x+\frac{c}{a}\right)$

$$=a\{x^2-(\alpha+\beta)x+\alpha\beta\} \text{ [Art. 307]}$$
$$=a(x-\alpha)(x-\beta)$$

Otherwise —By the Remainder Theorem [Art 141], the remainder is ah^2+bh+c, when $(ax^2+bx+c)\div(x-h)$

Now if $x-h$ is a factor of ax^2+bx+c, the remainder vanishes,

$$\text{i e}, \ ah^2+bh+c=0,$$

which shews that h is a root of the equation $ax^2+bx+c=0$

Similarly $x-k$ is a factor of ax^2+bx+c, when k is a root of the equation $ax^2+bx+c=0$

Thus $ax^2+bx+c\equiv\lambda(x-h)(x-k)$, where λ cannot contain x, since each side of the identity is of the second degree in x Hence by comparing the coefficients of x^2, we have $\lambda=a$

$$ax^2+bx+c=a(x-h)(x-k)$$

This Theorem enables us to **resolve any quadratic expression by solving the corresponding quadratic equation**

Ex. Resolve $15x^2+26x-288$ into factors

Now $15x^2+26x-288=15(x-\alpha)(x-\beta)$, where α and β are the roots of $15x^2+26x-288=0$.

Solving this equation, we get $x=\frac{18}{5}$, or $x=-\frac{16}{3}$

Hence $15x^2+26x-288=15(x-\frac{18}{5})(x+\frac{16}{3})$

$$=(5x-18)(3x+16)$$

309 *To form the quadratic equation whose roots are given*

Let $f(x)=0$ be the quadratic equation whose roots are α and β

Then the quadratic expression $f(x)=(x-\alpha)(x-\beta)$ [Art 308]

Hence the required equation is

$$(x-\alpha)(x-\beta)=0, \text{ or } x^2-(\alpha+\beta)x+\alpha\beta=0$$

Ex 1 Form the equations whose roots are (i) 4 and 8, (ii) 3 and -7, (iii) -1 and $-\frac{2}{3}$

The required equations are (i) $(x-4)(x-8)=0$, or $x^2-12x+32=0$, (ii) $(x-3)(x+7)=0$, or $x^2+4x-21=0$, (iii) $(x+1)(x+\frac{2}{3})=0$, or $3x^2+5x+2=0$

Ex 2 Form the quadratic whose roots are 2α and 2β, where α and β are the roots of the equation $ax^2+bx+c=0$

The required equation is

$$(x-2\alpha)(x-2\beta)=0, \text{ or } x^2-2(\alpha+\beta)x+4\alpha\beta=0$$

But by Art. 307, $\quad \alpha+\beta=-\frac{b}{a}$ and $\alpha\beta=\frac{c}{a}$,

$\therefore$ required equation is $x^2+\frac{2b}{a}x+\frac{4c}{a}=0$, or $ax^2+2bx+4c=0$.

Examples CLXIII

If α and β are the roots of the equation $ax^2+bx+c=0$, find in terms of a, b and c, the value of

1 $\alpha^2\beta+\alpha\beta^2$ 2 $\alpha^3+\beta^3$ 3 $(\alpha+2\beta)(2\alpha+\beta)$ 4 $\alpha-\beta$

Form the equation whose roots are

5 2 and 3 6 4 and -5 7 3 and $-\frac{1}{3}$ 8 $-\frac{1}{2}$ and $-\frac{3}{4}$

9 If α and β are the roots of $x^2-px+q=0$, construct the equation whose roots are $\alpha+\beta$ and $\alpha\beta$

10 If α and β are the roots of $ax^2-bx+c=0$, form the equation whose roots are (i) $\frac{\alpha}{\beta}$ and $\frac{\beta}{\alpha}$, and (ii) $2\alpha-\beta$ and $2\beta-\alpha$

Equations reducible to Quadratics

310 Equations of the form $ax^{2n}+bx^n+c=0$ When an equation after suitable *substitutions* is seen to contain *only two* powers of the variable such that *one is double* of the other, it can be solved as a quadratic

Thus by substituting y for x^n in the given equation, we get $ay^2+by+c=0$, which is a quadratic in y and can be solved as usual

Ex 1 Solve $8x^6-35x^3+27=0$

Assume $y=x^3$, thus

$$8y^2-35y+27=0, \text{ or } (y-1)(8y-27)=0,$$

whence $y=1$ or $\frac{27}{8}$, i e, $x^3=1$ or $\frac{27}{8}$, $\quad \therefore x=1$, or $\frac{3}{2}$

Ex 2 Solve $36x^{-4}-13x^{-2}+1=0$

Substitute y for x^{-2}, thus $36y^2-13y+1=0$,

whence $\quad (4y-1)(9y-1)=0$, or $y=\frac{1}{4}$ or $\frac{1}{9}$

Hence $\quad x^{-2}=\frac{1}{4}$ or $\frac{1}{9}$, i e, $x^2=4$ or 9, $\quad \therefore x=\pm 2$ or ± 3

Ex 3 Solve $3x-5\sqrt{x}=2$

Put $y=\sqrt{x}$, thus $3y^2-5y=2$, whence $y=2$ or $-\frac{1}{3}$

Thus $\sqrt{x}=2$ or $-\frac{1}{3}$, that is, $x=4$ or $\frac{1}{9}$

Ex 4 Solve $x^2-3x+5\sqrt{x^2-3x+6}=30$

Add 6 to both sides, thus

$$x^2-3x+6+5\sqrt{x^2-3x+6}=36$$

Assume $y=\sqrt{x^2-3x+6}$, and transpose, thus

$$y^2+5y-36=0, \text{ or } (y-4)(y+9)=0, \text{ i e}, y=4 \text{ or } -9$$

Hence $\sqrt{x^2-3x+6}=4$ or -9

Square, thus $x^2-3x+6=16$ or 81

Thus we have the two quadratics

$$x^2-3x-10=0, \text{ and } x^2-3x-75=0$$

From the first $x=5$ or -2, and from the second $x=\frac{1}{2}(3\pm\sqrt{309})$

Ex 5 Solve $(3x^2-2x-10)^2-9(3x^2-2x-10)=22$

Put $y=3x^2-2x-10$, thus we have $y^2-9y=22$

Solving this we get $y=11$ or -2 Thus we have the two equations

$$3x^2-2x-21=0, \text{ and } 3x^2-2x-8=0$$

The first gives $x=3$ or $-\frac{7}{3}$, and the second gives $x=2$ or $-\frac{4}{3}$

Ex 6 Solve $(x+1)(x+2)(x+3)(x+4)=360$

We have $(x+1)(x+4)\times(x+2)(x+3)=360$,

or $(x^2+5x+4)(x^2+5x+6)=360$,

put $y=x^2+5x$, thus $(y+4)(y+6)=360$,

multiply out and transpose, thus $y^2+10y-336=0$,

whence $y=14$ or -24

Thus we have the two quadratics

$$x^2+5x-14=0, \text{ and } x^2+5x+24=0,$$

the first of which gives $x=2$ or -7, and the second gives

$$x=\tfrac{1}{2}(-5\pm\sqrt{-71})$$

Ex 7 Solve $x^2+\frac{1}{x^2}+3\left(x+\frac{1}{x}\right)=8$

Add 2 to both sides, thus $\left(x^2+2+\frac{1}{x^2}\right)+3\left(x+\frac{1}{x}\right)=10$,

or $\left(x+\frac{1}{x}\right)^2+3\left(x+\frac{1}{x}\right)=10$, i e, $y^2+3y=10$, if $y=x+\frac{1}{x}$

From $y^2+3y=10$, we have $y=2$ or -5

Thus $x+\frac{1}{x}=2$ or $x+\frac{1}{x}=-5$, replacing y by $x+\frac{1}{x}$

From the first $x^2-2x+1=0$, or $(x-1)^2=0$, i.e., $x=1$, and from the second $x^2+5x+1=0$, or $x=\frac{1}{2}(-5\pm\sqrt{21})$.

Examples CLXIV

Solve the equations

1. $x^4+4x^2=5$
2. $3x^4-7x^2+2=0$
3. $x^6+4x^3=96$
4. $9x^{-4}+4x^{-2}=5$
5. $6x^{-2}-5x^{-1}+1=0$
6. $x^2+\dfrac{1}{x^2}=4\frac{1}{4}$
7. $x-4\sqrt{x}=5$
8. $2x+3\sqrt{x}=2$
9. $\dfrac{3}{\sqrt{x}}+2\sqrt{x}=7$
10. $7\sqrt{x}=\dfrac{1}{\sqrt{x}}-6$
11. $7x^{\frac{1}{3}}-20x^{\frac{1}{6}}=3$
12. $\sqrt[3]{x^2}-13\sqrt[3]{x}+36=0$
13. $2\sqrt{\dfrac{2}{x}}+3\sqrt{\dfrac{x}{2}}=7$
14. $4\sqrt{\dfrac{x}{3}}-3\sqrt{\dfrac{3}{x}}=11$
15. $x^2-6x-2\sqrt{x^2-6x+2}=1$
16. $2x^2-3x+6\sqrt{2x^2-3x+2}=14$.
17. $x^2=8x+6\sqrt{x^2-8x+9}$
18. $ax+2\sqrt{x^2-ax+a^2}=x^2+2a$
19. $2\sqrt{x^2-8x+8}=(x-4)^2-7$
20. $(x+1)^2=x+3\sqrt{3x^2+3x-11}$.
21. $x^2-3x+3-\sqrt{3x^2-7x+6}=\dfrac{x-x^2}{2}$
22. $\sqrt{x^2-6x+17}+2x^2=12x-13$
23. $x^2-x+5\sqrt{2x^2-5x+6}=\frac{3}{2}(x+11)$
24. $(x+5)(x-2)+3\sqrt{x(x+3)}=0$
25. $(x-7)(x+3)-\sqrt{(x+4)(x-8)}=67$.
26. $2(x^2-3x+6)^2-9(x^2-3x+6)+4=0$
27. $x(x-1)(x^2-x-1)=30$
28. $x(x-1)(x-2)(x-3)=120$
29. $(x+a)(x+2a)(x+3a)(x+4a)=24a^4$
30. $\dfrac{9}{1+x+x^2}=5-x-x^2$
31. $\dfrac{x(x-1)}{(x+1)(x+2)}=\dfrac{(x+3)(x+4)}{(x+5)(x+6)}$
32. $x^2+\dfrac{1}{x^2}-2\left(x+\dfrac{1}{x}\right)=\dfrac{22}{9}$
33. $4\left(x^2+\dfrac{1}{x^2}\right)-12\left(x-\dfrac{1}{x}\right)+1=0$

311 Reciprocal Equations *A reciprocal equation is one which remains unaltered, when the variable* x *is changed into its reciprocal* $\dfrac{1}{x}$ Thus $ax^2+bx+a=0$ is a reciprocal equation, for when

x is changed into $\frac{1}{x}$, it becomes $a\left(\frac{1}{x}\right)^2+b\left(\frac{1}{x}\right)+a=0$, which when reduced to an integral form by multiplying by x^2, is the *same* as the original equation. Similarly it will be seen that

$$ax^3+bx^2+bx+a=0,$$

$$ax^4+bx^3+cx^2+bx+a=0, \text{ \&c}$$

are other examples of reciprocal equations.

It will be seen that these equations do not change in form when x is changed into $\frac{1}{x}$ owing to their peculiar form of having the coefficients of terms equidistant from the beginning and the end equal. Hence we may define a reciprocal equation otherwise thus:— *A reciprocal equation is one in which the coefficients of terms equidistant from the beginning and the end are equal.*

Ex 1. Solve $2x^3+7x^2+7x+2=0$.

Bracket the terms having the *same* coefficients; thus

$$2(x^3+1)+7x(x+1)=0,$$

or $\quad (x+1)\{2(x^2-x+1)+7x\}=0,$

i.e., $\quad (x+1)(2x^2+5x+2)=0,$

whence $\quad x+1=0, \quad \ldots \text{(i)},$

or $\quad 2x^2+5x+2=0 \quad \ldots \text{(ii)}.$

From (i), $x=-1$; and from (ii), $x=\dfrac{-5\pm\sqrt{25-16}}{4}=-\frac{1}{2}$ or -2.

Ex 2. Solve $ax^4-bx^3+cx^2-bx+a=0$.

Divide by x^2; thus $ax^2-bx+c-\dfrac{b}{x}+\dfrac{a}{x^2}=0$.

Bracket the terms having the *same* coefficients; thus

$$a\left(x^2+\frac{1}{x^2}\right)-b\left(x+\frac{1}{x}\right)+c=0.$$

Put $y=x+\dfrac{1}{x}$; thus $x^2+\dfrac{1}{x^2}=y^2-2$; therefore

$$a(y^2-2)-by+c=0, \text{ or } ay^2-by+(c-2a)=0.$$

This is a quadratic in y and may be solved. Let $y=m$ or n where m and n are functions of a, b and c; thus $x+\dfrac{1}{x}=m$ or n.

Hence we have the two quadratics

$$x^2-mx+1=0 \text{ and } x^2-nx+1=0,$$

which can be solved in x.

Examples CLXV

Solve the equations

1. $x^3+x^2+x+1=0$
2. $2x^3+3x^2+3x+2=0$
3. $3x^3+13x^2-13x-3=0$
4. $ax^3-bx^2+bx=a$
5. $x^4+x^3-4x^2+x+1=0$.
6. $x^4+2x^3+x^2+2x+1=0$
7. $2x^4+3x^3+4x^2+3x+2=0$
8. $ax^4+bx^3+cx^2+bx+a=0$

312 Miscellaneous Equations We have seen [Arts 310 and 311] how equations of higher degree having peculiar forms can be solved like quadratics Other equations of higher degree can also be solved like quadratics if one or more of its roots can be found by inspection Hence, if one root of a cubic can thus be found, we can reduce it by the Factor Theorem of Art 291 to a quadratic and thus find all its roots [see Art 156]

Ex 1. Solve $(x-1)(x-2)(x-3)=2\,3\,4$

We have $(x-1)(x-2)(x-3)=4\,3\,2=(5-1)(5-2)(5-3)$

Thus 5 is a root of the proposed equation Hence $x-5$ is a factor of $(x-1)(x-2)(x-3)-2\,3\,4$, or of $x^3-6x^2+11x-30$

Now
$$x^3-6x^2+11x-30=x^2(x-5)-x(x-5)+6(x-5)$$
$$=(x-5)(x^2-x+6)$$

Hence
$$(x-5)(x^2-x+6)=0$$

Thus one root is 5, and the others are given by $x^2-x+6=0$

Ex 2 Solve $x^4-2x^3+2x^2+2x=3$

Transpose, thus
$$x^4-2x^3+2x^2+2x-3=0$$

Now the sum of the coefficients of the odd powers and that of the coefficients of the even powers in the left side of the equation severally vanish

Therefore x^2-1 is a factor of the left side [Art 292] Hence
$$x^4-2x^3+2x^2+2x-3=x^2(x^2-1)-2x(x^2-1)+3(x^2-1)$$
$$=(x^2-1)(x^2-2x+3)$$

Thus
$$(x^2-1)(x^2-2x+3)=0$$

$x^2-1=0$ or $x=\pm 1$, or $x^2-2x+3=0$, which gives $x=1\pm\sqrt{-2}$

Ex 3 Solve $(x-a)^3+(x-b)^3=c\{(x-a)^2-(x-b)^2\}$

From the given equation, we have
$$(2x-a-b)\{(x-a)^2-(x-a)(x-b)+(x-b)^2\}=c(2x-a-b)(b-a),$$
thus
$$(2x-a-b)\{x^2-(a+b)x+[a^2+(ca-ab-bc)+b^2]\}=0$$

Hence $\qquad 2x-a-b=0$, i e, $x=\frac{1}{2}(a+b)$,

or $\qquad x^2-(a+b)x+\{a^2+(ca-ab-bc)+b^2\}=0$,

a quadratic which will give the other values of x

Examples CLXVI.

Solve the equations

1 $(x-3)(x-4)(x-5)=1 \cdot 2 \cdot 3$ $\qquad$ 2 $(x-1)(x-2)(x-3)=24$

3 $x(x+1)(x+2)=a(a+1)(a+2)$ $\qquad$ 4 $2x^3-x^2-7x=4$

5 $(x-4)^3+(x-5)^3=31\{(x-4)^2-(x-5)^2\}$.

6 $(2x+a)^3-(x+a)^3=4a(7x^2+9ax+3a^2)$

7 $(a-x)^3+(b-x)^3=(a+b-2x)^3$ $\qquad$ 8 $2x^5+15x=17$

9 $x(5x^2+7x-2)=x^3$ $\qquad$ 10 $4x^3+6x^2+x=1$

11 $x^2+\frac{1}{x^2}=a^2+\frac{1}{a^2}$ $\qquad$ 12. $(1-x+x^2)^2=\frac{7}{13}(1+x^2+x^4)$

13 $\frac{x}{a}+\frac{b}{x}+\frac{b^2}{x^2}=1+\frac{b}{a}+\frac{b^2}{a^2}$ $\qquad$ 14 $\frac{x-a}{b}+\frac{x-b}{a}=\frac{b}{x-a}+\frac{a}{x-b}$

15 $x^4+2x^3-11x^2+4x+4=0$ $\qquad$ 16 $x^4+ax^3+bx^2+cx+\frac{c^2}{a^2}=0$

CHAPTER XXVI

PROBLEMS LEADING TO QUADRATIC EQUATIONS

313 Problems To solve a problem completely we (i) express its conditions symbolically which leads to an equation, (ii) solve this equation, and (iii) verify the solution

Each of the following problems will lead to a quadratic equation, and as a quadratic has *two* roots, two values of the unknown quantity will be found But these two values, though solutions of the equation, are not always solutions of the problem For a *numerical* solution is always limited by certain conditions expressed or implied in a problem, and therefore that solution only is to be accepted which on verification will be seen to satisfy all the conditions of the problem [See Examples 1—6]

The student should therefore verify each solution to see whether one or both are admissible as solutions of the problem A solution that does not satisfy the conditions of a problem is considered as a wrong solution and no credit is given for it

Ex 1 *What number is that 20 times which subtracted from its square leaves a remainder 1581 ?*

Let x = the required number,

then x^2-20x = 20 times x subtracted from x^2,

and by the condition of the problem, this = 1581

$$x^2-20x=1581$$

or $$x^2-20x-1581=0,$$

i e $$(x-51)(x+31)=0,$$

$$x=51 \text{ or } x=-31$$

Verification (i) $(51)^2-20\times 51=2601-1020=1581$

(ii) $(-31)^2-20(-31)=961+620=1581$

Thus the required number is either 51 or -31, for each of these numbers satisfies the problem

Here the **two values** of x are admissible

Ex 2 *By selling a horse for £24, a man lost as much per cent. as it cost him in pounds What was the cost price of the horse ?*

Let £ x be the cost price of the horse, then because the man's loss was x per cent, his loss by the transaction was $\frac{x}{100}$ of £x = £$\frac{x^2}{100}$.

Also this loss by the question was $(x-24)$£

$$\frac{x^2}{100}=x-24$$

Hence $$x^2-100x+2400=0,$$

or $$(x-40)(x-60)=0,$$

whence $$x=40 \text{ or } x=60$$

If the solution is verified, it will be seen that each of these roots satisfies the conditions of the problem

Thus the price of the horse was either £40 or £60

Here also the **two values** are admissible

Ex 3 *A number of labourers earned twice as many rupees each as there were men in the party If they had earned Rs 13 each, then this amount would have exceeded the amount actually earned by Rs 15 Find the number of labourers*

Let x = required number of labourers,

then $2x\times x$ or $2x^2$ = amount of rupees they earned

By supposition, $13x$ is the amount of rupees they would have earned

$$13x=2x^2+15$$

Thus $$2x^2-13x+15=0,$$
or $$(x-5)(2x-3)=0,$$
$$x=5 \text{ or } x=\tfrac{3}{2}$$

Here **one value**, *viz*, $x=5$ is only admissible, for though $x=\frac{3}{2}$ satisfies the *equation*, yet it does not satisfy the conditions of the problem which require that a number of men must be a *whole* number We therefore reject the value $x=\frac{3}{2}$

Thus the number of labourers required is 5

Ex 4 *Each of 13 men earned a sum of money, and the total amount they earned exceeded twice the square of the amount each earned by Rs 15 What is the amount each earned?*

Let x rupees be the amount each earned, then $13x$ is the total amount earned Hence

$$13x=2x^2+15, \text{ the same equation as in Ex 3,}$$
whence $$x=5 \text{ or } x=\tfrac{3}{2}$$

Here the **two values** are admissible, as an amount of money may be *whole* or *fractional*

Ex. 5 *A number of two digits is equal to twice the product of the digits If the units' digit is greater by 3 than the tens' digit, find the number*

Let x be the tens' digit, then $x+3$ is the units' digit,

and $10x+(x+3)$ is the required number

By the question therefore

$$10x+(x+3)=2x(x+3),$$
whence $$x=3 \text{ or } x=-\tfrac{1}{2}$$

The second value is inadmissible, because the digit of a number must be a positive integer not greater than 9

Hence 3 is the tens' digit and $3+3$ or 6 is the units' digit; thus the required number is 36

Here **one value** is admissible

Ex 6 *The sum of the ages of a father and his son is 50 years If 7 times the son's age is less by 420 years than the product of their ages, how old are they?*

Let x years be the son's age, then $50-x$ years is the father's age, and $x(50-x)$ years is the product of their ages

by the second condition of the problem

$$7x=x(50-x)-420$$

Solving this equation, we get $x=15$ or $x=28$

The first solution gives the son's age 15 years and the father's age 50 − 15 or 35 years The second solution gives son's age 28 years and father's age 50 − 28 or 22 years, an absurd result, since the father cannot be younger than his son Hence this value is inadmissible

Thus the son's age is 15 years and the father's age is 35 years

Here only **one value** is admissible, though the two values are positive

Examples CLXVII

1 The sum of two numbers is 84 and their product is 1728 *Find them*

2 The difference of two numbers is 6, and their sum multiplied by the greater is 756 What are the numbers ?

3 A number and 12 times its reciprocal amount to $9\frac{1}{2}$ What is the number ?

4 Divide 50 into two parts, such that the sum of their reciprocals may be $\frac{1}{12}$

5 Divide 20 into two parts such that the square of the greater added to the less gives 152

Why is one of the positive values rejected ?

6 A number is greater than its square root by 110 Find the number

Why is one of the solutions rejected ?

7 Find two numbers differing by 15, such that the difference of their squares is 735

8 By selling an article for Rs 16, a man lost as much per cent as *the article cost* him in rupees What was its cost price ?

9 By selling a horse for £96, a dealer gained as much per cent as the horse cost him in pounds Find the cost price of the horse

Why is there one answer ?

10 Five times the number of boys in a class is greater by 12 than one third of the square of that number Find the number of boys ?

11 Three times the square of a certain number of men is less by 8 than 14 times the number How many are they ?

12 If 5 times a certain integer is subtracted from thrice its square, the *remainder is 372* What is the number ?

13 The sum of the 2 digits of a number is 11, and their product is 24 Find the number

14 The tens' figure of a number formed by two consecutive digits is greater than the units' figure, and the number exceeds by 27 the square of the tens' figure Find the number

15 A number of two digits has the tens' digit greater by 4 than the units' digit, also the sum of the reciprocals of the digits is $\frac{2}{3}$ Find the number

16 A number consists of two digits whose sum is 6, and the number formed by reversing the digits is equal to 3 times the product of the digits What is the number?

17 The sum of two digits is 7, and the product of the two numbers formed by them is 976 Find the digits

18 Divide unity into 2 parts such that the sum of their cubes is 15

19 The denominator of a fraction exceeds the numerator by 3, and if 1 be added to both, then the fraction is increased by $\frac{1}{24}$ Find the fraction.

20 The numerator of a fraction is greater by 2 than the denominator, and if $\frac{16}{15}$ be subtracted from the fraction, the fraction will be inverted What is the fraction?

21 The sum of the ages of a father and his son is 100 years Five years ago, the product of their ages was 30 times the father's age as it then was. Find their ages

22 The ages of a father and his son are together 72 years, and 16 times the difference between the reciprocals of the numbers representing their ages is $\frac{1}{3}$ What are their ages?

314 Ex 1 *Wishing to buy gold, I find that if the price were Rs 3 less per tola, I could get 1 tola more for Rs 126 What is the price of gold per tola?*

Let x rupees = the price of gold per tola,

then $\frac{126}{x}$ = number of tolas that can be had for Rs 126

Also by the question,

$\frac{126}{x-3}$ = number of tolas that could be had for Rs 126

Now this number is greater than the first number by 1

$$\frac{126}{x-3}=\frac{126}{x}+1$$

Hence $x^2-3x-378=0$, or $(x-21)(x+18)=0$,

$$x=21 \text{ or } x=-18$$

Here only the positive value of x is admissible [why ?], we therefore reject the negative value

Thus the price of gold is Rs 21 per tola

Note The negative root $x=-18$ suggests a problem the conditions of which are *contrary in character* to those of the above problem, as we shall presently see

The value $x=-18$ is a root of the equation

$$\frac{126}{x-3}=\frac{126}{x}+1 \qquad \text{(i)}$$

Hence $x=+18$ is a root of the equation obtained by writing $-x$ for x in equation (i), i e, $x=+18$ is a root of

$$\frac{126}{-x-3}=\frac{126}{-x}+1,$$

which, when the signs of both sides are changed, becomes

$$\frac{126}{x+3}=\frac{126}{x}-1 \qquad \text{(ii)}$$

Now equation (ii) is evidently the symbolical expression of the following problem—

Wishing to buy gold, I find that if the price were Rs 3 more *per tola I could get 1 tola* less *for Rs 126 What is the price of gold per tola ?*

The student will see that the answer is Rs 18

Ex 2 *A trader bought a certain number of sheep for Rs 375, and after losing 4, sold the remainder for Rs 5 a head more than they cost him, thus gaining Rs 45 How many sheep did he buy and what was the cost price of each ?*

Let x be the number of sheep he bought,

then the cost price of each is $\frac{375}{x}$ rupees.

Hence by the question

$$\left(\frac{375}{x}+5\right)(x-4)=375+45=420,$$

or $$(375+5x)(x-4)=420x$$

Solving this equation, we get $x=25$ or $x=-12$

The negative value is inadmissible Thus the number of sheep required $=25$

Also the cost price of each sheep $=$ Rs $\frac{375}{25}=$ Rs 15

Question Interpret the negative result

Examples CLXVII (*Continued*)

23 A person wishes to distribute Rs 3 12*a* among a number of beggars and finds that if there had been 3 fewer, each would receive 1*a* more than he does Find the number of beggars

What problem is suggested by the negative root? Derive its equation from the equation of the given problem

24 A man bought some chairs for Rs 72 If he had obtained 6 more for the money, each chair would have cost Re 1 less How many did he buy and what was the price of each?

Interpret the negative result

25 A cyclist rode 36 miles at a uniform rate Had his rate been 3 miles an hour slower, it would have taken him 1 hour longer What was his rate of travelling?

What problem is suggested by the negative result?

26 A person buys a certain number of photographs for £1 Two get damaged, and by selling the remainder for 2*d* each more than they cost, he makes one shilling profit. How many did he buy?

27 A farmer rented some land for Rs 48 He cultivated 8 bighas himself, and subletting the rest for 12*a* per bigha more than he paid, received Rs 54 in rent How much land did he rent and what was the rent he paid per bigha?

28 A man bought a number of sheep for £37 10*s* Having lost 5, he sold the rest for 6*s* a head more than the cost price, and lost £1 10*s* by the transaction Find the number of sheep and the price of each

29 I bought for Rs 19 13*a* two pieces of cloth, the one being 3 yds longer than the other The pieces cost me as many annas per yard as there were yards in each Find the length of each piece

30 A man bought a certain number of sheep for Rs 243 He sold them at Rs 14 4*a* a head and gained as much on the whole as a single sheep cost him How many sheep did he buy?

31 A farmer buys sheep and lambs numbering 100, each lamb costing Rs 2 8*a* less than each sheep If he spends Rs 96 on sheep and Rs 114 on lambs, what is the price of each?

32 *A* and *B* each gives Rs 54 to a certain number of poor men, *B* relieves 24 men more than *A*, but *A* gives to each 3*a* more than *B* How many men did *A* relieve?

33 A party at a restaurant had to pay a bill for Rs 44 10*a* But one of the party also having offered to pay 4 times as much as any other person, the rest paid each 9*a* less How many men were there in the party?

34 A man walked $24\frac{3}{4}$ miles in a number of hours which is one more than the number of miles he walked per hour How long did he take to walk the distance ?

315 **Ex 1** *The area of a rectangle is 1800 sq yds If the length were 5 yds more and the breadth 4 yds less, the area would still be the same Find its dimensions*

Let x yds be the length, then, since 1800 sq yds is the area, the breadth is $\frac{1800}{x}$ yds

By the second condition, the length is $(x+5)$ yds and the breadth is $\left(\frac{1800}{x}-4\right)$ yds Hence the area of this rectangle being the same as, 1800 sq yds, we have

$$(x+5)\left(\frac{1800}{x}-4\right)=1800$$

Multiply by x, thus

$$(x+5)(1800-4x)=1800x,$$

whence $$x^2+5x-2250=0,$$

or $$(x-45)(x+50)=0,$$

$$x=45 \text{ or } x=-50$$

If the length and breadth are considered as *arithmetical* numbers, then $x=45$ is the only solution, length$=45$ yds and breadth $=\frac{1800}{x}=40$ yds If however the length and breadth are taken as *algebraical* quantities, then $x=-50$ is also a solution

Explain how the *negative* root suggests the following problem—

The area of a rectangle is 1800 sq yds If the length were 5 yds *less* and the breadth 4 yds *more*, the area would still be the same. Find its dimensions

Ex 2 *Two pipes together can fill a cistern in $8\frac{8}{9}$ min When working alone, one pipe takes 4 min more than the other to fill it. In what time would each pipe alone fill the cistern ?*

Let x min be the time which one pipe takes to fill the cistern, then $(x+4)$ min is the time which the other pipe takes to fill it

Hence in 1 min they together fill $\left(\frac{1}{x}+\frac{1}{x+4}\right)$ of the cistern

Also by the question, they together fill the cistern in $8\frac{8}{9}$ min., therefore in 1 min they fill $\frac{9}{80}$ of the cistern.

Hence $$\frac{1}{x}+\frac{1}{x+4}=\frac{9}{80}$$

Multiply by $80x(x+4)$, thus

$$80(x+4)+80x=9x(x+4),$$

whence $$9x^2-124x-320=0,$$

or $$(x-16)(9x+20)=0,$$

$$x=16 \quad \text{or} \quad x=-\tfrac{20}{9}.$$

The negative value is inadmissible Hence one pipe takes 16 min and the other 20 min to fill the cistern

Examples CLXVII. *(Continued.)*

35 The perimeter of a rectangle is 84 yds and its area is 432 sq yds. Find its dimensions

36 Two rectangles have the same area *viz*, 180 sq yds, their lengths differ by 5 yds, and their breadth by 3 yds What are their dimensions?

37 Two pipes can fill a cistern in 3 hrs One of them alone can fill it in 8 hrs more than the other In what time can each pipe fill the cistern separately?

38 A and B together can do a piece of work in 6 hrs 40 min When working alone A takes 3 hrs less than B to do it. In what time can each do it alone?

39 A number of men can be formed into a solid square and also into a hollow square four deep, a side of the latter containing 25 men more than a side of the solid square Find the number of men

40 Two persons start at the same time from P to go to Q 54 miles from P The one travelling 3 miles an hour faster, reaches Q 3 hrs before the other Find the rate of each

41 A person starts to walk 28 miles After walking the first half of the distance, he slackens his pace by half a mile an hr and is consequently half an hour late At what rate was he walking at first?

42 The circumference of one wheel is 2 ft more than that of the other, and one of them makes 220 revolutions more than the other in a mile. Find the circumference of each wheel

43 A boat goes 12 miles up a river and back again in 4 hrs 10 min If there had been no current, the boat can go $6\frac{1}{4}$ miles per hr. At what rate is the river flowing?

44 A waterman rows 30 miles down a river and back again in 12 hrs 48 min If the current flows at the rate of $1\frac{1}{4}$ miles an hr, at what rate does he row in still water ?

45 A boat's crew can row 8 miles down a river and back again in 4 hrs 40 min The rate of the crew in still water is 3 miles an hour slower than twice the rate of the current, At what rate does the current flow ?

46 A and B motored simultaneously to meet each other from two places 270 miles apart A travelled 3 miles a day quicker than B, and the number of days in which they met was equal to $\frac{2}{3}$ of the number of miles A motored in a day How far did each travel before they met ?

316 Ex 1 *What are oranges a score, when 16 more for a rupee lowers the price 2 pies per dozen ?*

Let x=number of oranges bought for 1 rupee,

then $\frac{192}{x}p$ =price of one orange . ... (1)

Also $\frac{192}{x+16}p$ =... ... . by supposition

Thus $\left(\frac{192}{x}-\frac{192}{x+16}\right)p$ =diff bet actual and supposed prices of one orange.

$12\left(\frac{192}{x}-\frac{192}{x+16}\right)p$ =diff bet the two prices of 1 doz $=2p$ by the question.

Hence
$$6\left(\frac{192}{x}-\frac{192}{x+16}\right)=1,$$

i e,
$$\frac{1152}{x}-\frac{1152}{x+16}=1,$$

whence
$$x^2+16x-1152\times 16=0,$$

or
$$(x-128)(x+144)=0,$$
$$x=128 \text{ or } x=-144$$

Rejecting the negative value, we have from (1)

$$\text{cost of a score}=20\times\frac{192}{x}p=20\times\frac{192}{128}p=30p=2a\ 6p.$$

Note The problem is more easily solved as above, than by representing by x the quantity we want to find, *viz*, the price of a score See Art 94.

Remark The student will see that the negative result $x=-144$ suggests the problem —*What are oranges a score when 16 less for a rupee raises the price 2 pies per dozen ?* [See Art 314, Ex. 1, *Note.*]

Ex 2 *A set out from Calcutta for Hughli and B at the same time from Hughli for Calcutta, both travelling uniformly A reaches Hughli in 3 hours and B reaches Calcutta in 5 hrs 20 min after they have met on the road Find the time each took to perform the journey*

Let x hours be the time in which they met Thus A took $(x+3)$ hrs and B took $(x+5\frac{1}{3})$ hrs to perform the journey

Now the distance remaining the same, their rates are inversely proportional to the times

$$\therefore \frac{A\text{'s rate}}{B\text{'s rate}}=\frac{x+5\frac{1}{3}}{x+3}=\frac{3x+16}{3x+9} \qquad \text{(i)}$$

Again by the question, A performs the distance in 3 hrs which B performs in x hrs

$$\therefore \frac{A\text{'s rate}}{B\text{'s rate}}=\frac{x}{3} \qquad \text{(ii)}$$

Hence from (i) and (ii),

$$\frac{3x+16}{3x+9}=\frac{x}{3};$$

whence $3x^2=48$, or $x=\pm 4$

The negative value is inadmissible, $\therefore x=4$

Thus A took 7 hours and B took 9 hours 20 min to perform the journey

Examples CLXVII *(Continued)*

47 What is the price of eggs per dozen, when two less in a shilling's worth raises the price one penny per dozen ?

48 What is the price of apples a score when 4 more for a rupee lowers the price 8*a* per dozen ?

49 What is the price of oranges per dozen when 2 less for a rupee raises the price 8 pies per score ?

50 Eight fewer bottles of wine can be had for £6, if the price is raised 15*s* per dozen Find the price

51 A servant was sent into the market to buy a rupee's worth of mangoes He having appropriated 4, his master had to pay for every score 4 annas more than the market price How many did his master get for his rupee ?

52 The area of a rectangle is 300 sq ft. If its length is diminished by 4 ft. and its breadth increased by 1 ft, it becomes a square Find its length and breadth

53 The area of a rectangle is 120 sq ft, and its diagonal is 17 ft Find its dimensions

54 The area of a rectangle is 234 sq ft. If the length is diminished by 2 ft. and the breadth increased by 3 ft, its area would be 253 sq ft What are its dimensions ?

55 A vintner sold 7 dozen of sherry and 12 dozen of claret for £50 He sold of sherry 3 dozen more for £10 than he did of claret for £6 Find the price of a dozen of each

56 Two men start at the same time to meet each other from towns 31 miles apart One of them takes 1 min longer than the other to walk a mile If they meet in 4 hrs, at what rate is each walking?

57 *A* starts to bicycle from Cambridge to London, and *B* at the same time from London to Cambridge, and they travel uniformly *A* reaches London 4 hrs and *B* reaches Cambridge 1 hr, after they have met on the road How long did each take to perform the journey?

58 Two persons start at the same time from *A* and *B* in order to meet each other. When they meet, it is found that the first has travelled 36 miles more than the second and he will reach *B* in 6 days while the other will reach *A* in $13\frac{1}{2}$ days, after they meet Determine the distance between *A* and *B*

59 *A* company of soldiers can be formed into a solid square A battalion consisting of 7 such equal companies can be formed into a hollow square four deep The hollow square formed by the battalion is 16 times as large as the solid square formed by the company. Find the number of men in the company

317 Hitherto the solutions of problems have been either integral or fractional We shall now give examples of problems which have (i) Irrational solutions and (ii) Impossible solutions

(i) Irrational Solutions

Ex 1 *Divide a line 16 inches long into 2 parts, so that the rectangle contained by the two parts may be equal to the square on a line 7 inches long*

Let x in. be one of the parts, so that the other part is $(16-x)$ in

Hence by the question,

$$x(16-x)=49,$$

whence $$x=8\pm\sqrt{15}$$

Now $\sqrt{15}$ is a surd number and its value cannot be finitely found. We can however find $\sqrt{15}$ sufficiently accurate for our purpose Thus correct to 2 decimal places $\sqrt{15}=3\cdot87$ Hence $x=11\cdot87$ or $x=4\cdot13$

Here the sum of the two values of x is 16 As there is only one mode of dividing 16, we have one of the parts $=11\cdot87$ and the other part $=4\cdot13$

(ii) Impossible Solutions

Ex 2 *Divide a line 16 inches long into 2 parts, so that the rectangle contained by the two parts may be equal to the square on a line 9 inches long*

As in Ex. 1, the problem leads to the equation

$$x(16-x)=81,$$

whence $$x=8\pm\sqrt{-17}$$

The solution shews that the problem is impossible In fact from Geometry we know that the greatest possible rectangle contained by the parts of a line is when the parts are equal, *i e*, in the present case the area of the greatest rectangle is 8^2 Hence no rectangle contained by the parts of this line can be equal to 9^2

Examples CLXVIII.

N B. *If a solution is impossible, explain why it is so.*

1. AB is the side subtending the obtuse angle of the triangle ABC If $AB=15$, $AC=10$ and the projection of AC on $BC=3$, find BC

2 In a triangle ABC, $AB=15$, $AC=10$ and $BC=8$, find the perpendicular from A on BC

✓3 In a triangle ABC, $AB=15$, $AC=8$ and $BC=6$ Find the perpendicular from A on BC.

✓4 The area of a field is 3000 sq yds., and the sum of the sides is 100 yds Find them

5 Find a number such that its square shall be equal to twice the number, increased by 5

✓6 Find a number such that its square shall be equal to twice the number, diminished by 5

✓7 The sum of two numbers is 12, and their product is 40 Find them

✓8 By selling a cow for Rs 29, a man loses as much per cent. as the cost of the cow was in rupees What was its cost price ?

✓9 The tap A can fill a tank in 5 hrs more than the tap B, and they running together can fill it in 4 hrs less than what A takes to fill Find the time in which each can fill the tank separately

10 In a rectangular picture-frame, 3 ft by 4 ft, one-eighth of the whole area is occupied by the frame which is of uniform width all round, and the remainder by the glass What is the width of the frame ?

CHAPTER XXVII

SIMULTANEOUS QUADRATICS

318 Definition Simultaneous Quadratic Equations are those in which one at least of the equations is of the second degree In this Chapter we shall give only a few examples of these equations which occur frequently No general rules can be given for their solution, except in the case of two classes of equations [See Arts 319 and 320 below]

319 Class I In these equations, one is of the *first* degree and the other is of the *second* degree The method of solution will be seen from the following examples

Ex 1. Solve $x+y=5$ ... ... (i),

$x^2+y^2=13$ ... ... (ii)

From (i) $y=5-x$ (iii)

Substitute in (ii), thus

$$x^2+(5-x)^2=13,$$

or $$x^2-5x+6=0,$$

whence $$(x-2)(x-3)=0,$$

$$\therefore x-2=0, \text{ or } x-3=0,$$

$$i\,e, \ x=2, \quad \text{or} \quad x=3$$

(1) If $x=2$, then from (iii), $y=5-2=3$

(2) If $x=3$, then from (iii), $y=5-3=2$

Thus the solutions are $\begin{cases} x=2 \text{ and } y=3, \\ \text{or } x=3 \text{ and } y=2 \end{cases}$

Hence the **Rule** —*From the equation of the first degree find one of the variables in terms of the other and substitute in the equation of the second degree*

Verify each of the solutions.

Note In the answer, the corresponding values of x and y *should be put together* Hence the answer should not be written thus $x=2$ or 3, $y=3$ or 2

Ex 2 Solve $x^2+3xy-2y=38$ ... ... (i),

$2x+5y=15$ ... ... (ii)

From (ii), $x=\frac{1}{2}(15-5y)$ ... ... (iii)

Substitute x in (i), thus

$$\tfrac{1}{4}(15-5y)^2+\tfrac{3}{2}(15-5y)y-2y=38\ ,$$

or $$5y^2+68y-73=0\ ,$$

whence $$(y-1)(5y+73)=0\ ,$$

thus $$y-1=0,\ \text{or}\ 5y+73=0\ ,$$

$$y=1,\qquad \text{or}\ \ y=-\tfrac{73}{5}.$$

(1) If $y=1$, then from (iii), $x=5$

(2) If $y=-\frac{73}{5}$, then from (iii), $x=44$

Solutions $\left.\begin{array}{l} x=5, \\ y=1\ , \end{array}\right\}$ or $\left\{\begin{array}{l} x=44, \\ y=-\frac{73}{5} \end{array}\right.$

Verify the solutions

Examples CLXIX

Solve the equations

1 $x+y=7,\ x^2+y^2=25$

2 $x-y=2,\ 2x^2+3y^2=77$

3 $x-2y=2,\ x^2+4y^2=100$

4 $x+2y=4,\ 3x^2+y^2=13$

5 $x-y=-3,\ x^2-2xy=5$

6 $x-3y=17,\ x^2+xy=40$

7 $3x-2y=4,\ 9x^2+4y^2=40$

8 $3x-4y+6=0,\ 37x^2-16y^2=4$

9 $x+y+1=0,\ 2x^2+xy=3y^2$

10 $5x+7=3y,\ x^2+2y^2=5xy+13$

11 $2x-3y=16,\ 2xy-y^2=7y-10$

12 $x=2(y+1),\ x^2-3xy+2x+4y=20$

320 Class II Homogeneous Equations There are two methods of solution

First Method

In this method, we *eliminate the constants*

Ex 1. Solve $x^2+2y^2=22$ . . . . (i),

$2xy+y^2=21$ . . . . (ii).

Multiply (i) by 21 and (ii) by 22, and subtract, thus

$$21x^2+42y^2-44xy-22y^2=0,$$

or $$21x^2-44xy+20y^2=0,$$

whence $$(3x-2y)(7x-10y)=0,$$

thus $3x-2y=0$, i e, $y=\frac{3}{2}x$. . . . . (iii),

or else $7x-10y=0$, i e, $y=\frac{7}{10}x$ . . . . (iv)

(1) If $y=\frac{3}{2}x$, we have from (i),

$$x^2+\frac{9}{2}x^2=22, \text{ whence } x=\pm 2,$$
$$\text{from (iii)}, \quad y=\frac{3}{2}x=\pm 3$$

(2) If $y=\frac{7}{10}x$, then from (i),

$$x^2+\frac{49}{50}x^2=22, \text{ whence } x=\pm\frac{10}{3},$$
$$\text{from (iv)}, \quad y=\frac{7}{10}x=\pm\frac{7}{3}$$

Thus the solutions $\begin{cases} x=\pm 2 \text{ and } y=\pm 3, \\ \text{or } x=\pm\frac{10}{3} \text{ and } y=\pm\frac{7}{3} \end{cases}$

Hence the **Rule** :—*Eliminate the constants and factorize the resulting equation*

Second Method

Assume $y=vx$, thus we have

from (i), $\qquad x^2(1+2v^2)=22 \quad \ldots (a),$

and from (ii), $\qquad x^2(2v+v^2)=21 \quad \ldots (b)$

Divide (a) by (b), thus

$$\frac{1+2v^2}{2v+v^2}=\frac{22}{21},$$

whence $\qquad 20v^2-44v+21=0,$

or $\qquad (2v-3)(10v-7)=0$

Hence $\qquad v=\frac{3}{2}$, or $v=\frac{7}{10}$,

$$y=\frac{3}{2}x, \text{ or } y=\frac{7}{10}x.$$

Now proceed as in the other method shewn above.

Ex 2 Solve $4x^2+3xy-2y^2=2 \quad \ldots$ (i),

$3x^2-8xy+4y^2=3 \quad \ldots$ (ii)

Multiply (i) by (ii) crosswise, thus

$$3(4x^2+3xy-2y^2)=2(3x^2-8xy+4y^2);$$

or $\qquad 6x^2+25xy-14y^2=0,$

whence $\qquad (2x-y)(3x+14y)=0$

Hence $\qquad 2x-y=0$, i e, $y=2x \quad \ldots$ (iii),

or else $\qquad 3x+14y=0$, i e, $y=-\frac{3}{14}x \quad \ldots$ (iv)

Now proceed as in Ex. 1

Solutions $\quad x=\pm 1, y=\pm 2$, or $x=\pm\dfrac{7}{4\sqrt{5}}, y=\mp\dfrac{3}{8\sqrt{5}}$.

Examples CLXX

Solve the equations

1 $x^2+3y^2=28,\ xy+y^2=12$ $\qquad$ 2 $x^2+xy=12,\ xy-y^2=2$

3. $4x^2+3xy=10,\ 2xy+3y^2=16$ $\qquad$ 4 $x^2+xy=3,\ y^2-xy=2$

Solve the equations

5 $x^2-xy=6,\ 2y^2-xy=20$ 6 $2xy+y^2=16,\ 2x^2-xy=12$

7 $y^2+xy=4,\ x^2-xy+2y^2=8$ 8 $x^2+2y^2=22,\ 3y^2-x^2-xy=17$

9. $x^2+2xy+3y^2=19,\ 3x^2+2xy+y^2=9$

10 $x^2+y^2=xy+7,\ x^2-y^2=xy-1$

321 Symmetrical Equations The solutions of these equations in most cases depend on known identities.

Ex 1 Solve $x+y=9$ ... (i),

$$xy=8 \qquad \text{(ii)}$$

Since $(x-y)^2=(x+y)^2-4xy$, we have from (i) and (ii),

$$(x-y)^2=9^2-32=49,$$

whence $$x-y=\pm 7$$

Hence we have *two* pairs of simple equations

$$\left.\begin{matrix}x+y=9\\x-y=7\end{matrix}\right\} \text{ (iii), and } \left.\begin{matrix}x+y=9\\x-y=-7\end{matrix}\right\} \text{ (iv)}$$

From (iii), $x=8$ and $y=1$, and from (iv), $x=1$ and $y=8$

Otherwise —From (ii), $y=\frac{8}{x}$, thus from (i),

$$x+\frac{8}{x}=9, \text{ or } x^2-9x+8=0,$$

i e, $$(x-8)(x-1)=0 \quad \text{whence } x=8, \text{ or } x=1$$

Thus from (ii), when $x=8$, $y=1$, or when $x=1$, $y=8$

Another method of course is to substitute y in (ii) from (i) [Art. 319]

Note The student will notice that in symmetrical equations, if the value of x be m or n, the corresponding value of y will be n or m

Ex 2 Solve $x+y=5$ (i),

$$x^2+y^2=13 \qquad \text{(ii)}$$

Square (i) and subtract (ii), thus

$$(x+y)^2-(x^2+y^2)=5^2-13=12,$$

whence $2xy=12$, or $xy=6$ (iii)

Take (i) and (iii), and proceed as in Ex 1

Ex 3 Solve $x^2+y^2=34$ (i),

$$xy=15 \qquad \text{(ii)}$$

Add and subtract twice (ii), thus

$$(x+y)^2=64 \text{ and } (x-y)^2=4$$

Hence $x+y=\pm 8$ and $x-y=\pm 2$

Thus we have *four* pairs of equations

$$\left.\begin{matrix} x+y=8 \\ x-y=2 \end{matrix}\right\}, \quad \left.\begin{matrix} x+y=8 \\ x-y=-2 \end{matrix}\right\}, \quad \left.\begin{matrix} x+y=-8 \\ x-y=2 \end{matrix}\right\}, \quad \left.\begin{matrix} x+y=-8 \\ x-y=-2 \end{matrix}\right\}.$$

Solving these we get *four* pairs of values of x and y viz,

$x=\pm 5$ and $y=\pm 3$, $x=\pm 3$ and $y=\pm 5$

Ex 4 Solve $x^3+y^3=152$ . . . . (i),

$x+y=8$ . . . . (ii)

First Method We have $x^3+y^3+3xy(x+y)=(x+y)^3$

Hence from (i) and (ii), $152+24xy=512$,

whence $xy=15$ . . . . (iii)

Take (ii) and (iii), and proceed as in Ex 1.

Solutions $x=5$ and $y=3$, or $x=3$ and $y=5$

Second Method. Divide (i) by (ii), thus

$x^2-xy+y^2=19$ . . (iv).

Square (ii) and subtract (iv), thus $xy=15$ (v)

Take (ii) and (v), and proceed as in Ex 1

Third Method Since $x+y$ is given, we assume $x-y=2v$, so that if v is found, x and y are found

We have $\left.\begin{matrix} x+y=8 \\ x-y=2v \end{matrix}\right\}$, hence $\left.\begin{matrix} x=4+v \\ y=4-v \end{matrix}\right\}$ . . . . (vi).

Substitute x and y in (i), thus $(4+v)^3+(4-v)^3=152$,

or $(64+48v+12v^2+v^3)+(64-48v+12v^2-v^3)=152$,

whence $24v^2=24$, or $v=\pm 1$

(1) If $v=1$, then from (vi), $x=5$ and $y=3$

(2) If $v=-1$, then from (vi), $x=3$ and $y=5$

Ex. 5 Solve $x^2+xy+y^2=7$ . . . . . . (i),

$x^4+x^2y^2+y^4=21$ . . . . (ii)

We have identically $(x^2+xy+y^2)(x^2-xy+y^2)=x^4+x^2y^2+y^4$

Thus from (i) and (ii), $7(x^2-xy+y^2)=21$,

or $x^2-xy+y^2=3$. . . . . . . (iii).

$\therefore$ from (i) and (iii), by addition and subtraction, we get

$x^2+y^2=5$, and $xy=2$

Now proceed as in Ex 3

Solutions $x=\pm 2$ and $y=\pm 1$, or $x=\pm 1$ and $y=\pm 2$

Examples CLXXI

Solve the equations

1 $x+y=17$, $xy=70$

2 $x+y=8$, $xy=15$

3 $x-y=13$, $xy=90$

4 $x-y=4$, $xy=192$

5 $5x-2y=26$, $xy=12$

6 $7x-10y=18$, $xy=4$

7 $8x+9y=5$, $12xy=1$

8 $16x-3y=9$, $2xy=15$

9. $5x-6y=16$, $3xy=8$

10 $x^2+y^2=45$, $x+y=3$

11 $x^2+y^2=29$, $x-y=7$

12 $x^2-xy+y^2=93$, $x+y=3$.

13 $x^2+xy+y^2=19$, $x-y=1$

14 $x^3-y^3=728$, $x-y=14$

15 $x^3+y^3=513$, $x+y=3$

16 $x^4+x^2y^2+y^4=741$, $x^2-xy+y^2=39$

17 $x^4+x^2y^2+y^4=931$, $x^2+xy+y^2=49$

322 Miscellaneous Equations In this Article are given some typical equations, whose solutions will suggest methods for the solution of other equations

Ex 1 Solve $yz=12$, $zx=24$, $xy=8$

Multiply the second and the third together and divide by the first, thus

$$\frac{(zx)(xy)}{yz}=\frac{24\cdot 8}{12}, \text{ or } x^2=16, \text{ i e }, x=\pm 4.$$

Similarly from $(xy)(yz)\div(zx)$ and $(yz)(zx)\div(xy)$,

we get $y=\pm 2$ and $z=\pm 6$

Otherwise.—Multiply the 3 equations together, thus

$$x^2y^2z^2=12\cdot 24\cdot 8, \text{ or } xyz=\pm 48$$

Now divide $xyz=\pm 48$ by the given equations in turn

Ex 2 Solve $x(y+z)=21$, $y(z+x)=25$, $z(x+y)=16$

Add the given equations together, thus

$$2(yz+zx+xy)=62, \text{ or } yz+zx+xy=31$$

From this equation, subtract the given equations in turn, thus

$$yz=10, zx=6, xy=15$$

Then proceed as in Ex. 1 Thus $x=\pm 3$, $y=\pm 5$, $z=\pm 2$

Ex. 3 Solve

$$x(x+y+z)=20-yz \quad \text{(i)},$$
$$y(x+y+z)=12-zx \quad \text{(ii)},$$
$$z(x+y+z)=15-xy \quad \text{(iii)}$$

From (i), $x^2+x(y+z)+yz=20$, or $(x+y)(x+z)=20$

Similarly $(y+z)(y+x)=12$ and $(z+x)(z+y)=15$

Solving for $y+z$, $z+x$ and $x+y$ as in Ex. 1, we get $y+z=\pm 3$, $z+x=\pm 5$ and $x+y=\pm 4$

Hence as in Art 244, we get x, y and z

Ex. 4 Solve $x(2x+y+3z)=30$ . . . .(i),

$y(2x+y+3z)=36$ . . . . (ii),

$z(2x+y+3z)=16$ . . . . (iii)

Multiply (i) by 2, the coefficient of x, and (iii) by 3, the coefficient of z, and add, thus

$(2x+y+3z)^2=144$, whence $2x+y+3z=\pm 12$ . . . .(iv);

Divide (i), (ii) and (iii) respectively by (iv), thus

$$x=\pm 2\tfrac{1}{2},\ y=\pm 3,\ z=\pm 1\tfrac{1}{3}$$

Ex 5. Solve $xyz=a(y+z)=b(z+x)=c(x+y)$

Obviously $x=0$, $y=0$ and $z=0$ is one solution

Divide by xyz, thus

$$\frac{1}{zx}+\frac{1}{xy}=\frac{1}{a},\quad \frac{1}{xy}+\frac{1}{yz}=\frac{1}{b},\quad \frac{1}{yz}+\frac{1}{zx}=\frac{1}{c}.$$

Now solve for $\frac{1}{yz}$, $\frac{1}{zx}$ and $\frac{1}{xy}$ We thus find yz, zx and xy, and finally x, y and z as in Ex 1

Examples CLXXII

Solve the equations

1 $yz=15$, $zx=40$, $xy=24$

2 $yz=a^2$, $zx=b^2$, $xy=c^2$.

3 $xyz=24$, $yzu=72$, $zux=48$, $uxy=36$

4 $x^2yz=60$, $xy^2z=90$, $xyz^2=150$

5 $y^2z=12$, $z^2x=36$, $x^2y=32$

6 $x(y+z)=50$, $y(z+x)=44$, $z(x+y)=54$

7. $\frac{1}{x}=y+z$, $\frac{1}{y}=z+x$, $\frac{1}{z}=x+y$

8 $2x(y+z)=15$, $2y(z+x)=9$, $z(x+y)=7$.

9 $x(x+y+z)=96$, $y(x+y+z)=216$, $z(x+y+z)=264$

10. $x(3x+2y+z)=34$, $y(3x+2y+z)=51$, $z(3x+2y+z)=85$.

11 $x(x+2y-3z)=6$, $y(x+2y-3z)=15$, $z(x+2y-3z)=9$.

Solve the equations

12 $xyz=4(y+z)=3(z+x)=6(x+y)$

13 $x(y+z)=yz+15,\ y(z+x)=zx-3,\ z(x+y)=xy+39.$

14 $yz+zx+xy=14-x^2=2-y^2=7-z^2$

15 $xyz=3(y+z-x)=2(z+x-y)=4(x+y-z)$

16 $xyz=a(y+z-x)=b(z+x-y)=c(x+y-z)$

CHAPTER XXVIII

GRAPHS OF QUADRATIC EQUATIONS

A Pair of Straight Lines

323 We have seen in Chapter XI that the graphs of Simple Equations are straight lines In this Chapter we shall shew that the graphs of Quardratic Equations are in general, **curves**, such as the circle, the parabola, &c We have said "in general," for in the particular case where a **quadratic Equation resolves into two linear equations**, its graph will be a pair of straight lines and not a curve, for the obvious reason that each linear equation has for its graph a straight line

We shall give below a few simple cases

324 **Case I.** *To draw the graph of* $x^2-16=0$

We have $x^2=16$, or $x=\pm 4$, *i e*, $x=4$, or $x=-4$

The first equation has for its graph the line PS [Fig 10, Art 115]

The graph of the second equation is a line parallel to the y axis and at a distance of 4 units from it on its left side Draw the graph.

Case II *To draw the graph of* $y^2-36=0$.

We have $y^2=36$, thus $y=6$, or $y=-6$

The graph of $y=6$ is the line PM [Fig 10, Art 115]

Similarly the graph of $y=-6$ is the line parallel to the x axis and at a distance of 6 units below it. Draw the praph.

Case III *To draw the graph of* $x^2-4x=0$.

We have $x(x-4)=0$, whence $x=0$, or $x=4$

The graph of $x=0$ is the y axis [Art. 113]

The graph of $x=4$ is the line PS [Fig. 10, Art. 115].

Case IV *To draw the grcph of* $y^2-6y=0$

One of the graphs is the x-axis [Art. 114], and the other is PM [Fig 10, Art. 115]

Case V *To draw the graph of* $x^2-x-12=0$

We have $(x-4)(x+3)=0$, whence $x=4$, or $x=-3$

Thus the graphs are PS and the line through N' parallel to the y axis [Fig 10, Art 115]

Case VI *To draw the graph of* $y^2+2y-48=0$

We have $(y-6)(y+8)=0$, hence $y=6$, or $y=-8$.

The graph of $y=6$ is PM [Fig 10, Art 115] Draw the other graph

Case VII *To draw the graph of* $xy-6x+3y=18$

Transpose 18, thus the given equation is equivalent to

$$(x+3)(y-6)=0,$$

whence $x=-3$, or $y=6$ Thus the graph of $xy-6x+3y=18$ is a pair of lines, the first of which is the line through N' parallel to the y axis [Fig 10, Art 115] and the second is the line PM parallel to the x axis

Case VIII *To draw the graph of*

$$5y^2-13xy+6x^2+11x-23y=10$$

Transpose 10, thus the given equation is equivalent to

$$(y-2x-5)(5y-3x+2)=0$$

Thus we have the two equations $y=2x+5$ and $5y=3x-2$, whose graphs are the line (ii) [Fig 12, Art 118], and the line in Fig 13, Art 122 respectively

[In drawing the graph of the given equation, the student must of course shew the pair of lines on the same diagram and not as we have done here]

325. We shall now shew that a **homogeneous equation** of the second degree when resolvable into factors, will have for its graph **a pair of straight lines passing through the origin**

Case I *To draw the graph of* $y^2-x^2=0$

The given equation is equivalent to $(y-x)(y+x)=0$,

whence $\qquad y-x=0 \quad \ldots\ldots\ldots\ldots \quad (i),$

or $\qquad y+x=0 \quad \ldots\ldots\ldots\ldots \quad (ii)$

Hence $y^2-x^2=0$ is satisfied by the co-ordinates of all points which satisfy (i) and also by the co-ordinates of all points which satisfy (ii)

Thus the graph of $y^2-x^2=0$ is the pair of straight lines OP and OP' passing through the origin [Fig 11, Art. 117], OP being the graph of (i) and OP' the graph of (ii)

Case II *To draw the graph of* $y^2-3xy+2x^2=0$.

The given equation is equivalent to $(y-x)(y-2x)=0$.

Reasoning as in Case I, we see that required graph is the pair of straight lines whose equations are $y-x=0$ and $y-2x=0$

The graph of $y-x=0$ is the line OP [Fig 11, Art 117] and the graph of $y-2x=0$ is the line (i) [Fig. 12, Art. 118]

Examples CLXXIII

Draw the graph of

1. $y^2-4x^2=0$	2 $x^2-4y^2=0$	3 $y^2-9x^2=0$
4 $x^2-9y^2=0$	5 $x^2-4=0$	6 $y^2-9=0$
7 $25x^2=16$	8 $16y^2=9$	9 $x^2+4x=0$
10 $y^2+6y=0$	11 $2x^2-5x=0$	12 $x^2-3x+2=0$
13 $x^2-10x+25=0$	14 $x^2+x=30$	15 $y^2-2y=3$
16 $2x^2-7x+3=0$	17 $xy+5y=4x+20$	18 $7x+3y-xy=21.$

19 $x^2-3xy-8x+12y+16=0$ 20 $x^2+2xy-3y^2-x+5y=2$

21 $12x^2+7xy-12y^2=0$, and shew that the lines are perpendicular to each other

22 $2y^2-11xy+15x^2=0$, and shew that one of the lines is parallel to the graph of $2y=5x+4$

23 $2x^2+3xy-2y^2=0$, and shew that one of the lines is perpendicular to the graph of $y=2x+3$

326 Lemma I The distance of a point (x, y) from the origin is $\sqrt{x^2+y^2}$

Let Q be a point whose co-ordinates are x and y [Fig 1], so that $ON=x$ and $QN=y$

Join OQ Since OQN is a right-angled triangle,

$$OQ^2=ON^2+QN^2=x^2+y^2, \quad \therefore \quad OQ=\sqrt{x^2+y^2}$$

327 Lemma II The distance between the two points (x', y') and (x'', y'') is $\sqrt{(x'-x'')^2+(y'-y'')^2}$

Let P, Q be the two points whose co-ordinates are x', y' and x'', y'' respectively Draw QR perpendicular to PM

Thus $QR=OM-ON=x'-x''$,

$PR=PM-QN=y'-y''$,

Now $PQ^2=QR^2+PR^2$

$$=(x'-x'')^2+(y'-y'')^2,$$

$$\therefore \quad PQ=\sqrt{(x'-x'')^2+(y'-y'')^2}$$

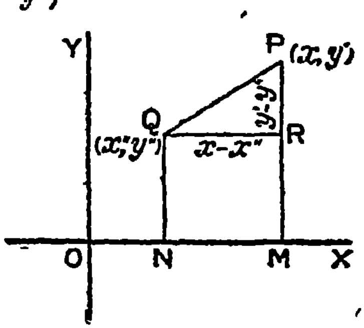


Fig 1.

Examples CLXXIV

1 Find the distance of the point $(12, -16)$ from the origin

2 Find the distance between the following pairs of points (i) $(7, 10)$, $(-5, -6)$, (ii) $(-3, 2)$, $(9, -3)$, (iii) $(-4, -3)$, $(4, -18)$

3 Shew, by drawing a circle with the origin as centre, that, the points $(5, -12)$, $(0, 13)$, $(-5, 12)$, $(13, 0)$, $(-12, -5)$, $(-13, 0)$, $(-12, 5)$, $(-5, -12)$, $(0, -13)$ and $(12, 5)$ all lie on its circumference Find the radius

4 Shew that the six points $(-8, 6)$, $(0, 10)$, $(6, 8)$, $(-10, 0)$, $(-6, -8)$, and $(8, -6)$ are equidistant from the origin, and find the distance

5 Shew that the points $(15, -20)$ and $(-7, 24)$ are equidistant from the origin Find the distance between them

Curves

328 *Draw the graph of the equation* $x^2+y^2=16$

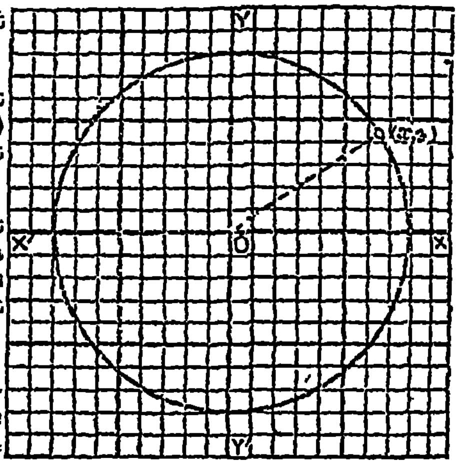
The equation can be written $\sqrt{x^2+y^2}=4$

We know that $\sqrt{x^2+y^2}$ represents the distance of a point (x, y) from the origin

Hence $\sqrt{x^2+y^2}=4$ shews that the distance of a point (x, y) from the origin is 4, a constant quantity

Thus wherever the point (x, y) may be, it is always at a constant distance of 4 units from O, and therefore it must be on a circle

Hence the graph of the equation $x^2+y^2=16$ is a circle whose centre is O and radius 4 units



Describe the circle

Fig 2

It is easy to see that (i) the co-ordinates of any point on the circle satisfy the given equation, and (ii) those of a point outside the circle do not satisfy it

Hence the graph required is the circle drawn (Two divisions of squared paper are taken as unit)

Note Generally the graph of an equation of the form $x^2+y^2=c^2$, where c is constant, is a circle. Hence also the graph of $y=\sqrt{c^2-x^2}$ is a circle.

329. *Draw the graph of the equation $x^2+y^2-4x-6y=23$.*

The equation can be put in the form $(x-2)^2+(y-3)^2=36$.

Thus $\sqrt{(x-2)^2+(y-3)^2}=6$

This shews that the distance of a point (x,y) from the point $(2, 3)$, is 6 units [Art. 327], *i e*, constant.

Thus wherever the point (x, y) may be, it is always at a *constant distance* of 6 units from the point $(2, 3)$ Hence the point is on a circle whose centre is $(2, 3)$ and radius 6 units.

Describe the circle

It will be seen that (i) the co-ordinates of any point, on the circle satisfy the given equation and that (ii) those of a point outside the circle do not satisfy it

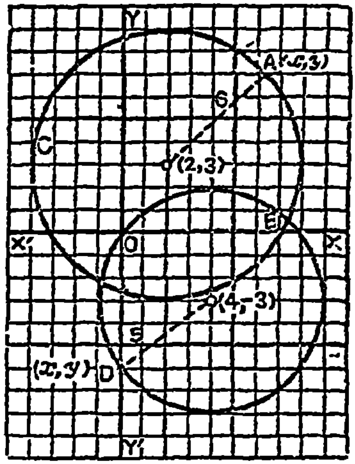


Fig 3

Hence the graph required is the circle *ABC*. (A side of a square is taken as unit.)

330 *Draw the graph of the equation $x^2+y^2-8x+6y=0$.*

The equation can be put in the form $(x-4)^2+(y+3)^2=25$

Thus as in the last example, the required graph is a circle whose centre is $(4, -3)$ and radius 5 units

Hence it is the circle *OBD* [Fig 3] (A side of a square is unit)

Observe that the circle *OBD* passes through the origin as it should do, for *there is no constant term in its equation*

Note Generally the graph of an equation of the form

$$a(x^2+y^2)+gx+fy+c=0,$$

is a circle, that is, an equation of the second degree, in which the coefficients of x^2 and y^2 are equal and there is no term involving xy, represents a circle.

331 Ex 1. Draw the graphs of $x^2+y^2=25$ and $x+y=7$, and measure the co-ordinates of their points of intersection [*Cal*, 1912]

The graph of the first equation is evidently a circle whose centre is O and radius 5 units

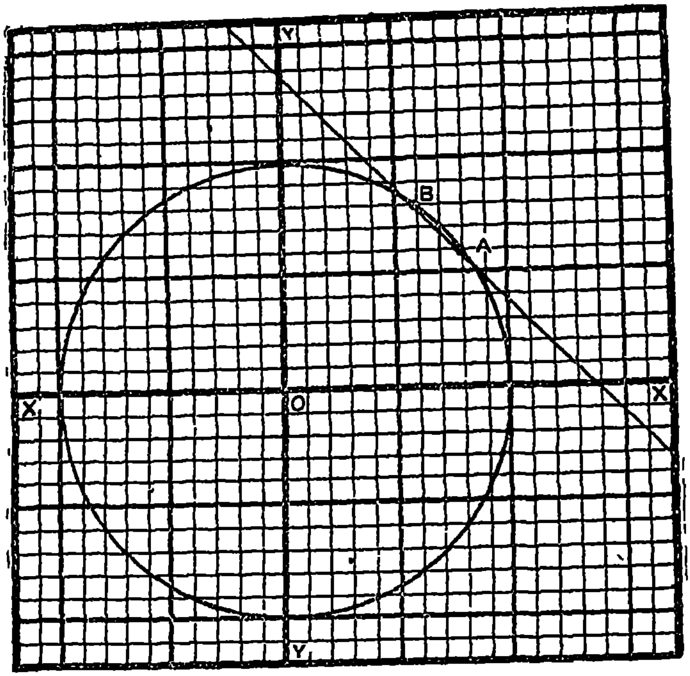


Fig 4

The graph of the second equation is a straight line which makes intercepts of 7 units on each axis, for $x=7$ when $y=0$, and $y=7$ when $x=0$ Thus the line passes through the points (7,0) and (0, 7),

Take 2 divisions of the paper as the unit of length, and draw the graphs

They are seen to intersect at the points A and B whose co-ordinates are 4, 3 and 3, 4 respectively

Note Since the graphs of the equations pass through the points A and B, the co-ordinates of A and B which are 4, 3 and 3, 4 will satisfy each of the equations Hence $x=4$ and $y=3$, or $x=3$ and $y=4$ are the solutions of the equations $x^2+y^2=25$ and $x+y=7$ Thus the equations are solved graphically.

Ex 2 Solve graphically the equations

$$x^2+y^2-4x-6y=23 \text{ and } x^2+y^2-8x+6y=0$$

The graphs of these equations have been shewn in Arts 329 and 330 B is one of the points of intersection whose co-ordinates are approximately 7 5 and 6 The co ordinates of the other point are roughly -9 and $-2\ 2$ Hence the solutions are $x=7\ 5$ and $y=6$, or $x=-9$ and $y=-2\ 2$

Examples CLXXV.

Draw the graphs of

1 $x^2+y^2=1$. 2 $x^2+y^2=9$ 3 $x^2+y^2=64$

4 $x^2+y^2=0$. 5 $x^2+y^2=2$ 6 $x^2+y^2=24$

7 $2x=\sqrt{9-4y^2}$ 8 $3y=\sqrt{20-9x^2}$

9 $(x-1)^2+(y-2)^2=25$ 10 $(x-4)^2+(y+4)^2=4$

11. $(x+2)^2+(y+3)^2=81$ 12 $x^2+y^2-6x-8y=0$

13 $x^2+y^2-4x-6y=0$ 14 $x^2+y^2-3x-10y=15$.

15. $x^2+y^2+8x+12y+16=0$ 16 $x^2+y^2-14x-33=0$

17 $x^2+y^2+10y=11$. 18 $\sqrt{6x-x^2}$

19. $\sqrt{x(8-x)}$ 20 $\sqrt{24-2x-x^2}$.

21 $\sqrt{8x-x^2-7}$ 22 $\sqrt{32+4x-x^2}$.

23 Draw the graphs of $x^2+y^2=25$ and $3x+4y=25$ Prove that the second graph touches the first and find the co ordinates of the point of contact [*Cal*, 1911]

24 Draw the graphs of the circle $x^2+y^2=9$, and of the lines $3x+4y=15$, $3x+4y=12$, and $3x+4y=18$ Shew that the circle touches the first line, and cuts the second in two real points and the third in no real points

25. Draw the graphs of $x^2+y^2-18x-16y+120=0$ and $3x+4y=59$ Find the co-ordinates of the points where they intersect and shew that one of them passes through the centre of the other

26 Solve graphically the equation $x^2+y^2=36$ and $x-y=4$.

27. Solve graphically the equations

$$x^2+y^2=16 \text{ and } x^2+y^2=6x+16$$

28 Solve graphically the equations

$$x^2+y^2=25 \text{ and } x^2+y^2-4x-6y=3.$$

332 *Draw the graph of the equation $y=x^2$.*

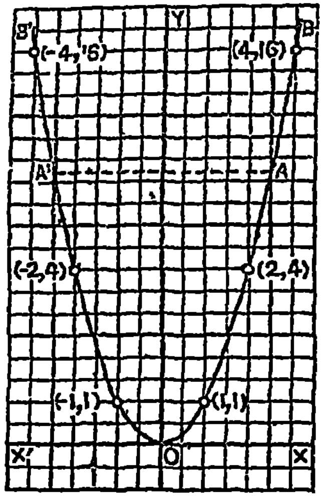


Fig 5

Tabulate the values of x and y

Values of x	-4	-3	-2	-1	0	1	2	3	4
Values of y	16	9	4	1	0	1	4	9	16

Now take twice the side of a square to denote the unit, plot these points and join them by a continuous line drawn freehand. We have thus for the required graph, the curve BOB'.

This curve is called a **parabola**

Observations on the shape of the curve

(i) For every value of y, there are two values of x *equal* in magnitude and *opposite* in sign, that is, for every point A on the curve to the *right* of OY, there is a corresponding point A' at the same distance to the *left* of OY. Thus **the curve is symmetrical about the axis of y.**

Hence after drawing the part OAB of the curve in the first quadrant, we can infer the form of the other part $OA'B'$ in the second quadrant without *actually plotting any points* in this quadrant

(ii) Whatever be the value of x, positive or negative, the value of x^2 and therefore of y, *is always positive* Hence all the ordinates lie above the

x axis and therefore the **curve itself lies, wholly above the x-axis,** i e, in the first and second quadrants.

(iii) Since there is no constant term in the equation $y=x^2$, **the curve passes through the origin** Also as x increases indefinitely, y also increases without limit, thus **the curve extends upwards to infinity**

(iv) From the above it is easy to see that the graph of $y=-x^2$ is a parabola, *exactly similar in shape and lying wholly below the x axis* i e, in the third and fourth quadrants The student will do well to draw the graph himself

Hence **the graph of $y=ax^2$ where a is any number is a parabola** having the same shape and lying above or below the x axis according as a is positive or negative

(v) Generally the graph of $y=ax^2+bx+c$, where a may have any value except o, and b and c any value whatever, is a parabola.

333 *Draw the graph of the equation* $9x^2+25y^2=225$.

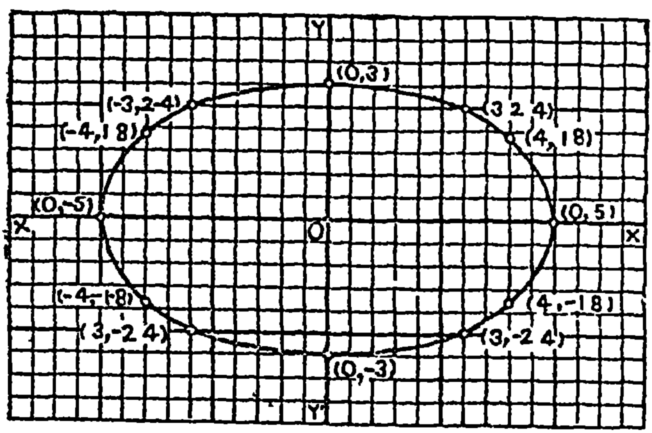


Fig 6

When $x=0$, we get $y=\pm 3$, thus the graph cuts the axis of y at the points (0, 3) and (0, −3)

When $y=0$, we get $x=\pm 5$, thus the graph cuts the axis of x at the points (5, 0) and (−5, 0)

Again when $x=\pm 3$, we have $y=\pm 2 4$, thus the graph passes through the four points (3, 2 4), (−3, 2 4), (−3, −2 4) and (3, −2 4).

Also when $x=\pm 4$, we have $y=\pm 1\ 8$, thus the graph passes through the 4 points $(4, 1\ 8)$, $(-4, 1\ 8)$, $(-4, -1\ 8)$ and $(4, -1\ 8)$

Take twice the side of a square as the unit and plot these 12 points Join them by a curve drawn freehand and we have the required graph as shewn in Fig 6

This curve is called an **ellipse.**

Observations on the shape of the curve

(i) The given equation can be put in the form

$$x=\pm \tfrac{5}{3}\sqrt{9-y^2}$$

Hence whatever be the value of y, x has *two* values *equal* in magnitude and *opposite* in sign Thus **the curve is symmetrical about the axis of y**

Also the greatest value which y can have is ± 3, for if $y^2>9$, x becomes imaginary Thus **the curve lies between the parallels** $y=\pm 3$.

(ii) Again from the given equation, we have

$$y=\pm \tfrac{3}{5}\sqrt{25-x^2}$$

Thus whatever value x may have, y has *two equal* and *opposite* values; hence **the curve is symmetrical about the axis of x** Moreover the greatest possible value of x is ± 5 for otherwise y would be imaginary, thus **the curve lies between the parallels** $x=\pm 5$

(iii) Hence an ellipes is a **closed curve.**

(iv) The graph of an equation of the form $ax^2+by^2=c$, where a and b have the same sign, is an ellipse

334. *Draw the graph of the equation* $9x^2-25y^2=225$.

From the given equation, we have

$$y=\pm \tfrac{3}{5}\sqrt{x^2-25}$$

Tabulate the values of x and y

Values of x	± 5	± 6	± 7	± 8	± 9	± 10	± 12	± 14	± 15
Values of y	0	± 2	$\pm 2\ 9$	$\pm 3\ 7$	$\pm 4\ 5$	$\pm 5\ 2$	$\pm 6\ 5$	$\pm 7\ 8$	$\pm 8\ 5$

Take 1 division of squared paper as the unit, and plot the points Join them by a continuous curve drawn freehand, and we have the graph as shewn in diagram [Fig 7]

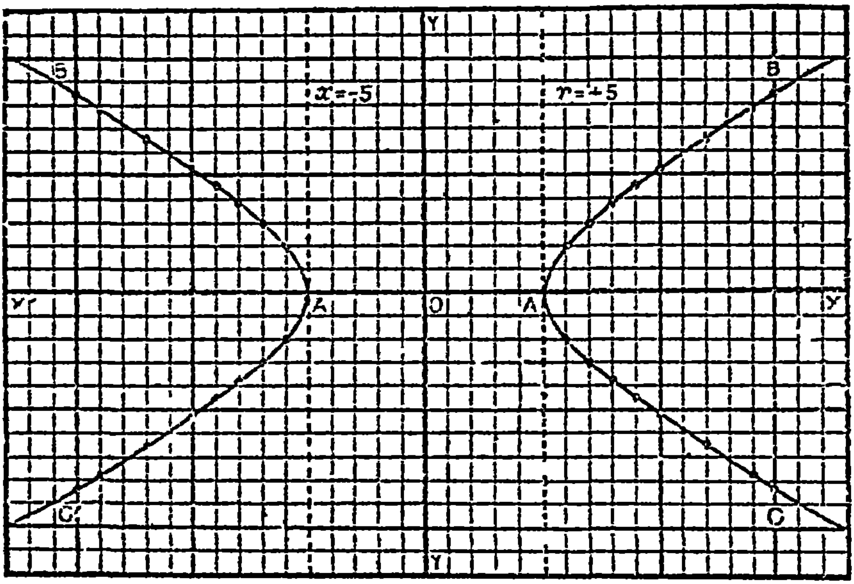


Fig 7

This curve is called a **hyperbola**

Observations on the shape of the curve

(i) The curve **consists of two branches** cutting the axis of x at the points A and A'

Also as x increases indefinitely, y too increases with limit, thus **each of the branches extends to infinity**

(ii) From the equation it is evident that any values of x between -5 and $+5$ make y imaginary Thus no portion of the curve can lie between the lines $x=5$ and $x=-5$, drawn through A and A', **one branch lying entirely on the right of $x=5$ and the other on the left of $x=-5$.**

(iii) To every abscissa a, there corresponds an abscissa $-a$, and corresponding to each of these abscissæ there is a pair of ordinates equal in magnitude and opposite in sign Thus one branch of the curve is **exactly similar** in shape to the other and each is symmetrical about the axes of x and y, and therefore **the curve itself is symmerical about these axes.**

(iv) The graph of an equation of the form $ax^2-by^2=c$, where a and b have opposite signs, is a hyperbola. If c is *positive*, the curve lies on the right and left of the y axis, if c is *negative*, it lies above and below the x-axis.

335 *Draw the graph of the equation $xy=1$.*

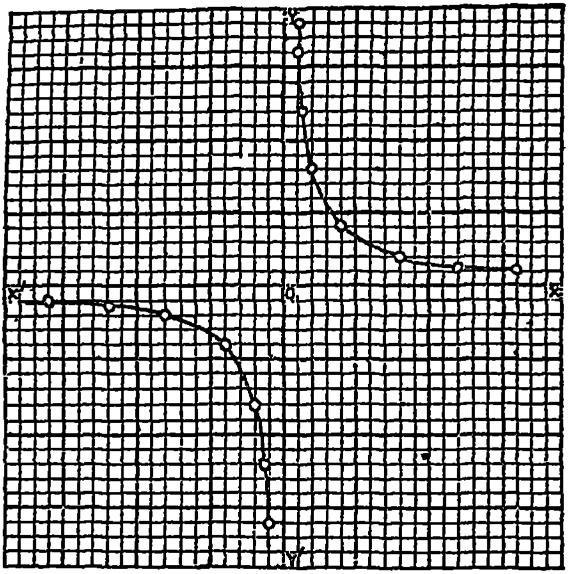


Fig 8

From the given equation, we have $y=\frac{1}{x}$

Tabulate the values of x and y

Values of x	$\pm\frac{1}{4}$	$\pm\frac{1}{3}$	$\pm\frac{1}{2}$	± 1	± 2	± 3	± 4	...
Values of y	± 4	± 3	± 2	± 1	$\pm\frac{1}{2}$	$\pm\frac{1}{3}$	$\pm\frac{1}{4}$	..

Take four times the length of the side of a square as the unit, plot these points and join them freehand Thus we have for the required graph the curve, the two branches of which are seen in the *first* and *third* quadrants

This curve is called a **rectangular hyperbola.**

Observations on the shape of the curve

(i) Since $xy=1$ as well as $(-x)(-y)=1$, we see that whenever the absscissa of any point on the curve is positive, its ordinate is also positive; and whenever the abscissa is negative, the ordinate is also negative. Thus **the curve lies in the first and third quadrants.**

(ii) Since $xy=(-x)(-y)$, we see that if (x, y) is any point on the curve, then the point $(-x, -y)$ is also on the curve. Thus the branch of the

curve in the third quadrant is **exactly similar** to the branch in the first quadrant

(iii) From $y=\frac{1}{x}$, it is evident that as x increases, y diminishes, and therefore the curve continually approaches nearer and nearer to the x-axis but never actually meets it, as y cannot be 0 for any *finite* value of x But when $x=\infty$, then $y=0$, thus the curve meets the axis of x *at infinity*

Similarly from $x=\frac{1}{y}$, it may be shewn that the curve continually approaches nearer and nearer to the y-axis but never actually meets it

Hence **the curve, though it continually approaches nearer and nearer to the axes of x and y, does not actually meet them**

(iv) The graph of an equation of the form $xy=c$, where c is constant is a hyperbola If c is *positive*, the hyperbola lies in the first and third quadrants, if c is *negative*, the curve lies in the second and fourth quadrants

Definition When a curve continually approaches nearer and nearer to a straight line but never actually meets it, the straight line is called an **asymptote** of the curve Thus each of the axes is an asymptote of the hyperbola $xy=1$ Since the asymptotes are at right angles, the curve drawn is a *rectangular hyperbola*

336 The General Equation of the second degree

To recognise an equation, as the equation of a particular graph by merely looking, at it and without actually drawing its graph, is very helpful in tracing graphs We have already noticed some particular cases We shall now give the conditions under which the general equation of the second degree in x and y, viz.,

$$ax^2+2hxy+by^2+2gx+2fy+c=0,$$

will represent (i) a circle, (ii) a parabola, (iii) an ellipse, (iv) a hyperbola and (v) a pair of straight lines.

The equation will represent—

(i) **A circle**, if $a=b$ and $h=0$, i e, if the coefficients of x^2 and y^2 are equal and the term involving xy is absent,

(ii) **A parabola**, if $h^2=ab$, i e., if the terms of the second degree form a perfect square,

(iii) **An ellipse**, if $h^2<ab$, i e, if the terms of the second degree do not break up into real factors,

(iv) **A hyperbola**, if $h^2>ab$, i e, if the terms of the second degree break up into two real factors,

(v) **A pair of straight lines**, if

$$abc+2fgh-af^2-bh^2-ch^2=0.$$

For this is the condition under which the general expression of the second degree breaks up into two linear fractors.

For example, $x^2+3xy+2y^2-4x-7y+3=0$ represents a pair of straight lines, for here

$$abc+2fgh-af^2-bg^2-ch^2$$
$$=1\cdot 2\cdot 3+2(-\tfrac{7}{2})(-2)(\tfrac{3}{2})-1(-\tfrac{7}{2})^2-2(-2)^2-3(\tfrac{3}{2})^2=0 \text{ identically}$$

Thus the given equation resolves into two linear equations, which will be found to be $x+2y-1=0$ and $x+y-3=0$

Examples CLXXVI.

Trace the graphs of

1. $y+x^2=0$ 2 $y=2x^2$ 3 $2y=x^2$ 4. $y^2=x$
5. $x+y^2=0$ 6 $-4y^2=0$ 7 $x^2+9y^2=9$ 8 $x^2+y^2=16$.
9 $x^2-y^2=1$. 10 $y^2-x^2=16$ 11 $16x^2+y^2=16$
12 $4x^2+y^2=4$ 13 $4x^2-y^2=4$ 14 $y^2-4x^2=4$
15 $2xy=1$ 16. $xy=4$ 17 $xy=16$ 18 $y=\frac{2}{x}$
19 $2x+x^2$ 20 $1+\frac{}{x}$ 21 $2x-\frac{x^2}{4}$.
22. $4x^2+9y^2=36$ 23 $4x^2-9y^2=36$ 24. $9y^2-4x^2=36$
25. $4x^2+25y^2=100$. 26 $4x^2-25y^2=100$. 27 $25y^2-4x^2=100$
28. $16x^2+9y^2=144$ 29 $16x^2-9y^2=144$ 30. $9y^2-16x^2=144$
31 $y=(1-x)(2-x)$ 32 $y^2=(1-x)(1+x)$ 33 $y^2=(x-1)(x+1)$.

337 Graphical Solution of Quadratic Equations. The following example will explain the method

Ex Solve graphically the equation $2x^2-x-6=0$

The equation may be written

$$x^2=\frac{x}{2}+3 \quad \ldots\ldots\ldots \text{(i)}.$$

Draw, as in Art 332, the graph of

$$y=x^2 \quad \ldots\ldots \text{(ii)}$$

Thus the graph is the parabola APB [The unit is twice the side of a square]

In the same diagram and with the same unit of length, draw the graph of

$$y=\frac{x}{2}+3 \quad \ldots\ldots\ldots \text{(iii)},$$

which is the straight line AB

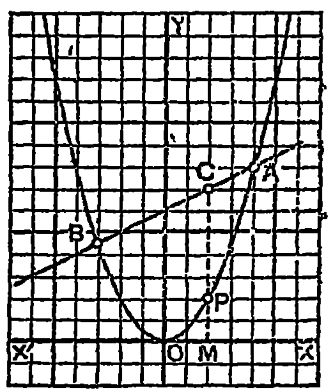


Fig. 9.

The graphs are seen to intersect at two points A and B.

Because A is on both the graphs, its co-ordinates satisfy both the equations of the graphs Hence at A the y of (ii) is the same as the y of (iii), and the x of (ii) is also the same as the x of (iii) Thus at A, the equations (ii) and (iii) hold simultaneously, and therefore

$$x^2=\frac{x}{2}+3. \quad \ldots \ldots \ldots \ldots \text{(iv)}$$

Thus the abscissa of A satisfies (iv), and (iv) is the same as (i), *i e.*, the given equation

Hence the abscissa of A (viz, 2) is a root of $2x^2-x-6=0$

Similarly the abscissa of B (viz, $-1\cdot5$) is also a root of the equation

The roots required therefore are 2 and $-1\cdot5$

Thus to solve $2x^2-x-6=0$, plot the graphs of $y=x^2$ and $y=\frac{x}{2}+3$ or $2y-x-6=0$, and *read off the abscissæ of the points of intersection* These abscissæ will be the roots required

Obs. The method adopted here is the safest for a beginner.

Note 1 To solve the general quadratic $ax^2+bx+c=0$, plot the graphs of $y=x^2$ and $ay+bx+c=0$, and read off the abscissæ of the points of intersection

Note 2 From Fig 9, we see that between the points A and B, the ordinate y_1 of any point C on the line AB is greater than the ordinate y_2, of the corresponding point P on the curve Thus y_2-y_1 is negative as long as the point C lies between A and B

Now from (ii) and (i), $y_2=x^2$ and $y_1=\frac{x}{2}+3$,

$$\therefore \quad y_2-y_1=x^2-\left(\frac{x}{2}+3\right),$$

or $$2(y_2-y_1)=2x^2-x-6;$$

i e, the expression $2x-x-6$ is negative

Hence the expression $2x^2-x-6$ is *negative* as long as any point (x, y) lies between A and B, *i e.*, for all values of x between 2 and $-1\cdot5$.

Similarly by taking any point (x, y) on the line AB beyond A and B, we shall see that the expression $2x^2-x-6$ is *positive* for all values of x beyond the limits 2 and $-1\cdot5$, *i e*, for values greater than 2 and less than $-1\cdot5$

Hence the expression $2x^2-x-6$ (i) **vanishes** when x has the values 2 and $-1\cdot5$, (ii) is **negative** when the values of x are between 2 and $-1\cdot5$, and (iii) is **positive** for all values of x greater than 2 and less than $-1\cdot5$.

Examples CLXXVII.

Solve graphically the equations

1. $x^2+x-6=0$
2. $x^2-x-6=0$
3. $2x^2+x-10=0$
4. $x^2-5x+4=0$
5. $x^2-x-4=0$
6. $2x^2-11x+12=0$
7. $2x^2-3x+5=0$
8. $2x^2-3x-7=0$.
9. $4x^2+x-5=0$
10. $3x^2-2x+1=0$
11. $2x^2-3x=0$.
12. $x^2-4x=0$
13. $3x^2-5x+2=0$
14. $x^2-2x+3=0$.

CHAPER XXIX

ARITHMETICAL PROGRESSION

338 *Definitions* A succession of quantities which are formed in order according to some fixed law is called a **series**, and the fixed law is called the **law** of the series Each of the quantities forming a series is called a **term** of the series, and the aggregate of all the terms is called the **sum** of the series

Thus 1, 3, 5, 7, 9, form a series, for each term exceeds the one preceding it by 2

Again 1, 2, 4, 8, 16, ... form a series in which each term is double of the one preceding it

The student will notice that the word 'term' is here used in a different sense from that in Art 9

A series of quantities is said to be in **Arithmetical Progression** (A P), when throughout the series the difference between any term and the next preceding term is constant

Thus 1, 3, 5, 7, 9, . . is an arithmetic seires, for $3-1=2$, $5-3=2$, $9-7=2$, &c

Hence a, b, c, d, &c, are in arithmetical progression, if

$$b-a=c-b=d-c=\&c$$

The difference between any two consecutive terms of an A P is called the **common difference** The common difference is found by subtracting any term from the term that next *follows* it

The following are other examples of arithmetic series

$$1, 2, 3, 4, 5, \ldots\ldots$$

$$8, 5, 2, -1, -4, -7, \ldots$$

$$a, a+b, a+2b, a+3b, \ldots\ldots$$

$$a, a-2d, a-4d, a-6d, \ldots\ldots$$

in which the common differences are respectively 1, -3, b and $-2d$

From the definition, the two following conclusions are obvious

Note 1 *If each of the terms of an A P be increased or decreased by a constant quantity, the resulting quantities are in A P with the same common difference as before*

Note 2 *If each of the terms of an A P be multiplied or divided by a constant quantity, the resulting quantities are in A P with a new common difference*

339 Formation of Terms If a be the first term and d the common difference of an A. P, then by definition

$$\begin{aligned}
&\text{the 2nd term} = a + d = a + (2-1)d, \\
&\text{,, 3rd.} \quad = a + 2d = a + (3-1)d, \\
&\text{,, 4th.} \quad = a + 3d = a + (4-1)d,
\end{aligned}$$

and so on, the coefficient of d in any term being *one* less than the number of the term in the series Thus

$$\text{the } n\text{th term} = a + (n-1)d \quad \ldots\ldots \quad (\text{i})$$

Hence *we can write down any term of an* A P, *when the first term and the common difference are given*

Thus if the first term $= 1$ and the common difference $= 3$,

$$\begin{aligned}
&\text{the 5th term} \quad = 1 + (5-1)3 = 1 + 4 \times 3 = 13, \\
&\text{the 100th term} = 1 + (100-1)3 = 1 + 99 \times 3 = 298
\end{aligned}$$

Again, if l denotes the last term and n the number of terms of an A P, whose first term is a and common difference d, then from (i)

$$l = a + (n-1)d \quad \ldots\ldots \quad (\text{ii})$$

Examples CLXXVIII

1. The first term of an A P is 8 and the common difference is 4, find its 10th, 15th and 54th terms

2 The first term of an A P is 50 and the common difference is -5, find its 5th, 11th and 100th terms

3 The first term of an A P is -114 and the common difference is 6; find its 20th, 112th and 150th terms

4 Find the 50th term of the series 15, 22, 29, .

5. Find the 100th term of the series 48, 37, 26,

6 Find the 25th term of the series $-3, -5, -7, -9, \ldots$

340 Any two terms of an A P determine the series. An arithmetical progression is determined when *any two* of its terms are given. The method will be seen from the following example

Ex Find the A P whose 5th term is 7 and whose 17th term is -2 Find also its 76th term

Let a be the first term and d the common difference

Thus the 5th term $=a+4d$ and the 17th term $=a+16d$

Hence
$$a+4d=7,$$
$$a+16d=-2$$

Solving these equations, we get $a=10$ and $d=-\frac{3}{4}$

Thus the series is $10, 9\frac{1}{4}, 8\frac{1}{2}, 7\frac{3}{4}, 7, 6\frac{1}{4}, \ldots\ldots$.

Also the 76th term $=10+75(-\frac{3}{4})=-46\frac{1}{4}$

Examples CLXXIX.

1 The first two terms of an A P are 8 and 15, find the 15th and nth terms

2 Find the 10th term of the A. P whose first and 50th terms are 5 and 348 respectively

3 What is the 15th term of the A. P whose first term is 42 and whose 12th term is 9 ?

4 The sum of the fifth and twelfth terms of an A P is 81, and the sum of the eighth and fifteenth terms is 111, find the first term

5 Find the series in A P in which 7 and 5 are the 5th and 7th terms respectively What is its 12th term ?

6 Find the tenth term of a series in A P whose 3rd and 20th terms are respectively $-1\frac{5}{8}$ and $-21\frac{2}{3}$ Find also the series

7. Find the sum and difference of the 80th and 150th terms of an A P whose 50th and 100th terms are respectively 20 3 and 34 65

8 If the mth term of an A P is n and the nth term is m, find the $(m+n)$th and $(m+n-1)$th terms

9 If the mth term of an A P be $p+q$ and the nth term be $p-q$, find the common difference

10 Shew that the sum of the $(m-n)$th and $(m+n)$th terms of an A P is equal to twice the mth term

341 General Term The rth term of a series when written as *a function of r* is called the **general term** of the series

The general term of a series being given, all its terms can be found, by giving different values to r, and thus the whole series can be determined.

Ex 1. Find the series whose rth term is $2r-1$.

Put $r=1$, thus the 1st term $=2-1=1$,

..... $r=2$, ... 2nd . $=4-1=3$,

. $r=3$, . . 3rd ... $=6-1=5$,

and so on. Hence the series is 1, 3, 5, 7, 9, 11, 13, . . .

Ex 2 Which term of the series 2, 7, 12, 17, . . . is 247 ?

Let the nth term of the series be 247

The first term $=2$ and the common difference $=7-2=5$

Hence the nth term $=2+(n-1)5=5n-3$

$\therefore$ $5n-3=247$, whence $n=50$

Thus the 50th term of the series $=247$

Examples CLXXX

1. Find the 1st term of the series whose rth term is $2r+1$.
2. .. 5th . . .. mth . . $m+5$
3. . . 12th . . . . . . nth . . $4n+1$
4. . 20th ... . rth . . . $3r-5$.
5. . . .15th . . . .. . nth .. ... $3n-1$.
6. . . .. . 13th ... . nth . . $\frac{4n-7}{5}$
7. . .. . 11th . . . . $(n+1)$th . $2n-3$.
8. .... 8th . . .. . . $(m+2)$th $9m-5$
9. ... 7th . .. . $(n+3)$th .. $3n-4$.
10. . ... . 2nd .. . . $(m+3)$th $m+3$.
11. . . . . 2nd . . .. . $(n+2)$th $5n+1$.
12. . 1st . . . $(n+5)$th $2n-1$.
13. Find the common difference of the series in Examples 1, 4, 6, 7, 10 and 12
14. Find the series in Examples 2, 3, 5, 8, 9 and 11
15. Which term of the series 3, 11, 19, 27, . .. is 155 ?
16. Which term of the series 15, 13, 11, 9, . . is -83 ?
17. If the 4th and 13th terms of an arithmetic series are 6 and -21 respectively, which term of the series is 0 ?

342. Sum of an A P To find the sum of n terms of an A. P. having given the first term and the common difference

Let a be the first term, d the common difference, l the last term and S the sum of the n terms.

Then $S=a+(a+d)+(a+2d)+\ldots+(l-2d)+(l-d)+l \ldots\ldots\ldots (a)$.

Also by writing the same series in the reverse order, we get

$$S=l+(l-d)+(l-2d)+\ldots+(a+2d)+(a+d)+a \ldots\ldots (b)$$

Hence by adding the corresponding terms, we get

$$2S=(a+l)+(a+l)+(a+l)+\ \text{to } n \text{ terms}=n(a+l) \ \ldots (c)$$

$$\therefore S=\frac{n}{2}(a+l) \quad \ldots\ldots (\text{i})$$

Again since l is the last or nth term, we have by Art 339

$$l=a+(n-1)d \quad \ldots\ldots (\text{ii})$$

Hence using (ii), we have

$$S=\frac{n}{2}\left\{2a+(n-1)d\right\} \quad \ldots\ldots (\text{iii})$$

From (iii), it is clear that any *three* of the four quantities a, d, n and S being given, we can find the fourth

The result (i) is expressed in words thus —*The sum of a series of quantities in A P is equal to the sum of the first and last terms into half the number of terms*

Ex 1 Find to 20 terms the sum of the series

$$5+17+29+41+$$

Here $a=5$, $d=17-5=12$, $n=$ C

Thus sum required $=\frac{20}{2}\{2\times 5+(20-1)12\}$ from (iii)

$$=10(10+228)=2380$$

Ex. 2 Sum to 35 terms the series $8\frac{3}{4}$, $7\frac{1}{2}$, $6\frac{1}{4}$, 5, .

Here $a=8\frac{3}{4}$, $d=7\frac{1}{2}-8\frac{3}{4}=-1\frac{1}{4}$, $n=35$

sum required $=\frac{35}{2}\{2\times 8\frac{3}{4}+(35-1)(-1\frac{1}{4})\}$

$$=\frac{35}{2}(\frac{35}{2}-\frac{85}{2})=-437\frac{1}{2}$$

Ex 3 Find the sum of the first n natural numbers

$$1, 2, 3, 4, 5,$$

Sum required $=1+2+3+4+\ \ldots\ +n$

$$=\frac{n}{2}(1+n)=\tfrac{1}{2}n(n+1) \text{ [Art 342, (i)]}$$

Ex 4. Shew that the sum of the first n consecutive odd numbers is a square number

The first n consecutive odd numbers are 1, 3, 5, 7, 9, 11, ...

Here $a=1$ and $d=2$, thus

sum required $=\frac{n}{2}\{2\times 1+(n-1)2\}=\frac{n}{2}\times 2n=n^2$

Ex 5. Sum to n terms the series whose r^{th} term is $3r-1$

Since the r^{th}, or general term, is $3r-1$, put $r=1$, thus

$$\text{the first term} = 3\times 1 - 1 = 2$$

Similarly by putting $r=n$, the nth term $=3n-1$.

Hence required sum $=\frac{n}{2}\{2+(3n-1)\}=\frac{n}{2}(3n+1)$

Otherwise —As above, put r equal to 1 and 2 successively, thus the first term $=2$, and the second term $=5$,

$$\text{common difference} = 5-2=3$$

Hence sum required $=\frac{n}{2}\{4+(n-1)3\}=\frac{n}{2}(3n+1)$

Examples CLXXXI

1 Sum to 15 terms 6, 9, 12, 15, .

2 Sum to 16 terms 15, $13\frac{1}{2}$, 12, $10\frac{1}{2}$,

3 Sum to 20 terms -7, -3, 1, 5, .

4 Sum to 18 terms 1, $2\frac{1}{4}$, $3\frac{1}{2}$, $4\frac{3}{4}$,

5 Sum to 24 terms $\frac{1}{2}$, $-\frac{3}{4}$, -2, $-3\frac{1}{4}$, ..

6 Sum to 9 terms $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{6}$,

7 Sum to 21 terms $\frac{1}{2}$, $-\frac{2}{3}$, $-\frac{11}{6}$, .

8. Sum to 40 terms 117, 111, 105,

9 Sum to 19 terms $-\frac{3}{4}$, $-\frac{2}{3}$, $-\frac{7}{12}$,

10 Sum to 12 terms $2a-b$, $2a+b$, $2a+3b$, . .

11 Sum to 11 terms $5m-6$, $4m-5$, $3m-4$,

12 Sum to n terms 3, 9, 15, 21,

13 Sum to n terms -20, -24, -28, .

14 Sum to n terms $3\frac{1}{3}$, $2\frac{1}{2}$, $1\frac{2}{3}$, .

15. Sum to n terms $\frac{n-1}{n}$, $\frac{n-2}{n}$, $\frac{n-3}{n}$, .

16 Sum to 15 terms the series whose n^{th} term is $3n-1$

17 Sum to 20 terms the series whose r^{th} term is $3r-5$

18 Sum to 50 terms the series whose $(n+1)$th term is $2n-3$.

19 Sum to n terms the series whose rth term is $2r+1$

20 Sum to n terms the series whose $(m+2)$th term is $9m-5$.

21 Sum to n terms the series whose $(n+3)$th term is $3n-4$

22 The nth term of an A P is $\frac{1}{6}(3n-1)$, prove that the sum of n terms is $\frac{n}{12}(3n+1)$, and find the series.

343 Theorem *In an A P, the sum of any two terms equidistant from the beginning and end is equal to the sum of the first and last terms.*

This follows at once from (c) [Art 342], for each of the terms $(a+l)$ in (c) is the sum of the first and last terms and is obtained by adding the corresponding terms in (a) and (b), i e, the pair of terms that are equidistant from the beginning and the end

Otherwise :—The mth term from beginning $=a+(m-1)d$. The mth term from end is the $(n-m+1)$th term from beginning. Hence the mth term from end $=a+(n-m)d$

Thus their sum $=2a+(n-1)d=a+\{a+(n-1)d\}$;

i e, sum of first and last terms.

344 Theorem *In an A P consisting of an odd number of terms, twice the middle term is equal to the sum of the first and last terms*

Let g be the middle term and f and h the preceding and succeeding terms. Then by definition

$$g-f=h-g, \text{ or } 2g=f+h=a+l \text{ [Art 343]}$$

Cor Hence the sum of the series to n terms

$$=\frac{n}{2}(a+l)=\frac{n}{2}\times 2g=gn$$

Thus the *sum of an A P of odd number of terms is equal to the middle term into the number of terms*

345 The theorems of the last two articles enable us to find rather easily the sum (i) of an A P containing an *even* number of terms if the two middle terms are given, or (ii) of an A P. containing an *odd* number of terms if the middle term is given.

Ex 1. Find the sum of $1+\frac{5}{4}+\frac{3}{2}+\frac{7}{4}+2+$ to 8 terms.

Here the two middle terms are $\frac{7}{4}$ and 2, hence the required sum

$$=\frac{8}{2}(\tfrac{7}{4}+2)=7+8=15$$

Ex 2 Sum to 7 terms the series $5+\frac{9}{2}+4+\frac{7}{2}+$.

Required sum $=\frac{7}{2}\times 7=\frac{49}{2}=24\frac{1}{2}$

346 Inverse Questions. If any *three* of the five quantities a, d, n, l and S are given, we can find the remaining *two* from the Formulæ (i), (ii) and (iii) of Art 342

Ex 1. The sum of 30 terms of an A P. whose first term is 4, is -1185 Find the common difference and the last term

Here $a=4$, $n=30$, $S=-1185$, to find d and l.

We have $S=\frac{n}{2}\{2a+(n-1)d\}$[Art. 342 (iii)],

$$i.e., \quad -1185=\tfrac{30}{2}\{2\times 4+29d\},$$

whence $\qquad 15\times 29d=-1305$, or $d=-3$

From Art 342, (ii), $l=a+(n-1)d=-83$.

Ex 2. The first term, common difference and sum of an A.P are 3, 2 and 255 respectively, find the number of terms

We have $S=\frac{n}{2}\{2a+(n-1)d\}$, i e, $255=\frac{n}{2}\{6+(n-1)2\}$.

Thus we have a *quadratic* equation to determine n.

When reduced, the equation becomes

$$n^2+2n-255=0,$$

or $$(n-15)(n+17)=0,$$

whence $$n=15 \text{ or } n=-17$$

Thus the required number of terms is 15

Note. We have taken the value 15, for a number of terms must necessarily be a *positive integer* The value -17 is to be rejected, for though -17 is a *root* of the equation, it is not a *solution* of the problem, as *negative number of terms* has no meaning [See Art 313]

Ex 3 Of how many terms of the series $16+14+12+\ldots$will the sum be 60?

Here $a=16$, $d=-2$ and $S=60$.

Hence $$60=\frac{n}{2}\{32-(n-1)2\};$$

or $$n^2-17n+60=0,$$

whence $$(n-5)(n-12)=0, \qquad n=5 \text{ or } n=12$$

Note Here both values of n are *positive integers* and must therefore be accepted In fact both values satisfy the condition of the problem, for the series to 5 terms is $16+14+12+10+8$, and to 12 terms is $16+14+12+10+8+6+4+2+0-2-4-6$, the sum of each of which is 60

Ex 4 How many terms of the series $25+21+17+\ldots$ must be taken that the sum may be 85?

Here $a=25$, $d=-4$ and $S=85$, therefore

$$85=\frac{n}{2}\{50-(n-1)4\},$$

or $$2n^2-27n+85=0,$$

whence $$(n-5)(2n-17)=0,$$

$$n=5 \text{ or } n=8\tfrac{1}{2}$$

Note Here the value $8\frac{1}{2}$ is to be rejected as it is not an *integer* But $n=8\frac{1}{2}$ shews that n lies between 8 and 9, hence 85 lies in value between the sums of 8 and 9 terms of the series, *i e*, 85 is greater than one of the sums but less than the other as the student can easily verify

Examples CLXXXII

1 The 4th term of an A P is 10, and the 7th term is 19, of how many terms is the sum 590 ?

2 What is the last term of the series $12+15+18+$. when its sum is 342 ?

3 The sum of 10 terms of an A P is -2 and the common difference is $-\frac{2}{5}$, find the first and last terms

4 Find the sum of 25 consecutive odd numbers of which the last is 123

5 What is the common difference of an A P whose first and last terms are 1 and 41 and sum 378 ?

6 How many terms of the series $31+27+23+$ will amount to 133 ?

7 The sum of a certain number of terms of the series $21+19+17+$ is 120 Find the last term and the number of terms

8 The common difference of an A P is $-\frac{1}{3}$ and its sum to 7 terms is $30\frac{1}{2}$, find the last term

9 The sum of an A P is 136, the common difference is 4 and the last term is 31 Find the first term and the number of terms

10 The sum of 6 terms of an A P is 6, and the sum of 14 terms is 126, find the series

347 Definition If three quantities are in A P, the middle one is called the **arithmetic mean** of the other two

If any number of quantities are in A P, all the terms intermediate between the first and last are called the **arithmetic means** of the two extreme terms

Thus in the A P 1, 2, 3, 4, 5, 6 and 7, the terms 2, 3, 4, 5 and 6 are the *arithmetic means* of 1 and 7

348 Arithmetic Means

If three quantities a, A and c are in A P, we have

$$A-a=\text{common difference}=c-A, \qquad A=\tfrac{1}{2}(a+c)$$

Thus *the arithmetic mean of two given quantities is half the sum of the two quantities*

349 *To insert* n **arithmetic means** *between two given quantities* a *and* c

Here the n arithmetic means together with a and c will evidently form an A P of $n+2$ terms, the first term of which is a and the last or $(n+2)$th term is c

Hence if d be the common difference, we have

$$c=a+(n+1)d\,,$$

whence $$d=\frac{c-a}{n+1}.$$

Thus the series is

$$a,\ a+\frac{c-a}{n+1},\ a+2\frac{c-a}{n+1},\ a+3\frac{c-a}{n+1},\ \ldots$$

and the required means are

$$a+\frac{c-a}{n+1},\ a+2\frac{c-a}{n+1},\ a+3\frac{c-a}{n+1},\ \ldots,\ a+n\frac{c-a}{n+1},$$

or $$\frac{na+c}{n+1},\ \frac{(n-1)a+2c}{n+1},\ \frac{(n-2)a+3c}{n+1},\ \ldots,\ \frac{a+nc}{n+1}.$$

Cor Hence if *two* means x and y are inserted between a and c, then $$x=\tfrac{1}{3}(2a+c),\ y=\tfrac{1}{3}(a+2c)$$

Ex To insert 5 arithmetic means between 15 and 29

Here 15 is the first term and 29 the last term of an A P of $5+2$ or 7 terms Thus if d=the common difference, the required means are $$15+d,\ 15+2d,\ 15+3d,\ 15+4d,\ 15+5d\,,$$
and they are known if d is found

Since 29 is the last or 7th term, we find d from the equation

$$29=15+(7-1)d=15+6d,$$

whence $$d=2\tfrac{1}{3}$$

Hence the means are $17\frac{1}{3}$, $19\frac{2}{3}$, 22, $24\frac{1}{3}$, and $26\frac{2}{3}$

Examples CLXXXIII.

1 Find the arithmetic mean of (i) 10 and 15, (ii) -14 and 4, (iii) $\frac{1}{2}$ and $\frac{1}{3}$, (iv) $a+b$ and $a-b$, (v) $a-2b$ and $2a-b$, (vi) $(x+y)^2$ and $-(x-y)^2$

2 Insert 7 arithmetic means between 1 and 41

3 Insert 8 arithmetic means between 7 and 13

4 Insert 17 arithmetic means between 5 and -22

5. Insert 11 arithmetic means between -1 and 2, and find their sum

6 Insert 10 arithmetic means between $5x-6y$ and $5y-6x$

350 Notation We shall henceforth denote by $t_1, t_2, t_3, \quad , t_n$, the first, second, third, , nth terms respectively of a series, and by $s_1, s_2, s_3, \quad , s_n$, the sum of 1, 2, 3, , n terms respectively.

We work out below some **Miscellaneous Examples**

Ex 1 Find the sum of 39 terms of an A P whose 20th term is 15

Let a=the first term and d=the common difference, then

$$t_{20}=a+19d = 15 \quad . . \quad ...... \quad . .. \quad .(A),$$

$$S_{39}=\tfrac{39}{2}\{2a+38d\}=39(a+19d)=39\times 15 \text{ [from } (A)]=585$$

Ex 2 Sum the series $1+8+15+ \quad . \quad +6994$

The com diff is 7 If the series consists of n terms, then the nth term is the last term, thus

$$6994=1+(n-1)7, \text{ whence } n=1000$$

$$\text{sum required}=\tfrac{1000}{2}(1+6994)=3497500$$

Ex 3 If $2n+n^2$ represent the sum of n terms of an A P find the 20th term of the series

The 20th term is the *difference* between the sums of 20 and 19 terms

We have $\quad S_{20}=2\times 20+(20)^2$, by putting $n=20$,

and $\quad S_{19}=2\times 19+(19)^2, \quad n=19$

Hence $\quad t_{20}=S_{20}-S_{19}=2(20-19)+(20)^2-(19)^2$

$$=2+(20+19)(20-19)=2+39=41$$

Otherwise —Put $n=1$, thus $s_1=2+1=3$, $\quad t_1=3$

Put $n=2$, thus $s_2=4+4=8$, $\quad t_2=s_2-s_1=5$

Thus the first term is 3 and the second term is 5, hence the common difference$=5-3=2$

$$t_{20}=3+19\times 2=41$$

Ex 4 If $2n+n^2$ represent the sum of n terms of an A P, find the nth term of the series

This is the same as Ex 3 put in the general form

Let S_n denote the sum of n terms, then will S_{n-1} denote the sum of $n-1$ terms. Then the nth term of the series is S_n-S_{n-1}

Hence $t_n = S_n - S_{n-1} = 2n + n^2 - \{2(n-1) + (n-1)^2\} = 2n + 1$

Note This method enables us to find **any term** when the sum of n terms is given as a function of n Also the nth term found being a function of n, the **whole series can be found** [Art 341]

Ex. 5 If a, b, c be respectively the pth, qth, and rth terms of an A P, shew that

$$a(q-r)+b(r-p)+c(p-q)=0$$

Let α=the first term, and β=the common difference of the A P;

then $$p\text{th term} = \alpha + (p-1)\beta = a \quad \ldots\ldots \quad (\text{i}),$$
$$q\text{th term} = \alpha + (q-1)\beta = b \quad \ldots\ldots \quad (\text{ii}),$$
$$r\text{th term} = \alpha + (r-1)\beta = c \quad \ldots\ldots \quad (\text{iii})$$

As the given relation does not contain α and β, it is evident that it will be obtained by eliminating the two quantities α and β from the three equations (i), (ii) and (iii)

Subtract (ii) from (i), and (iii) from (ii), thus

$$(p-q)\beta = a-b,$$

and $$(q-r)\beta = b-c,$$

hence $$(a-b)(q-r)\beta = (b-c)(p-q)\beta$$

Divide by β, and bracket the co-efficients of a, b and c, thus

$$a(q-r)+b(r-p)+c(p-q)=0$$

Ex 6 The sum of m terms of an A P is n, and the sum of n terms is m, shew that the sum of $(m+n)$ terms is $-(m+n)$, and the sum of $(m-n)$ terms is $(m-n)\left(1+\dfrac{2n}{m}\right)$

Let a=the first term and d=the common difference, then the sum of m terms is n, we have

$$\frac{m}{2}\{2a+(m-1)d\}=n$$

Similarly $$\frac{n}{2}\{2a+(n-1)d\}=m$$

Thus $$2a+(m-1)d=\frac{2n}{m} \quad \ldots\ldots \quad (\text{i}),$$

and $$2a+(n-1)d=\frac{2m}{n} \quad \ldots\ldots \quad (\text{ii})$$

From (i) and (ii) by subtraction and division by $m-n$, which is not 0, we get $$d=-\frac{2(m+n)}{mn} \quad \ldots\ldots \quad (\text{iii})$$

$$\text{Now } S_{m+n}=\frac{m+n}{2}\{2a+(m+n-1)d\}$$

$$=\frac{m+n}{2}\{2a+(m-1)d+nd\}$$

$$=\frac{m+n}{2}\left\{\frac{2n}{m}-\frac{2(m+n)}{m}\right\} \text{ from (i) and (iii),}$$

$$=\frac{m+n}{2}\left(-\frac{2m}{m}\right)=-(m+n)$$

$$\text{Also } S_{m-n}=\frac{m-n}{2}\{2a+(m-n-1)d\}=\frac{m-n}{2}\{2a+(m-1)d-nd\}$$

$$=\frac{m-n}{2}\left\{\frac{2n}{m}+\frac{2(m+n)}{m}\right\} \text{ from (i) and (iii),}$$

$$=\frac{m-n}{2}\,2\left(1+\frac{2n}{m}\right)=(m-n)\left(1+\frac{2n}{m}\right).$$

REMARK. If we had obtained a and d from (i) and (ii), and then found *the required sum, the work would have been long and tedious*

Ex 7 There are two series in A P, the sums of which to n terms are as $13-7n$ $3n+1$ Prove that their first terms are as 3 2, and their second terms as -4 5

Let a, a' be the first terms and d, d' the common differences of the two series respectively

$$\text{Thus} \qquad \frac{\frac{n}{2}\{2a+(n-1)d\}}{\frac{n}{2}\{2a'+(n-1)d'\}}=\frac{13-7n}{3n+1},$$

$$\text{or} \qquad \frac{2a+(n-1)d}{2a'+(n-1)d'}=\frac{13-7n}{3n+1}.$$

This being an identity holds for any value of n.

Let $n=1$, thus $\frac{2a}{2a'}=\frac{6}{4}$, or $\frac{a}{a'}=\frac{3}{2}$

Let $n=3$, thus $\frac{2a+2d}{2a'+2d'}=\frac{-8}{10}$, or $\frac{a+d}{a'+d'}=\frac{-4}{5}$

Ex 8 If a series of terms in A P, be collected into groups of n terms, and the terms in each group be added together the results form an A P whose com diff the original com diff as n^2 1.

Let a=the first term and d=the com diff of the series; then the sum of the first group of n terms

$$P=\frac{n}{2}\{2a+(n-1)d\}$$

The second group begins with the $(n+1)$th term; thus the first term of this group $=a+nd$, and therefore the sum of the second group of n terms

$$Q=\frac{n}{2}\{2(a+nd)+(n-1)d\}$$

$$=\frac{n}{2}\{2a+(n-1)d\}+n^2d \quad \ldots\ldots \quad \text{(ii)}.$$

The third group begins with the $(2n+1)$th term, and therefore its first term $=a+2nd$, thus the sum of the third group of n terms

$$R=\frac{n}{2}\{2(a+2nd)+(n-1)d\}$$

$$=\frac{n}{2}\{2a+(n-1)d\}+2n^2d. \quad \ldots\ldots \quad \text{(iii)}$$

And so on

From (i), (ii) and (iii), we see that

$$Q-P=n^2d=R-Q$$

Hence $P, Q, R, \ldots$ are in A P, and their common difference is n^2d

com. diff of new series original com diff. $=n^2d \quad d=n^2 \quad 1$

Ex 9 Three numbers are in A P, their sum is 24 and the sum of their square is 242 Find the numbers

Let $a-\beta$, and $a+\beta$ be the numbers We can make this assumption, as $a-\beta$, a and $a+\beta$ are in A P

$$(a-\beta)+a+(a+\beta)=24,$$

whence $$a=8 \quad \ldots\ldots \quad \text{(i)}$$

Also $$(a-\beta)^2+a^2+(a+\beta)^2=242,$$

or $$3a^2+2\beta^2=242,$$

from (i), $$\beta=\pm 5. \quad \ldots\ldots \quad \text{(ii)}.$$

Thus from (i) and (ii), the numbers are 3, 8, 13, or 13, 8, 3

Examples CLXXXIV

1. Find the sum of 53 terms of an A P whose 27th term is 11

2 If the sum of 31 terms of an A P is 372, what is the 16th term?

3. The sum of the first 8 terms of an A P is 8 and the 9th term is 10, find the series

4 Shew that the sum of any number of terms of the series 4, 12, 20, 28, . . . is the square of an even number.

5 Shew that if unity be added to the sum of any number of terms of the series $8+16+24+ \ldots$, the result is the square of an odd number

6 Find the sum of the series $-25-21-17-\ldots+131$

7 The $(p+q)$th term of an A P is m and the $(p-q)$th term is n, shew that the pth term is $\frac{1}{2}(m+n)$ and the qth term is $m-(m-n)\dfrac{p}{2q}$

8 If $4n^2+3n$ represent the sum of n terms of an A P, find its 10th term

9 If $5n-n^2$ represent the sum of n terms of an A P, find the series

10 If $pn+qn^2$ represent the sum of a series to n terms, find the mth term and also the series

11 The first term of an A P is unity and the sum of the first 6 terms the sum of the next 6 terms $=11$ 35 Find the series

12 If the sum of the first p terms of an A.P is 0, then the sum of the next q terms is $-\dfrac{aq(p+q)}{p-1}$, a being the first term

13 If the sum of m terms of an A P be always to the sum of n terms in the ratio of m^2 n^2, and the first term be unity, shew that

the mth term the nth term as $2m-1$ $2n-1$.

14 The $(n+1)$th term of a series in A P is $\dfrac{ma-nb}{a-b}$, find the sum of the series to $(2n+1)$ terms

15 If a^2, b^2, c^2 are in A P, then $\dfrac{1}{b+c}, \dfrac{1}{c+a}, \dfrac{1}{a+b}$ are in A P

16 If the mth term of an A P is n and the nth term is m, of how many terms will the sum be $\frac{1}{2}(m+n)(m+n-1)$, and what will be the last of them ?

17 If from a series of n terms in A P another series be formed such that each term is the arithmetic mean between the consecutive terms of the original series, and if s, s' be the sums of these series respectively, prove that s . $s'=n$ $n-1$

18 If s_1, s_2, s_3 be the sums of n, $2n$ and $3n$ terms respectively of an A P, shew that $s_3=3(s_2-s_1)$

19 If $s_1, s_2, s_3,$ s_m are the sums of m arithmetical progressions to n terms, whose first terms are 1, 2, 3, . . . and common differences are 1, 3, 5, , shew that

$$s_1+s_2+s_3+\ldots+s_m=\frac{mn}{2}(mn+1)$$

20 Sum the series $1-3+5-7+\ldots$ to $2n$ terms and to $2n+1$ terms

21 Sum the series $1-2+3-4+\ldots$ to $2n$ terms and to $2n+1$ terms

22 Sum the series $(a+b)+(a-2b)+(a+3b)+(a-4b)+\ldots$ to 40 terms

23 Three numbers are in A P, the difference of the first and third is 14 and their product is 312, find the numbers

24 Four numbers whose sum is 32 are in A P, and the product of the first and second is less than the product of the other two by 64, what are the numbers?

[Let the numbers be $a-3\beta$, $a-\beta$, $a+\beta$, $a+3\beta$]

25 There are 5 numbers in A P, such that the product of the extreme terms is 297, and the sum of all the others is 63 Find the numbers.

[Let the numbers be $a-2\beta$, $a-\beta$, a, $a+\beta$, $a+2\beta$]

26 The sides of a right-angled triangle are in A P, shew that they are proportional to the numbers 3, 4, 5

27 A man arranges to pay off a debt of £3600 by 40 annual instalments which form an A P When 30 instalments are paid he dies, leaving a third of the debt unpaid. Find the value of the first instalment

28 One hundred stones are placed in a straight line at intervals of 2 ft. from each other How far does a man walk in gathering them up, supposing that he fetches each stone singly and deposits it in a basket which is in the same straight line produced 20 yards distant from the nearest stone and that he starts from the basket?

CHAPTER XXX

GEOMETRICAL PROGRESSION

351 *Definitions* A series of quantities is said to be in **Geometrical Progression** (G P) when, throughout the series, the ratio of any term to the next preceding term is constant

The ratio of any term to the next preceding term of a G P is called the **common ratio** The common ratio is obtained by dividing *any term* by the one that *next precedes* it, for each term is obtained by *multiplying the preceding term by the common ratio*

Thus each of the following series is in G P.

$1, 3, 9, 27, 81,$

$1, -2, 4, -8, 16, \ldots$

$\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16},$

$a, ar, ar^2, ar^3, ar^4,$

In the first series the common ratio is 3, in the second it is -2, in the third $\frac{1}{2}$, and in the fourth r

Note From the definition it is obvious that

(i) *If each of the terms of a* G P *be multiplied or divided by a constant quantity, the resulting quantities will be in* G P *with the same common ratio as before*

(ii) *If each of the terms of a* G P *be raised to any the same power, the resulting quantities will be in* G P *with a new common ratio*

352 Formation of Terms If a is the first term and r the common ratio of a G P, then by definition

the second term $= ar = ar^{2-1}$,

" third $= ar^2 = ar^{3-1}$,

" fourth $= ar^3 = ar^{4-1}$,

and so on, the index of r in any term being *one* less than the number of the term Thus

$$\text{the } n\text{th term} = ar^{n-1}$$

If therefore a G P consists of n terms, the nth term is the last term, thus of this series the

$$\text{last term} = ar^{n-1}$$

Hence *any term of a* G P *can be written down when the first term and the common ratio are given*

Thus if the first term of a G P is 3 and the common ratio is 2, then the 5th term $= 3 \cdot 2^{5-1} = 48$,

and the 100th term $= 3 \cdot 2^{100-1} = 3 \cdot 2^{99}$

Examples CLXXXV

1 The first term of a G P is 2 and the common ratio is 3, find the 10th term

2 Find the 5th term of a G P whose first term is $2\frac{1}{2}$ and common ratio is $\frac{3}{5}$

3 Find the 8th term of the series $1, -2, 4, -8, \ldots$.

4 Find the 11th term of the series $\frac{4}{5}, -\frac{1}{2}, \frac{1}{5}, \ldots$

5 If n be an even number, find the $(n-1)$th and $(n+1)$th terms of the series $-\frac{2}{3}, \frac{1}{2}, -\frac{3}{8}, \ldots$

353 Any two terms of a G P determine the series The following examples will illustrate the method.

Ex 1. Find the G P whose first and second terms are $\frac{3}{4}$ and $-\frac{1}{2}$.

The common ratio $= -\frac{1}{2} \div \frac{3}{4} = -\frac{2}{3}$

Hence the series is $\frac{3}{4}, -\frac{1}{2}, \frac{1}{3}, -\frac{2}{9}, \ldots$

Ex 2 Find the series in G P whose 5th term is 48 and 8th term is 384

Let a = the first term, and r = the common ratio.

Thus the 5th term $= ar^{5-1} = ar^4$, and the 8th term $= ar^{8-1} = ar^7$,

$\therefore$ $ar^4 = 48$, and $ar^7 = 384$,

hence $ar^7 \div ar^4 = 384 \div 48$, or $r^3 = 8$, i e, $r = 2$,

and therefore from $ar^4 = 48$, we get $a = 3$

Thus the series is $3, 6, 12, 24, 48, \ldots$

Examples CLXXXVI

1 The first term of a G P is 2 and the common ratio is 3, find the 6th term

2 Find the 20th term of a G P whose 4th term is 8 and 8th term is 128.

3 Find the G P whose first and second terms are $\frac{3}{4}$ and $\frac{1}{2}$

4 The third and sixth terms of a series in G P. are 3 and 81 respectively, find the first term and the common ratio

5 The third term of a G P is 1 and the 6th term is $-\frac{1}{8}$, find the 10th term

354 General Term If the *general term* [see Art. 341] of a G P be given as a function of r, all the terms of the series and therefore the series itself can be found by giving different values to r

Ex Find the G P whose rth term is 3^r What is its 8th term?

Put $r=1$, thus the 1st term $= 3^1 = 3$,

" $r=2$, ... " 2nd " ... $= 3^2 = 9$,

" $r=3$, ... 3rd ... $= 3^3 = 27$, and so on

Hence the series is $3, 9, 27, \ldots$

Thus the com ratio $= 9 \div 3 = 3$, and the 8th term $= 3.3^7 = 3^8 = 6561$.

Examples CLXXXVII.

1. Find the 4th term of a series whose nth term is 5^n
2. ... 5th ... ... 2^n-3.
3. ... $(n-2)$th ... ... 4^n-1.
4. ... 1st term of a series and also the series whose $(n+1)$th term is 3^n
5. ... the series whose nth term is $3+(-2)^n$.

355 Sum of a G P. To find the sum of n terms of a Geometrical Progression, having given the first term and the common ratio

Let a be the first term, r the common ratio, S the required sum of a G P

Thus $$S=a+ar+ar^2+ar^3+\ldots+ar^{n-1}$$

Multiply by r, thus $$Sr=ar+ar^2+ar^3+\ldots+ar^{n-1}+ar^n.$$

Subtract, thus $$S-Sr=a-ar^n,$$
$$S(1-r)=a(1-r^n),$$
$$S=\frac{a(1-r^n)}{1-r}\ldots\ldots(\text{i}),$$
$$\text{or}=\frac{a(r^n-1)}{r-1}\ldots\ldots(\text{ii})$$

If l denote the last (i e, the nth) term, we have $l=ar^{n-1}$, thus
$$S=\frac{a-rl}{1-r}\ldots\ldots(\text{iii}),$$
$$\text{or}=\frac{rl-a}{r-1}\ldots\ldots(\text{iv})$$

Ex. 1 Sum to 8 terms the series $1\frac{1}{2}+2\frac{1}{4}+3\frac{3}{8}+\ldots$.

Here $a=1\frac{1}{2}=\frac{3}{2}$, $r=2\frac{1}{4}\div1\frac{1}{2}=\frac{3}{2}$, $n=8$

Sum required $=\dfrac{\frac{3}{2}\{(\frac{3}{2})^8-1\}}{\frac{3}{2}-1}=\dfrac{\frac{3}{2}\{\frac{6561}{256}-1\}}{\frac{1}{2}}=3\times\frac{6305}{256}=73\frac{227}{256}$

Ex 2. Sum the series $2-4+8-16+\ldots$ to 8 terms and also to 9 terms.

Here $a=2$, $r=-4\div2=-2$

(i) When $n=8$, sum reqd $=\dfrac{2\{(-2)^8-1\}}{-2-1}=\dfrac{2(2^8-1)}{-3}$
$$=\frac{2\times255}{-3}=-170$$

(11) When $n=9$, sum reqd. $=\dfrac{2\{(-2)^9-1\}}{-2-1}=\dfrac{2(-2^9-1)}{-3}$

$$=\frac{2(-513)}{-3}=342.$$

Note Mark that $(-2)^8=2^8$ and $(-2)^9=-2^9$; and generally $(-1)^n=+1$, when n is *even*, and $(-1)^n=-1$ when n is odd.

Examples CLXXXVIII.

1. Sum to 15 terms $1+2+4+8+\ldots\ldots$
2. Sum to 9 terms $1-3+9-27+\ldots\ldots$
3. Sum to 6 terms $1\frac{3}{5}+2\frac{2}{3}+4\frac{4}{9}+\ldots\ldots$
4. Sum to 7 terms $-2+2\frac{1}{2}-3\frac{1}{8}+\ldots\ldots$
5. Sum to 10 terms $\dfrac{1}{\sqrt{2}}-1+\sqrt{2}-\ldots\ldots$
6. Sum to 20 terms $a+a^2+a^3+a^4+\ldots\ldots$
7. Sum to $2n$ terms $1-2+4-8+\ldots\ldots$
8. Sum to n terms $3-\frac{3}{2}+\frac{3}{4}-\ldots\ldots$

Note. The student will notice that the *limiting value* of the sum is $\frac{a}{1-r}$, which means that however large the number of terms may be, the sum of the series cannot exceed the value $\frac{a}{1-r}$.

Illustration Consider the series $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots$ Here $a=1$ and $r=\frac{1}{2}$ Hence the sum of n terms

$$=\frac{1}{1-\frac{1}{2}}\left(1-\frac{1}{2^n}\right)=2\left(1-\frac{1}{2^n}\right)=2-\frac{1}{2^{n-1}}$$

By taking n very large, 2^{n-1} can be made very large, and therefore $\frac{1}{2^{n-1}}$ can be made very small Thus by taking n large enough, the sum of n terms of the series can be made to differ from 2 *by as small a quantity as we please* Hence the sum of the series *cannot exceed the value* 2, even if we take an infinite number of terms This is briefly stated thus *the sum of an infinite number of terms of the series is* 2

Ex 1 Sum the G P $\frac{2}{3}-\frac{1}{2}+\frac{3}{8}-\ldots$ to infinity

"To sum to infinity" means "to find the sum of an infinite number of terms".

Here $a=\frac{2}{3}$, $r=-\frac{3}{4}$, a proper fraction

Hence sum reqd $=\dfrac{\frac{2}{3}}{1+\frac{3}{4}}=\frac{2}{3}\times\frac{4}{7}=\frac{8}{21}$

Ex. 2 Find the values of (i) $\dot{5}$ and (ii) $8\dot{3}\dot{7}$.

(i) $\quad \dot{5}=5555$

$$=5+05+005+0005+\ldots$$

$$=\frac{5}{10}+\frac{5}{10^2}+\frac{5}{10^3}+\frac{5}{10^4}+\ \text{to infinity}$$

$$=\tfrac{5}{10}\div\left(1-\tfrac{1}{10}\right)=\tfrac{5}{9}$$

(ii) $\quad 8\dot{3}\dot{7}=8373737\ \ldots$

$$=8+037+00037+\ \ldots$$

$$=\frac{8}{10}+\frac{37}{10^3}+\frac{37}{10^5}+\ \ldots\ \text{to infinity}$$

Here all the terms except the first form a G P whose first term $=\frac{37}{10^3}$, and common ratio $=\frac{1}{10^2}$. Hence

$$8\dot{3}\dot{7}=\frac{8}{10}+\frac{37}{10^3}\div\left(1-\frac{1}{10^2}\right)=\frac{8}{10}+\frac{37}{10^3}\times\frac{10^2}{99}$$

$$=\frac{8}{10}+\frac{37}{990}=\frac{829}{990}.$$

We thus see that **Recurring Decimals** are good examples of infinite geometric series The reason for the Arithmetical rule for converting a recurring decimal into a vulgar fraction is now obvious

To convert $\cdot 8\dot{3}\dot{7}$ *into vulgar fraction*

Let $s = \cdot 8373737 \ldots$,

$10s = 8\cdot 373737 \ldots$,

and $1000s = 837\cdot 373737 \ldots$,

hence by subtraction $(1000-10)s = 837-8$,

$$s = \frac{837-8}{1000-10} = \frac{837-8}{990}$$

Examples CLXXXIX.

Sum to infinity the series

1 $\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+ \ldots$

2 $9-6+4- \ldots$

3 $3\frac{3}{8}-2\frac{1}{4}+1\frac{1}{2}- \ldots$

4 $-8-6-4\frac{1}{2}- \ldots$

5 $-3\frac{1}{3}+1\frac{2}{3}-\frac{5}{6}+ \ldots$

6 $1+\frac{1}{\sqrt{2}}+\frac{1}{2}+\frac{1}{2\sqrt{2}}+ \ldots$

7. $1+2x+4x^2+8x^3+ \ldots$

8 $\frac{a}{x}+\frac{b}{x^2}+\frac{a}{x^3}+\frac{b}{x^4}+ \ldots$

9. Find by the method of summing up a geometric series the values of

(i) $\cdot 37\dot{8}$ (ii) $\cdot 08\dot{7}83$ (iii) $1\cdot 05\dot{3}\dot{6}$ (iv) $3\cdot\dot{3}8461\dot{5}$

357 Theorem. In a G P continued to infinity, where the common ratio is less than unity, any term will bear a constant ratio to the sum of all the terms which follow it.

Let the G P be $a, ar, ar^2, ar^3, \ldots$

Take any term, say, the nth term

The nth term is ar^{n-1}, and the sum of all the terms which follow it is

$$ar^n+ar^{n+1}+ar^{n+2}+ \ldots \text{ to infinity } = \frac{ar^n}{1-r}.$$

Thus the ratio of the nth term to the sum of all the succeeding terms is

$$ar^{n-1} \div \frac{ar^n}{1-r} = \frac{1-r}{r},$$

which, being independent of n, is constant whatever n may be

Ex Find the common ratio of an infinite G P in which the ratio of any term to the sum of all the suceeding terms is $\frac{2}{3}$

Let the G P be a, ar, ar^2, ar^3, . , r being supposed less than unity

Then whatever n may be, the ratio of any term to the sum of all the succeeding terms is $\dfrac{1-r}{r}$

$$\frac{1-r}{r}=\frac{2}{3} \text{ by the question , whence } r=\tfrac{3}{5}$$

358 *Definitions* If three quantities are in G P, the middle one is called the **geometric mean** (G M) of the other two

Thus if a, b and c are in G P, b is the G M. of a and c

Again if any number of quantities are in G P, all the terms intermediate between the first and last are called the **geometric means** of the two extreme terms

Thus in the G P 1, 2, 4, 8 and 16, the terms 2, 4 and 8 are the three geometric means of 1 and 16

359 Let three quantities a, G and c be in G P Then by definition

$$\frac{G}{a}=\text{the common ratio}=\frac{c}{G},$$

$$G^2=ac, \quad \text{or} \quad G=\sqrt{(ac)}$$

Thus the geometric mean of two quantities is the **square root** of their product

360 *To insert n* **geometric means** *between two quantities a and c*

Here the n means together with the first and last terms form a G P of $n+2$ terms, whose first term is a and last term is c

Hence if r = the common ratio of the required G. P, the last term

$$c=ar^{n+1} \text{ [Art 352] , whence } r=\left(\frac{c}{a}\right)^{\frac{1}{n+1}}.$$

Thus if the series is $a, ar, ar^2 ar^3, \quad , c,$

then the required means are $ar, ar^2, ar^3, \ldots,$

that is, $a\left(\dfrac{c}{a}\right)^{\frac{1}{n+1}},\ a\left(\dfrac{c}{a}\right)^{\frac{2}{n+1}},\ a\left(\dfrac{c}{a}\right)^{\frac{3}{n+1}},\ \ldots$

Cor Hence if g_1, g_2 be *two* geometric means between a and c, then $ac=g_1g_2$

Ex Insert 4 geometric means between 2 and 486

Let the series be 2, $2r$, $2r^2$, $2r^3$, $2r^4$, 486, so that the means are $2r$, $2r^2$, $2r^3$, $2r^4$, where r is the common ratio There are 6 terms, therefore $486=$ the sixth term $=2r^{6-1}$, $\qquad r^5=243$, or $r=3$

Hence $2r=2\cdot3=6$, $2r^2=2\cdot3^2=18$, $2r^3=2\cdot3^3=54$, $2r^4=2\cdot3^4=162$. Thus the means are 6, 18, 54 and 162

Examples CXC

1. Insert 2 geometric means between $\frac{3}{2}$ and $\frac{4}{9}$
2. Insert 3 geometric means between 4 and $20\frac{1}{4}$
3. Insert 3 geometric means between $\frac{1}{9}$ and 9
4. Insert 5 geometric means between 1 and $\frac{1}{64}$
5. Insert 4 geometric means between $\frac{3}{4}$ and -24
6. Insert 6 geometric means between 14 and $-\frac{7}{64}$.

361 Miscellaneous Examples

Ex. 1 Sum to n terms the series $1+5+13+29+\ldots\ldots$

Reqd sum $=1+5+13+29+\ldots\ldots$ to n terms

$$=(2^2-3)+(2^3-3)+(2^4-3)+(2^5-3)+\ldots \text{ to } n \text{ terms}$$
$$=(2^2+2^3+2^4+\ldots \text{to } n \text{ terms})-(3+3+3+\ldots \text{ to } n \text{ terms})$$
$$=\frac{2^2(2^n-1)}{2-1}-3n=4(2^n-1)-3n$$

Ex. 2. Sum to n terms the series whose nth term is $5n+3^n$

Put $n=1$, thus $t_1=5\cdot1+3^1$

$n=2$, $\;\therefore\; t_2=5\cdot2+3^2$

$n=3$, $\;\therefore\; t_3=5\cdot3+3^3$

.. ..

$$\text{sum reqd}=5(1+2+3+\ldots \text{ to } n \text{ terms})$$
$$+(3+3^2+3^3+\ldots \text{ to } n \text{ terms})$$
$$=5\cdot\tfrac{1}{2}n(n+1)+\frac{3(3^n-1)}{3-1}=\tfrac{5}{2}n(n+1)+\tfrac{3}{2}(3^n-1)$$

Ex 3 Sum the series $6+66+666+\ldots$ to n terms

Reqd. sum $=6(1+11+111+\ldots$ to n terms)

$$=\tfrac{6}{9}\times9(1+11+111+\ldots \text{ to } n \text{ terms})$$
$$=\tfrac{6}{9}(9+99+999+\ldots \text{ to } n \text{ terms})$$
$$=\tfrac{6}{9}\{(10-1)+(100-1)+(1000-1)+\ldots \text{ to } n \text{ terms}\}$$
$$=\tfrac{6}{9}\{(10-1)+(10^2-1)+(10^3-1)+\ldots \text{ to } n \text{ terms}\}$$
$$=\tfrac{6}{9}\{(10+10^2+10^3+\ldots \text{ to } n \text{ terms})-n\}$$
$$=\frac{2}{3}\cdot\frac{10(10^n-1)}{10-1}-\frac{2}{3}n=\frac{20}{27}(10^n-1)-\frac{2}{3}n.$$

Ex 4 If a, b, c, d are in G P, shew that

$$(a+d)(b+c)-(a+c)(b+d)=(b-c)^2$$

By def, $\frac{b}{a}=\frac{c}{b}=\frac{d}{c}$, each being equal to the common ratio

Hence $b^2=ac,\ c^2=bd,\ bc=ad.$ .... (A).

Now $(a+d)(b+c)-(a+c)(b+d)$

$$=(ab+ac+bd+cd)-(ab+bc+ad+cd)$$
$$=ac+bd-bc-ad$$
$$=b^2+c^2-bc-bc, \text{ using (A)}$$
$$=b^2+c^2-2bc=(b-c)^2$$

Ex 5 If a, b, c, d are in G P, shew that $a^2-b^2,\ b^2-c^2,\ c^2-d^2$ are also in G P

Let r denote the common ratio

Thus $a=a,\ b=ar,\ c=ar^2$ and $d=ar^3$

Now $a^2-b^2=a^2-a^2r^2=a^2(1-r^2)$,

$b^2-c^2=a^2r^2-a^2r^4=a^2r^2(1-r^2)$,

and $c^2-d^2=a^2r^4-a^2r^6=a^2r^4(1-r^2)$

Evidently $a^2(1-r^2)$, $a^2r^2(1-r^2)$, and $a^2r^4(1-r^2)$ are in G P, (the common ratio being r^2), therefore also $a^2-b^2,\ b^2-c^2,\ c^2-d^2$ are in G P

Ex 6 If the $(p+q)$th term of a G P $=m$, and the $(p-q)$th term $=n$, shew that the pth term $=\sqrt{mn}$, and the qth term $=m\left(\frac{n}{m}\right)^{\frac{p}{2q}}$

Let $a=$ the first term and $r=$ the common ratio of the G P,

then the $(p+q)$th term $=ar^{p+q-1}=m.$ ...... (i),

and the $(p-q)$th term $=ar^{p-q-1}=n.$ .. (ii)

Multiply (i) by (ii), thus $a^2r^{2p-2}=mn.$

Extract the sq root, thus $ar^{p-1}=\sqrt{mn}$

Again divide (i) by (ii), thus

$r^{2q}=\frac{m}{n}$, or $r=\left(\frac{m}{n}\right)^{\frac{1}{2q}}$, whence $r^p=\left(\frac{m}{n}\right)^{\frac{p}{2q}}$ ....(iii)

From (i), $m=ar^{p+q-1}=ar^{q-1}r^p=ar^{q-1}\left(\frac{m}{n}\right)^{\frac{p}{2q}}$ from (iii),

$$ar^{q-1}=m\left(\frac{n}{m}\right)^{\frac{p}{2q}}.$$

Ex 7 The sum of 3 numbers in G P is 57, and their product is 343, what are the numbers?

Let $\frac{a}{\beta}$, a and $a\beta$ represent the numbers, for they are in G P

Then by the question

$$\frac{a}{\beta}+a+a\beta=57. \qquad (i),$$

and
$$\frac{a}{\beta}\,a\,a\beta=343. \qquad (ii)$$

From (ii), $a^3=343$, whence $a=7$

Hence from (i), $a\left(\frac{1}{\beta}+1+\beta\right)=57$,

$$7(1+\beta+\beta^2)=57\beta,$$
$$7\beta^2-50\beta+7=0,$$
$$(\beta-7)(7\beta-1)=0,$$
$$\beta=7 \text{ or } \tfrac{1}{7}$$

Thus the required numbers are 1, 7, 49 or 49, 7, 1

Examples CXCI.

Sum to n terms the series

1 $1+3+7+15+\ldots$ 2 $3+5+9+17+\ldots$

3 $2+8+26+80+\ldots$ 4 $5+11+29+83+\ldots$

5 $3+6+11+20+\ldots$ 6 $2+7+24+77+\ldots$

7 Sum to n terms the series whose nth term is

(i) $2n+2^n$ (ii) 3^n-2n (iii) $an+r^n$

8. Sum to n terms the series

(i) $9+99+999+\ldots$ (ii) $5+55+555+\ldots$

9 What value must a have so that the sum of the series

$2a+a\sqrt{2}+a+\frac{a}{\sqrt{2}}+\ldots$ to infinity may be 8?

10 Shew, by the method of summation of a geometrical series, that

$$\sqrt{(444\qquad)} = 666$$

11 If a, b, c, d are in G P, shew that

(i) $(b+c)(b+d)=(c+a)(c+d)$

(ii) $(a+b+c)(a-b)=a(a-d)$

(iii) $(b-c)^2+(c-a)^2+(d-b)^2=(a-d)^2$

12 If a, b, c, d are in G P, prove that

$$(a+b+c+d)^2=(a+b)^2+2(b+c)^2+(c+d)^2$$

13 If a, b, c, d are in G P, shew that

(i) $a^2+b^2,\ b^2+c^2,\ c^2+d^2$ are in G P

(ii) $(a+b)^2,\ (b+c)^2,\ (c+d)^2$ are in G P

(iii) $(a-b)^2,\ (b-c)^2,\ (c-d)^2$ are in G P

(iv) $\dfrac{1}{a^2-b^2},\ \dfrac{1}{b^2-c^2},\ \dfrac{1}{c^2-d^2}$ are in G P

14 The arithmetical mean of two numbers is 15 and the geometrical mean is 9 Find the numbers

15 Insert between 6 and 16 two numbers, such that the first 3 may be in A P, and the last 3 in G P

16 If the arithmetic mean between x and y is twice their geometrical mean, then

$$\frac{x}{y}=\frac{2+\sqrt{3}}{2-\sqrt{3}} \text{ or } \frac{2-\sqrt{3}}{2+\sqrt{3}}$$

17 The continued product of 3 numbers in G P is 216, and the sum of the products of them in pairs is 126, find the numbers

18 Three numbers whose sum is 15 are in A P, if 1, 4 and 19 be added to them respectively, the results are in G P Determine the numbers

19 If P be the product of n terms in G P, s their sum and s' the sum of their reciprocals, then $P^2=\left(\dfrac{s}{s'}\right)^n$

20 If P be the pth term and Q the qth term of a G P, shew that the nth term $=\left(\dfrac{P^{n-q}}{Q^{n-p}}\right)^{\frac{1}{p-q}}$.

21 If a, b, c be respectively the pth, qth and rth terms of a G P, shew that

$$a^{q-r}b^{r-p}c^{p-q}=1$$

22 Shew that the product of n quantities in G. P whose first term is a and last term c, is $(ac)^{\frac{n}{2}}$

23 If a, b, c are in A P, and $a, b, c+1$ are in G P, prove that
$$a=(a-b)^2=(b-c)^2$$

24 If a, b, c are in G P, and x, y are the arithmetic means between a, b and b, c respectively, prove that
$$\frac{2}{b}=\frac{1}{x}+\frac{1}{y} \text{ and } 2=\frac{a}{x}+\frac{c}{y}.$$

25 Find the common ratio of an infinite geometrical series in which each term is twice the sum of all the terms which follow it

26 If S_p denote the sum of the series $1+r^p+r^{2p}+\ldots$ to infinity and s_p the sum of the series $1-r^p+r^{2p}-\ldots$ to infinity, prove that
$$S_p+s_p=2S_{2p}$$

27 If S_1, S_2, S_3 be the sums to n, $2n$, $3n$ terms respectively of a G P, prove that $S_1(S_3-S_2)=(S_2-S_1)^2$

28 If S_n represent the sum of n terms of a G P whose first term is a and common ratio r, find $S_1+S_2+S_3+\ldots+S_n$

CHAPTER XXXI

MISCELLANEOUS SERIES

362 We shall now give some easy examples of the *Summation of Series*, which are neither arithmetical, nor geometrical To sum such a series, the method is to find the **general term**, which we shall call the **rth term**, for then the law of the series will be apparent and therefore the series can be expressed as the *algebraic sum* of two or more known series, and thus its sum can be found

In connection with the summation of these series, the results established in the next three articles should be carefully remembered.

The numbers 1, 2, 3, 4, 5, 6, ... are called *natural* numbers

363 *The sum of the first n* **natural numbers**

We have already seen [Ex 3, Art 342] that if s denote the sum required, then
$$s=\frac{n(n+1)}{2}.$$

364 *The sum of the* squares *of the first n natural numbers*

Let s = the required sum, thus

$$s = 1^2 + 2^2 + 3^2 + \ldots + n^2.$$

We have the identity

$$a^3 - (a-1)^3 = 3a^2 - 3a + 1,$$

which will be satisfied by any value of a. Put for a, the values 1, 2, 3, ..., n, in succession, thus we have

$$1^3 - 0^3 = 3 \cdot 1^2 - 3 \cdot 1 + 1,$$
$$2^3 - 1^3 = 3 \cdot 2^2 - 3 \cdot 2 + 1,$$
$$3^3 - 2^3 = 3 \cdot 3^2 - 3 \cdot 3 + 1,$$

$$(n-1)^3 - (n-2)^3 = 3(n-1)^2 - 3(n-1) + 1,$$
$$n^3 - (n-1)^3 = 3n^2 - 3n + 1.$$

Hence, by addition, $n^3 = 3\{1^2 + 2^2 + \ldots + n^2\} - 3\{1 + 2 + \ldots + n\} + n$

$$= 3s - 3\,\frac{n(n+1)}{2} + n,$$

whence

$$3s = n^3 - n + \frac{3n(n+1)}{2} = n(n^2-1) + \frac{3n(n+1)}{2}$$

$$= n(n+1)(n - 1 + \tfrac{3}{2}) = \frac{n(n+1)(2n+1)}{2},$$

$$s = \frac{n(n+1)(2n+1)}{6}.$$

365 *The sum of the* cubes *of the first n natural numbers*

Let s = the required sum, thus

$$s = 1^3 + 2^3 + 3^3 + \ldots + n^3$$

We have the indentity

$$a^4 - (a-1)^4 = 4a^3 - 6a^2 + 4a - 1,$$

which is satisfied by any value of a. Put for a the values 1, 2, 3, ..., n, in succession, thus

$$1^4 - 0^4 = 4 \cdot 1^3 - 6 \cdot 1^2 + 4 \cdot 1 - 1,$$
$$2^4 - 1^4 = 4 \cdot 2^3 - 6 \cdot 2^2 + 4 \cdot 2 - 1,$$
$$3^4 - 2^4 = 4 \cdot 3^3 - 6 \cdot 3^2 + 4 \cdot 3 - 1,$$

$$(n-1)^4 - (n-2)^4 = 4(n-1)^3 - 6(n-1)^2 + 4(n-1) - 1,$$
$$n^4 - (n-1)^4 = 4n^3 - 6n^2 + 4n - 1.$$

Hence, by addition,

$$n^4=4(1^3+2^3+3^3+\ldots+n^3)-6(1^2+2^2+3^2+\ldots\ldots+n^2)$$
$$+4(1+2+3+\ldots+n)-n$$
$$=4s-6\times\tfrac{1}{6}n(n+1)(2n+1)+4\times\tfrac{1}{2}n(n+1)-n\,,$$
$$4s=(n^4+n)+n(n+1)(2n+1)-2n(n+1)$$
$$=n(n+1)(n^2-n+1)+n(n+1)(2n+1)-2n(n+1)$$
$$=n(n+1)\{(n^2-n+1)+(2n+1)-2\}$$
$$=n(n+1)(n^2+n)=n^2(n+1)^2\,,$$
$$s=\frac{n^2(n+1)^2}{4}=\left\{\frac{n(n+1)}{2}\right\}^2$$

Thus *the sum of the cubes of the first n natural numbers is equal to the square of the sum of the numbers*

366 Important Example Sum to n terms the series whose rth term is $3r^2+2r$

Here the general term is already given

Let $r=1$, thus $t_1=3\cdot1^2+2\cdot1$,
... $r=2$, . $t_2=3\cdot2^2+2\cdot2$;
. $r=3$, . $t_3=3\cdot3^2+2\cdot3$,
. ...
. $r=n$, $t_n=3\cdot n^2+2\cdot n$

Add the first column and the second column of numbers separately

thus sum reqd $=(3\cdot1^2+3\cdot2^2+3\cdot3^2+\ldots+3\cdot n^2)+(2\cdot1+2\cdot2+2\cdot3+\ldots+2\cdot n)$

$$=3(1^2+2^2+3^2+\ldots+n^2)+2(1+2+3+\ldots+n)$$
$$=3\,\frac{n(n+1)(2n+1)}{6}\text{[Art. 364]}+2\cdot\frac{n(n+1)}{2}\text{[Art 363]}$$
$$=\frac{n(n+1)(2n+1)}{2}+n(n+1)=n(n+1)\left\{\frac{2n+1}{2}+1\right\}$$
$$=\tfrac{1}{2}n(n+1)(2n+3)$$

As every series can be summed, when its rth, *i e*, general term is known, this example is very useful in the summation of series, and should be carefully remembered

367 We have seen [Art 362] that a series can be summed, if its rth or general term is known The following examples will shew how the rth term, and therefore the sum, of a series can be found

Ex 1 Sum to n terms the series $1\cdot2+2\cdot3+3\cdot4+\ldots\ldots$

Here the general or rth term evidently $=r(r+1)=r^2+r$.

$$\begin{array}{lll} \text{Put } r=1, & \text{then } t_1=1^2+1, \\ \quad r=2, & \quad t_2=2^2+2, \\ \quad r=3, & \quad t_3=3^2+3, \\ \ldots & \ldots \\ \quad r=n, & \quad t_n=n^2+n \end{array}$$

Sum reqd $=(1^2+2^2+3^2+\ldots+n^2)$ adding nos. in 1st column
$+(1+2+3+\ldots+n)$ adding nos in 2nd column,

$$=\frac{n(n+1)(2n+1)}{6}[\text{Art } 364]+\frac{n(n+1)}{2}[\text{Art } 363]$$

$$=\frac{n(n+1)}{2}\left\{\frac{2n+1}{3}+1\right\}=\frac{n(n+1)(n+2)}{3}$$

In this example the rth term is found by inspection

Ex 2 Sum to n terms $1\cdot2\cdot3+2\cdot3\cdot4+3\cdot4\cdot5+4\cdot5\cdot6+\ldots$

Evidently the rth term $=r(r+1)(r+2)=r^3+3r^2+2r$.

Here, as in Ex 1

$$\begin{aligned} t_1&=1^3+3\cdot1^2+2\cdot1, \\ t_2&=2^3+3\cdot2^2+2\cdot2, \\ t_3&=3^3+3\cdot3^2+2\cdot3, \\ &\ldots \\ t_n&=n^3+3\,n^2+2\,n \end{aligned}$$

Sum reqd $=(1^3+2^3+3^3+\ldots+n^3)$ taking 1st column
$+3(1^2+2^2+3^2+\ldots+n^2)$ taking 2nd column
$+2(1+2+3+\ldots+n)$ taking 3rd column

$$=\frac{n^2(n+1)^2}{4}[\text{Art } 365]+3\,\frac{n(n+1)(2n+1)}{6}[\text{Art } 364]$$
$$+2\frac{n(n+1)}{2}\ [\text{Art } 363]$$
$$=\tfrac{1}{2}n(n+1)\{\tfrac{1}{2}n(n+1)+(2n+1)+2\}$$
$$=\tfrac{1}{4}n(n+1)(n^2+5n+6)=\tfrac{1}{4}n(n+1)(n+2)(n+3)$$

Here also the rth term is found by inspection

Ex 3. Sum to n terms $2^2+5^2+8^2+11^2+\ldots$

Each term of this series is the square of the corresponding term of the arithmetic series $2+5+8+11+\ldots$

Hence the rth term of the given series

$$=\{2+(r-1)3\}^2=(3r-1)^2=9r^2-6r+1.$$

Thus, as in Art 366, we have by putting $r=1, 2, 3$, &c.,

$$t_1=9\cdot 1^2-6\cdot 1+1,$$
$$t_2=9\cdot 2^2-6\cdot 2+1,$$
$$t_3=9\cdot 3^2-6\cdot 3+1,$$
$$\cdots\quad\cdot\quad\cdot$$
$$t_n=9\cdot n^2-6\cdot n+1$$

Sum reqd

$$=9(1^2+2^2+3^2+\ldots+n^2)-6(1+2+3+\ldots+n)$$
$$+(1+1+1+\ldots\text{ to } n \text{ terms})$$
$$=9\,\frac{n(n+1)(2n+1)}{6}-6\,\frac{n(n+1)}{2}+n$$
$$=\tfrac{1}{2}n\{3(2n^2+3n+1)-6(n+1)+2\}=\tfrac{1}{2}n(6n^2+3n-1)$$

Ex 4 Sum to n terms $1+7+18+34+55+\ldots$

Here the rth term is found as follows.

Let $s=$ the sum of r terms of the series thus

$$s=1+7+18+34+55+\ldots,$$

also $$s=\quad 1+\ 7+18+34+\ldots+t_r$$

By subtraction

$$0=1+6+11+16+21+\ldots\text{ to } r \text{ terms } -t_r,$$
$$t_r=1+6+11+16+21+\ldots\text{ to } r \text{ terms}$$
$$=\frac{r}{2}\{2+(r-1)5\}=\tfrac{5}{2}r^2-\tfrac{3}{2}r$$

Hence the rth term being $\frac{5}{2}r^2-\frac{3}{2}r$, the sum required can be found as in Art 366 Thus $s=\frac{1}{6}n(n+1)(5n-2)$

Ex 5 Sum to n terms the series

$$1+(1+2)+(1+2+3)+(1+2+3+4)+\ldots$$

Evidently the rth term$=(1+2+3+\ldots+r)=\frac{1}{2}r(r+1)=\frac{1}{2}r^2+\frac{1}{2}r$

Hence, as in Art 366, the sum reqd $=\dfrac{1}{2}\cdot\dfrac{n(n+1)(2n+1)}{6}+\dfrac{1}{2}\cdot\dfrac{n(n+1)}{2}$

$$=\tfrac{1}{6}n(n+1)(n+2)$$

Ex 6 Find the sum of the series

$$\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\frac{1}{4\cdot 5}+\frac{1}{5\cdot 6}+\ldots\ldots\text{to } n \text{ terms.}$$

The rth term of the series $=\frac{1}{(r+1)(r+2)}=\frac{1}{r+1}-\frac{1}{r+2}$

Now $\qquad t_1=\frac{1}{2\cdot3}=\frac{1}{2}-\frac{1}{3},$

$$t_2=\frac{1}{3\cdot4}=\frac{1}{3}-\frac{1}{4},$$

$$t_3=\frac{1}{4\cdot5}=\frac{1}{4}-\frac{1}{5},$$

$$\cdots \quad \cdot \quad \cdot \quad \cdots$$

$$t_n=\frac{1}{(n+1)(n+2)}=\frac{1}{n+1}-\frac{1}{n+2}.$$

Thus if $t_1, t_2, t_3, \ldots$ be added, we see that all the terms cancel except the first and last. Hence

$$\text{sum reqd} \quad =\frac{1}{2}-\frac{1}{n+2}=\frac{n}{2(n+2)}$$

Ex 7. Sum to n terms the series

$$a+(a+d)x+(a+2d)x^2+(a+3d)x^3+\ldots$$

Let $\quad s=$ the sum required, thus

$$s=a+(a+d)x+(a+2d)x^2+(a+3d)x^3+\ldots+(a+\overline{n-1}\,d)x^{n-1}$$

$$sx= \qquad ax+(a+d)x^2+(a+2d)x^3+\ldots+(a+\overline{n-2}\,d)x^{n-1}$$
$$+(a+\overline{n-1}\,d)x^n,$$

$$\therefore (1-x)s=a+dx+dx^2+dx^3+\ldots+dx^{n-1}-(a+\overline{n-1}\,d)x^n$$

$$=a+\frac{dx(1-x^{n-1})}{1-x}-ax^n-(n-1)dx^n$$

$$=a(1-x^n)-(n-1)dx^n+\frac{dx(1-x^{n-1})}{1-x},$$

$$s=\frac{a(1-x^n)-(n-1)dx^n}{1-x}+\frac{dx(1-x^{n-1})}{(1-x)^2}$$

Cor If $x<1$, the sum of the given series to infinity

$$=\frac{a}{1-x}+\frac{dx}{(1-x)^2},$$

for then x^n and x^{n-1} are each 0.

Examples CXCII

Sum the series to n terms

1 $(1^2+1)+(2^2+2)+(3^2+3)+(4^2+4)+\dots$

2 $(1^3-1)+(2^3-2)+(3^3-3)+(4^3-4)+\dots$

3 $2\cdot 3+3\cdot 4+4\cdot 5+5\cdot 6+\dots$

4 $1\cdot 2+2\cdot 5+3\cdot 8+4\cdot 11+\dots$

5 $1\cdot 3+3\cdot 5+5\cdot 7+7\cdot 9+\dots$

6 $1\cdot 2\cdot 4+2\cdot 3\cdot 5+3\cdot 4\cdot 6+\dots$

7 $1^2+3^2+5^2+7^2+\dots$

8 $1^2+4^2+7^2+10^2+\dots$

9 $1\cdot 3\cdot 5+3\cdot 5\cdot 7+5\cdot 7\cdot 9+\dots$

10 $5^2+11^2+17^2+23^2+\dots$

11 $2+7+15+26+40+\dots$

12 $1+8+21+40+65+\dots$

13 $8+15+24+35+48+\dots$

14 $1+3+7+15+31+63+\dots$

15 $1+4+13+40+121+\dots$

16 $1^2+(1^2+2^2)+(1^2+2^2+3^2)+\dots$

17 $5+6+9+18+45+126+\dots$

18 $\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\frac{1}{4\cdot 5}+\dots$

19 $\frac{1}{1\cdot 3}+\frac{1}{3\cdot 5}+\frac{1}{5\cdot 7}+\frac{1}{7\cdot 9}+\dots$

20 $\frac{1}{2\cdot 5}+\frac{1}{5\cdot 8}+\frac{1}{8\cdot 11}+\frac{1}{11\cdot 14}+\dots$

21 Sum the series $1+2^2+3+4^2+5+6^2+\dots$ to n terms, where n is an odd number

22 Find the sum of the series

$$1+(1+a)x+(1+a+a^2)x^2+(1+a+a^2+a^3)x^3+\dots$$

ad infinitum, where a and x are less than unity

23 If $(x-1)+2(x-2)+3(x-3)+\dots$ to n terms be equal to $\frac{1}{3}n(n+1)$, find x

24 The series of natural numbers is divided into groups as follows, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, and so on Prove that the sum of the numbers in the nth group is $\frac{1}{2}n(n^2+1)$

25 On the ground are placed n stones, the distance between the first and second is 1 yd, between the second and third 3 yds, between the third and fourth 5 yds, and so on How far will a person have to travel who shall bring them, one by one, to a basket placed at the first stone?

APPENDIX

In the Appendix a few examples of Algebraical Artifices are given in addition to those of Chapter XXIV

Ex 1 If $x=(b-c)(a-d)$, $y=(c-a)(b-d)$, $z=(a-b)(c-d)$, find the value of $x^3+y^3+z^3-3xyz$

We have $x+y+z=(b-c)(a-d)+(c-a)(b-d)+(a-b)(c-d)$
$=a(b-c)+b(c-a)+c(a-b)-d\{(b-c)+(c-a)+(a-b)\}=0$ [Art 149], thus $x+y+z$ which is a factor of the proposed expression, is 0; therefore the proposed expression is 0

Ex. 2 Prove that

(i) $\{(x-y)^2+(y-z)^2+(z-x)^2\}^2=2\{(x-y)^4+(y-z)^4+(z-x)^4\}$,

(ii) $(x-y)^4+(y-z)^4+(z-x)^4$
$$=2\{(x-y)^2(y-z)^2+(y-z)^2(z-x)^2+(z-x)^2(x-y)^2\}$$

Put $a=y-z$, $b=z-x$, $c=x-y$, thus $a+b+c=0$,

whence $a^2+b^2+c^2+2bc+2ca+2ab=0$ [Art 148],

transpose and square, thus
$$(a^2+b^2+c^2)^2=4(bc+ca+ab)^2$$
$$=4(b^2c^2+c^2a^2+a^2b^2),\qquad a+b+c=0,$$

expand, thus $a^4+b^4+c^4=2(b^2c^2+c^2a^2+a^2b^2)$ (α),

add $a^4+b^4+c^4$ to both sides, thus
$$2(a^4+b^4+c^4)=a^4+b^4+c^4+2(b^2c^2+c^2a^2+a^2b^2)$$
$$=(a^2+b^2+c^2)^2 \text{ [Art 148]} \qquad (\beta),$$

Thus (α) proves the *second* identity and (β) proves the *first* identity

Ex. 3 If $(by-cx)^2=(b^2-ac)(y^2-cz)$,

prove that $(bx-ay)^2=(b^2-ac)(x^2-az)$

From the proposed relation, we get
$$b^2y^2-2bcxy+c^2x^2=b^2y^2-acy^2-b^2cz+ac^2z,$$
whence cancelling b^2y^2, and dividing by c, we have
$$-2bxy+cx^2=-ay^2-b^2z+acz,$$
multiply by a, and add b^2x^2 to both sides, thus
$$b^2x^2-2abxy+acx^2=b^2x^2-a^2y^2-ab^2z+a^2cz,$$
transpose, thus $b^2x^2-2abxy+a^2y^2=b^2x^2-ab^2z-acx^2+a^2cz$,
$$\therefore (bx-ay)^2=b^2(x^2-az)-ac(x^2-az)$$
$$=(b^2-ac)(x^2-az).$$

Ex. 4 Find the H C F of

$$(ax+by)^2-(a-b)(x+z)(ax+by)+(a-b)^2xz$$

and $$(ax-by)^2-(a+b)(x+z)(ax-by)+(a+b)^2xz$$

First expression

$$=(ax+by)^2-\{(a-b)x+(a-b)z\}(ax+by)+(a-b)x\times(a-b)z$$
$$=\{(ax+by)-(a-b)x\}\{(ax+by)-(a-b)z\} \text{ [Art 74]}$$
$$=b(x+y)(ax+by)-(a-b)z\}$$

Similarly second expression

$$=\{(ax-by)-(a+b)x\}\{(ax-by)-(a+b)z\}$$
$$=-b(x+y)\{(ax-by)-(a+b)z\}$$

The H C F required evidently $=b(x+y)$

Note The second expression differs from the first only in the *sign* of b, therefore its factors might have been obtained by writing $-b$ for b in the factors of the first expression

Ex. 4 Assuming that $\dfrac{a+b-c}{a+b}=\dfrac{b+c-a}{b+c}=\dfrac{c+a-b}{c+a}$, and that $a+b+c$ is not $=0$ shew that $a=b=c$ [*Cal*, 1873]

From the given relations, we have

$$2-\frac{a+b-c}{a+b}=2-\frac{b+c-a}{b+c}=2-\frac{c+a-b}{c+a},$$

or $$\frac{a+b+c}{a+b}=\frac{a+b+c}{b+c}=\frac{a+b+c}{c+a},$$

divide by $a+b+c$, which, by supposition, is not 0, therefore

$$\frac{1}{a+b}=\frac{1}{b+c}=\frac{1}{c+a},$$

whence $$a+b=b+c=c+a,$$

that is, $$a=b=c$$

Ex 6. Shew that if $\dfrac{a-b}{c}+\dfrac{b-c}{a}+\dfrac{c+a}{b}=1$ and $a-b+c$ is not $=0$,

then $$\frac{1}{a}=\frac{1}{b}+\frac{1}{c}. \quad \text{[Cal, 1875]}$$

We have from the given relation,

$$\frac{a-b}{c}+1+\frac{b-c}{a}-1+\frac{c+a}{b}-1=0,$$

or

$$\frac{a-b+c}{c}+\frac{b-c-a}{a}+\frac{c+a-b}{b}=0,$$

whence

$$(a-b+c)\left\{\frac{1}{c}-\frac{1}{a}+\frac{1}{b}\right\}=0\,,$$

thus either $a-b+c=0$, or $\frac{1}{b}+\frac{1}{c}-\frac{1}{a}=0$ [Art 282], but by supposition, $a-b+c$ is not $=0$, therefore

$$\frac{1}{b}+\frac{1}{c}-\frac{1}{a}=0, \text{ or } \frac{1}{a}=\frac{1}{b}+\frac{1}{c}$$

Ex 7 Shew that $(1+x)(1+x^2)(1+x^4)(1+x^8)=\dfrac{1-x^{16}}{1-x}$.

Let

$$y=(1+x)(1+x^2)(1+x^4)(1+x^8),$$

$$(1-x)y=(1-x)(1+x)(1+x^2)(1+x^4)(1+x^8)$$

$$=(1-x^2)(1+x^2)(1+x^4)(1+x^8)$$

$$=(1-x^4)(1+x^4)(1+x^8)=(1-x^8)(1+x^8)=1-x^{16}\,,$$

$$\therefore\ y=\frac{1-x^{16}}{1-x}$$

Ex. 8 Express $\alpha\beta+\sqrt{(\alpha^2-1)(\beta^2-1)}$ in terms of x and y, when $2\alpha=x+x^{-1}$ and $2\beta=y+y^{-1}$

Proposed expression

$$=\frac{1}{2}\left(x+\frac{1}{x}\right)\times\frac{1}{2}\left(y+\frac{1}{y}\right)+\sqrt{\left\{\frac{1}{4}\left(x+\frac{1}{x}\right)^2-1\right\}\left\{\frac{1}{4}\left(y+\frac{1}{y}\right)^2-1\right\}}\,,$$

$$=\frac{1}{4}\left(x+\frac{1}{x}\right)\left(y+\frac{1}{y}\right)+\sqrt{\frac{1}{4}\left(x-\frac{1}{x}\right)^2\times\frac{1}{4}\left(y-\frac{1}{y}\right)^2}$$

$$=\frac{1}{4}\left(x+\frac{1}{x}\right)\left(y+\frac{1}{y}\right)+\frac{1}{4}\left(x-\frac{1}{x}\right)\left(y-\frac{1}{y}\right)$$

$$=\frac{1}{4}\left\{\left(xy+\frac{y}{x}+\frac{x}{y}+\frac{1}{xy}\right)+\left(xy-\frac{y}{x}-\frac{x}{y}+\frac{1}{xy}\right)\right\}$$

$$=\frac{1}{4}\left(2xy+\frac{2}{xy}\right)=\frac{1}{2}\left(xy+\frac{1}{xy}\right)$$

Ex. 9 Find the first 4 terms of the square root of a^2+x^2, and from the result deduce the square root of 101 correct to 6 places of decimals. [*Cal*, 1877]

$$
\begin{array}{r|l}
 & a^2+x^2\left(a+\frac{x^2}{2a}-\frac{x^4}{8a^3}+\frac{x^6}{16a^5}\right. \\
 & a^2 \\ \hline
2a+\frac{x^2}{2a} & x^2 \\
 & x^2+\frac{x^4}{4a^2} \\ \hline
2a+\frac{x^2}{a}-\frac{x^4}{8a^3} & -\frac{x^4}{4a^3} \\
 & -\frac{x^4}{4a^2}-\frac{x^6}{8a^4}+\frac{x^8}{64a^6} \\ \hline
2a+\frac{x^2}{a}-\frac{x^4}{4a^3}+\frac{x^6}{16a^5} & \frac{x^6}{8a^4}-\frac{x^8}{64a^6} \\
 & \frac{x^6}{8a^4}+\frac{x^8}{16a^6}-\frac{x^{10}}{64a^8}+\frac{x^{12}}{256a^{10}} \\ \hline
\end{array}
$$

$101=100+1=(10)^2+1^2$, thus here $a=10$, $x=1$, therefore required

square root $=10+\dfrac{1}{2\times 10}-\dfrac{1}{8\times(10)^3}+\dfrac{1}{16\times(10)^5}$

$=10+\dfrac{5}{10}-\dfrac{\cdot 125}{(10)^3}+\dfrac{\cdot 0625}{(10)^5}$

$=10+\cdot 05-\cdot 000125+\cdot 000000625$

$=10+\cdot 05-\cdot 000125$ (rejecting the last term as it cannot affect the required result, for there are 6 zeros after the decimal point)

$=10\cdot 049875$

Ex 10 Divide $x^{2^n}-y^{2^n}$ by $x^{2^{n-1}}+y^{2^{n-1}}$ [*Cal*, 1879]

Let $2^{n-1}=a$, thus

$(x^{2^{n-1}})^2=(x^a)^2=x^{2a}$ [Art. 195] $=x^{2\cdot 2^{n-1}}=x^{2^{n-1+1}}=x^{2^n}$.

Hence $x^{2^n}=\left(x^{2^{n-1}}\right)^2$ Similarly $y^{2^n}=\left(y^{2^{n-1}}\right)^2$.

$$\text{Dividend}=\left(x^{2^{n-1}}\right)^2-\left(y^{2^{n-1}}\right)^2=\left(x^{2^{n-1}}+y^{2^{n-1}}\right)\left(x^{2^{n-1}}-y^{2^{n-1}}\right),$$

the second factor = quotient required

Ex 11. A man receives $\dfrac{x}{y}$ths of 10 rupees and afterwards $\dfrac{y}{x}$ ths of 10 rupees He then gives away 20 rupees. Shew that he cannot lose by the transaction. [*Cal*, 1881]

He receives $\left(10\frac{x}{y}+10\frac{y}{x}\right)$ rupees and gives away 20 rupees, therefore he has $\left(10\frac{x}{y}+10\frac{y}{x}\right)$ rupees $-$ 20 rupees

$$=10\left(\frac{x}{y}+\frac{y}{x}-2\right) \text{ rupees}=10\frac{x^2+y^2-2xy}{xy} \text{ rupees}=\frac{10(x-y)^2}{xy} \text{ rupees}$$

Now x and y being supposed positive, $(x-y)^2$ is always positive whatever values x and y may have, hence $\frac{10(x-y)^2}{xy}$ is a positive quantity. Therefore $\left(10\frac{x}{y}+10\frac{y}{x}\right)$ rupees is *greater than* 20 rupees [Art. 26]; and consequently the man cannot lose

Ex 12. Solve $\dfrac{x}{\sqrt{1+x}+\sqrt{1-x}}=\dfrac{2a^3}{1-\sqrt{1-x^2}}$.

Rationalize the denominator of left side, and multiply numerator and denominator of right side by 2, thus

$$\frac{x(\sqrt{1+x}-\sqrt{1-x})}{2x}=\frac{4a^3}{2-2\sqrt{1-x^2}}=\frac{4a^3}{(\sqrt{1+x}-\sqrt{1-x})^2}.$$

$$(\sqrt{1+x}-\sqrt{1-x})^3=8a^3,$$
$$\sqrt{1+x}-\sqrt{1-x}=2a,$$
$$2-2\sqrt{1-x^2}=4a^2,$$
$$\sqrt{1-x^2}=1-2a^2,$$
$$1-x^2=1-4a^2+4a^4,$$
$$x^2=4a^2(1-a^2),$$
$$x=2a\sqrt{1-a^2}$$

Ex 13 Solve $\sqrt{a^2-x^2}+x\sqrt{a^2-1}=a^2\sqrt{1-x^2}$.

We have the identical relation $(a^2-x^2)-x^2(a^2-1)=a^2(1-x^2)$ (1); divide (1) by the proposed equation, thus

$$\sqrt{a^2-x^2}-x\sqrt{a^2-1}=\sqrt{1-x^2} \quad (2);$$

subtract (2) from the given equation, thus

$$2x\sqrt{a^2-1}=(a^2-1)\sqrt{1-x^2},$$
$$2x=\sqrt{a^2-1}\sqrt{1-x^2},$$
$$4x^2=(a^2-1)(1-x^2),$$
$$x^2(a^2+3)=a^2-1,$$
$$\therefore\ x=\sqrt{\frac{a^2-1}{a^2+3}}$$

Ex. 14. Solve $(a+x)\sqrt{1+a}+(a-x)\sqrt{1-a}=2\sqrt{a^2+x^2}$.

Since $a^2+x^2=\frac{1}{2}(a+x)^2+\frac{1}{2}(a-x)^2$ [Art 143, (iii)], we have by squaring

$$(a+x)^2(1+a)+(a-x)^2(1-a)+2(a^2-x^2)\sqrt{1-a^2}=2(a+x)^2+2(a-x)^2,$$

whence by transposition,

$$(a+x)^2(1-a)+(a-x)^2(1+a)-2(a^2-x^2)\sqrt{1-a^2}=0,$$

or $$\{(a+x)\sqrt{1-a}-(a-x)\sqrt{1+a}\}^2=0,$$

thus $$(a+x)\sqrt{1-a}-(a-x)\sqrt{1+a}=0$$

whence $$\frac{a+x}{a-x}=\frac{\sqrt{1+a}}{\sqrt{1-a}},$$

$$\therefore \quad \frac{x}{a}=\frac{\sqrt{1+a}-\sqrt{1-a}}{\sqrt{1+a}+\sqrt{1-a}} \text{ [Art. 277]}=\frac{(\sqrt{1+a}-\sqrt{1-a})^2}{2a}, \text{ \&c.}$$

Ex. 15 Solve $\dfrac{1+x-\sqrt{2x+x^2}}{1+x+\sqrt{2x+x^2}}=a^3\,\dfrac{\sqrt{2+x}+\sqrt{x}}{\sqrt{2+x}-\sqrt{x}}$.

Multiply numerator and denominator of first side by 2, thus

$$\frac{2+2x-2\sqrt{2x+x^2}}{2+2x+2\sqrt{2x+x^2}}=a^3\,\frac{\sqrt{2+x}+\sqrt{x}}{\sqrt{2+x}-\sqrt{x}},$$

$$\frac{(\sqrt{2+x}-\sqrt{x})^2}{(\sqrt{2+x}+\sqrt{x})^2}=a^3\,\frac{\sqrt{2+x}+\sqrt{x}}{\sqrt{2+x}-\sqrt{x}},$$

$$\frac{(\sqrt{2+x}-\sqrt{x})^3}{(\sqrt{2+x}+\sqrt{x})^3}=a^3,$$

$$\frac{\sqrt{2+x}-\sqrt{x}}{\sqrt{2+x}+\sqrt{x}}=a,$$

$$\frac{\sqrt{2+x}}{\sqrt{x}}=\frac{1+a}{1-a},$$

$$\frac{2+x}{x}=\left(\frac{1+a}{1-a}\right),$$

$$\frac{2}{x}=\left(\frac{1+a}{1-a}\right)^2-1=\frac{4a}{(1-a)^2};\ \therefore \text{ \&c.}$$

Ex. 16. Solve
$$cx+(a+b)y+(a-b)z=a \qquad (1),$$
$$ax+(b+c)y+(b-c)z=b \qquad (2),$$
$$bx+(c+a)y+(c-a)z=c \qquad (3).$$

Adding the equations together, we get

$$(a+b+c)x+2(a+b+c)y=a+b+c,$$

or $$x+2y=1 \qquad (4).$$

Again multiply (1) by $a-b$, (2) by $b-c$, (3) by $c-a$, and add, thus

$$\{(a-b)^2+(b-c)^2+(c-a)^2\}z=a^2+b^2+c^2-bc-ca-ab,$$

whence $$z=\tfrac{1}{2}$$

Substitute the value of z in (1), thus

$$cx+(a+b)y=a-\tfrac{1}{2}(a-b)=\tfrac{1}{2}(a+b) \quad (5).$$

Multiply (4) by c, and subtract from (5), thus

$$(a+b-2c)y=\tfrac{1}{2}(a+b)-c=\tfrac{1}{2}(a+b-2c),$$

$$y=\tfrac{1}{2}$$

Hence from (4), $$x=1-2y=0$$

Ex 17 Solve

$$(b+c)x+by+cz=(c+a)y+cz+ax=(a+b)z+ax+by \quad (1),$$

$$xyz-(c+a-b)yz-(a+b-c)zx-(b+c-a)xy=2xyz\,\frac{a^2+b^2+c^2}{(a+b+c)^2} \quad (2)$$

From equations (1) by transposition, we get

$$(b+c-a)x=(c+a-b)y=(a+b-c)z=k \text{ suppose} \quad (3)$$

From (2) by division by xyz, we have

$$1-(c+a-b)\frac{1}{x}-(a+b-c)\frac{1}{y}-(b+c-a)\frac{1}{z}=\frac{2(a^2+b^2+c^2)}{(a+b+c)^2},$$

or $$(c+a-b)\frac{1}{x}+(a+b-c)\frac{1}{y}+(b+c-a)\frac{1}{z}=1-\frac{2(a^2+b^2+c^2)}{(a+b+c)^2},$$

$$=\frac{2bc+2ca+2ab-a^2-b^2-c^2}{(a+b+c)^2}$$

substitute the values of x, y, z in terms of k from (3), thus

$$(c+a-b)\frac{b+c-a}{k}+(a+b-c)\frac{c+a-b}{k}+(b+c-a)\frac{a+b-c}{k}$$

$$=\frac{2bc+2ca+2ab-a^2-b^2-c^2}{(a+b+c)^2},$$

$$\frac{1}{k}\{c^2-(a-b)^2+a^2-(b-c)^2+b^2-(c-a)^2\}$$

$$=\frac{2bc+2ca+2ab-a^2-b^2-c^2}{(a+b+c)^2},$$

whence $$k=(a+b+c)^2, \quad (4).$$

Thus from (3) and (4), the values of x, y, z are known.

Ex. 18 Solve
$$(b+c)(y+z)=a(x+1) \qquad (1),$$
$$(c+a)(z+x)=b(y+1) \qquad (2),$$
$$(a+b)(x+y)=c(z+1) \qquad (3).$$

By transposition
$$-ax+(b+c)y+(b+c)z=a,$$
$$(c+a)x-by+(c+a)z=b,$$
$$(a+b)x+(a+b)y-cz=c,$$

whence by addition,
$$(a+b+c)x+(a+b+c)y+(a+b+c)z=a+b+c,$$
or
$$x+y+z=1 \qquad (4).$$

From (1) and (4), $(b+c)(1-x)=a(x+1)$,
$$\therefore \quad x=\frac{-a+b+c}{a+b+c}$$

Similarly from (2) and (3), the values of y and z may be found.

Note Observe that (1), (2) and (3) are collaterally symmetrical

Ex. 19. The expression $ax-3b$ is equal to 30 when x is 3, and to 42 when x is 7, what is its value when $x=4\dot{3}$ and for what value of x is it zero? [*Cal*, 1874]

By the first condition, $3a-3b=30$ (1),

By the second condition, $7a-3b=42$ (2).

Subtract (1) from (2), thus $4a=12$ or $a=3$ Hence from (2), $b=-7$.

Therefore when $x=4\dot{3}=4\frac{1}{3}$, the required value
$$=3\times4\tfrac{1}{3}-3\times-7=34$$

Again, $3x-3\times-7=0$ gives, when solved, $x=-7$

Ex. 20 If $x+y+z=0$, then $\dfrac{x^2}{2x^2+yz}+\dfrac{y^2}{2y^2+zx}+\dfrac{z^2}{2z^2+xy}=1$

From the given relation, $x=-(y+z)$, $\therefore x^2=-x(y+z)$, hence
$$2x^2+yz=x^2+x^2+yz=x^2-x(y+z)+yz=(x-y)(x-z)$$

Similarly $2y^2+zx=(y-z)(y-x)$, $2z^2+xy=(z-x)(z-y)$
$$\text{Left side}=\frac{x^2}{(x-y)(x-z)}+\frac{y^2}{(y-z)(y-x)}+\frac{z^2}{(z-x)(z-y)}=1$$

[See Art 190, Ex. 1]

Ex 21. If $\dfrac{bx-ay}{cy-az}=\dfrac{cx-az}{by-ax}=\dfrac{z+y}{z+x}$, then each of these ratios $=\dfrac{x}{y}$, unless $b+c=0$.

Let each ratio $=k$, thus

$$k=\frac{bx-ay}{cy-az}=\frac{cx-az}{by-ax}=\frac{a(z+y)}{a(z+x)} \text{ [Art. 181]}$$
$$=\frac{(bx-ay)+(cx-az)+(az+ay)}{(cy-az)+(by-ax)+(az+ax)} \text{ [276, Cor. 1]}$$
$$=\frac{(b+c)x}{(b+c)y}=\frac{x}{y} \qquad b+c \text{ is not } =0$$

Note If $b+c=0$, then $k=\frac{0\times x}{0\times y}=\frac{0}{0}$, thus the value of k would be *indeterminate* [See Art 283]

Ex. 22 If $\frac{x}{l(mb+nc-la)}=\frac{y}{m(nc+la-mb)}=\frac{z}{n(la+mb-nc)}$,

then $$\frac{l}{x(by+cz-ax)}=\frac{m}{y(cz+ax-by)}=\frac{n}{z(ax+by-cz)}$$

Divide the terms of the first fraction by l, of the second fraction by m, and of the third fraction by n, thus

$$\frac{\frac{x}{l}}{mb+nc-la}=\frac{\frac{y}{m}}{nc+la-mb}=\frac{\frac{z}{n}}{la+mb-nc}=\lambda \text{ say},$$

$$\therefore\quad \lambda=\frac{\frac{y}{m}+\frac{z}{n}}{2la}=\frac{\frac{z}{n}+\frac{x}{l}}{2mb}=\frac{\frac{x}{l}+\frac{y}{m}}{2nc}$$
$$=\frac{ny+mz}{2lmna}=\frac{lz+nx}{2lmnb}=\frac{mx+ly}{2lmnc},$$

hence $$2\lambda lmn=\frac{ny+mz}{a}=\frac{lz+nx}{b}=\frac{mx+ly}{c}.$$

now multiply the terms of the first fraction by x, of the second fraction by y, and of the third fraction by z, thus

$$2\lambda lmn=\frac{nxy+mxz}{ax}=\frac{lyz+nxy}{by}=\frac{mxz+lyz}{cz}$$
$$=\frac{2lyz}{by+cz-ax}=\frac{2mxz}{cz+ax-by}=\frac{2nxy}{ax+by-cz}.$$

Next multiply the terms of the first, second and third members of the last equality by x, y and z respectively, thus

$$\frac{2lxyz}{x(by+cz-ax)}=\frac{2mxyz}{y(cz+ax-by)}=\frac{2nxyz}{z(ax+by-cz)};$$

divide by $2xyz$, thus the required relations follow.

Ex 23 If $a=b\frac{y}{z}+c\frac{z}{y}$, $b=c\frac{z}{x}+a\frac{x}{z}$, $c=a\frac{x}{y}+b\frac{y}{x}$,

prove that $$\frac{1}{x^3}+\frac{1}{y^3}+\frac{1}{z^3}+\frac{1}{xyz}=0$$

From the first two equations by transposition, we get

$$-a+b\frac{y}{z}+c\frac{z}{y}=0,\ a\frac{x}{z}-b+c\frac{z}{x}=0,$$

whence by Cross Multiplication

$$\frac{a}{\frac{y}{z}\times\frac{z}{x}+\frac{z}{y}}=\frac{b}{\frac{z}{y}\times\frac{x}{z}+\frac{z}{x}}=\frac{c}{1-\frac{xy}{z^2}}=k \text{ suppose},$$

$$\therefore\ a=k\left(\frac{y}{x}+\frac{z}{y}\right),\ b=k\left(\frac{x}{y}+\frac{z}{x}\right),\ \ c=k\left(1-\frac{xy}{z^2}\right),$$

substitute a, b, c, in the third equation, thus

$$k\left(1-\frac{xy}{z^2}\right)=k\left(\frac{y}{x}+\frac{z}{y}\right)\frac{x}{y}+k\left(\frac{x}{y}+\frac{z}{x}\right)\frac{y}{x};$$

divide by k and transpose, thus

$$\frac{yz}{x^2}+\frac{zx}{y^2}+\frac{xy}{z^2}+1=0,$$

divide now by xyz, thus the required relation follows.

Ex 24 If $a+b+c=0$ and

$$a(by+cz-ax)=b(cz+ax-by)=c(ax+by-cz),$$

then will $\quad x+y+z=0$

From the first and second members, by transposition,

$$a(a+b)x-b(a+b)y+c(b-a)z=0.$$

From the second and third members, by transposition,

$$a(b-c)x-b(b+c)y+c(b+c)z=0$$

Hence by Cross Multiplication

$$\frac{x}{-bc(a+b)(b+c)+bc(b+c)(b-a)}=\frac{y}{ca(b-c)(b-a)-ca(a+b)(b+c)}$$

$$=\frac{z}{-ab(a+b)(b+c)+ab(a+b)(b-c)}$$

or after reduction,

$$\frac{x}{-2abc(b+c)}=\frac{y}{-2abc(c+a)}=\frac{z}{-2abc(a+b)},$$

whence $\qquad \dfrac{x}{b+c}=\dfrac{y}{c+a}=\dfrac{z}{a+b}=k$ suppose,

thus $\qquad x=k(b+c),\ y=k(c+a),\ z=k(a+b)$,

therefore $\qquad x+y+z=k(b+c)+k(c+a)+k(a+b)$

$$=2k(a+b+c)=0, \quad \therefore \quad a+b+c=0.$$

Miscellaneous Examples

1. If $a=1$, $b=\frac{2}{3}$, $x=7$, $y=8$, find the value of

$$5(a-b)\sqrt[3]{(a+x)y^2}-b\sqrt{(a+x)y}$$

2. Multiply $(x^2+x-6)^2$ by x^2+4x+4.

3 Shew that $(3x^2-4x+2)^2-(2x^2+9x+3)^2$ is divisible by x^2+x+1 without performing the operation of division, and find the quotient

4 Find the H C F of $a(a-1)x^2+(2a^2-1)x+a(a+1)$ and $(a^2-3a+2)x^2+(2a^2-4a+1)x+a(a-1)$

5. If $2s=a+b+c$, prove that

$$(s-a)(s-b)+(s-b)(s-c)+(s-c)(s-a)=s^2-\tfrac{1}{2}(a^2+b^2+c^2).$$

6 Shew that

$$a^3(b-c)^3+b^3(c-a)^3+c^3(a-b)^3=3abc(b-c)(c-a)(a-b).$$

7 Shew that the sum of

$$\frac{x}{x+a}+\frac{x}{x+b}+\frac{x}{x+c} \text{ and } \frac{a}{x+a}+\frac{b}{x+b}+\frac{c}{x+c}$$

is independent of x

8 If $x\pm\dfrac{1}{x}=y$, shew that $x^3\pm\dfrac{1}{x^3}=y^3\mp3y$

9. If $\dfrac{1-2ax+a^2}{1-a^2}=\dfrac{1-a^2}{1+2ay+a^2}$, then $\dfrac{x-y}{1-xy}=\dfrac{2a}{1+a^2}$.

10 Find the square root of $x^2+8x-64x^{-1}+64x^{-2}$

11 Find the expression whose square is $3x-1+2\sqrt{2x^2+x-6}$.

12 Solve $\dfrac{6x-7}{9x+6}-\dfrac{5(x-1)}{12x+8}=\dfrac{1}{12}$

13. Solve $\dfrac{5x+6}{10}-\dfrac{11y-5}{21}=11$, $\dfrac{1}{25}(55y-12)=\dfrac{7x}{5}-37$.

14. If $a \,.\, b=c \quad d$, prove that

$$a^2d-b^2c+b^2d \quad a^2c+b^2d=d \,.\, c+d$$

15. Of the candidates in a certain examination 45 per cent. passed. If there had been 30 more candidates of whom 19 failed, the number of successful candidates would have been 44.8 per cent. How many candidates were there?

16. Find the value of

$$\frac{4y}{5}(y-x)-35\left[\frac{3x-4y}{5}-\frac{1}{10}\{3x-\tfrac{5}{7}(7x-4y)\}\right]$$

when $x=-\frac{1}{2}$, $y=2$.

17. Multiply

$$\sqrt{(2x)}+\sqrt{2(2x-1)}-\frac{1}{\sqrt{(2x)}} \text{ by } \frac{1}{\sqrt{(2x)}}+\sqrt{2(2x-1)}-\sqrt{(2x)}.$$

18. Divide $\dfrac{\left(\dfrac{3x+x^3}{1+3x^2}\right)^2-1}{\dfrac{3x^2-1}{x^3-3x}+1}$ by $\dfrac{\dfrac{9}{x^2}-\dfrac{33-x^2}{3x^2+1}}{\dfrac{3}{x^2}-\dfrac{2(x^2+3)}{(x^3-x)^2}}$

19. If
$$a^2=(x+y-2z)(y+z-2x),$$
$$b^2=(y+z-2x)(z+x-2y),$$
$$c^2=(z+x-2y)(x+y-2z),$$
then
$$(a^2+b^2+c^2)(x+y+z)=3(3xyz-x^3-y^3-z^3).$$

20. If $2s=a+b+c$, then $(s-a)^3+(s-b)^3+(s-c)^3+3abc=s^3$.

21. If
$$\frac{a}{x}(b-c)+\frac{b}{y}(c-a)+\frac{c}{z}(a-b)=0,$$
then
$$\frac{x}{a}(y-z)+\frac{y}{b}(z-x)+\frac{z}{c}(x-y)=0.$$

22. If $\left(a+\frac{1}{a}\right)^2=3$, shew that $a^3+\frac{1}{a^3}=0$.

23. Find the value of $27x+48x^2-8x^4$ when $x=\frac{1}{4}(\sqrt{21}-3)$.

24. Find the square root of $57-12\sqrt{15}$.

25. Solve $\dfrac{x}{2}+\dfrac{5x^2-15x-8}{10(x-3)}=\dfrac{5x-9}{5}+1$.

26. Solve $\dfrac{ax}{a+x}+\dfrac{by}{b+y}=\dfrac{(a+b)c}{a+b+c}$, $x+y=c$.

27. If $x:y=a:b$, shew that $(x^2-a^2)(y^2-b^2)=(xy-ab)^2$.

28. A and B shoot by turns at a target; A puts in 3 arrows out of 7 and B 2 arrows out of 5; how many must each shoot, that they may put in 29 arrows between them?

29. Find the value of x^2-6x+7, when $x=3-\sqrt{3}$.

30. Shew that $a(a-x)(a-2x)$

$$=(a-b)(a-b-x)(a+2b-2x)+b(b-x)(3a-2b-2x).$$

31. Find the coefficients of x^2, x^3 and x^4 in $(x+a)^3(x-a)^5$.

32. Find the value of

$$\left(a+\frac{1}{a}\right)\left(b+\frac{1}{b}\right)\left(c+\frac{1}{c}\right)-\left(a-\frac{1}{a}\right)\left(b-\frac{1}{b}\right)\left(c-\frac{1}{c}\right)$$

33. Find the square root of $(a^2+b^2)(x^2+y^2)-(ax+by)^2$.

34. Given $2s=a+b+c+d$, shew that

$$4(bc+ad)^2-(b^2+c^2-a^2-d^2)^2=16(s-a)(s-b)(s-c)(s-d).$$

35. Prove that

$$\left(x+\frac{1}{x}\right)^2+\left(y+\frac{1}{y}\right)^2+\left(xy+\frac{1}{xy}\right)^2=4+\left(x+\frac{1}{x}\right)\left(y+\frac{1}{y}\right)\left(xy+\frac{1}{xy}\right).$$

36. If $\dfrac{a-b}{1+ab}=\dfrac{x-y}{1+xy}$, shew that

$$\frac{a-x}{1+ax}=\frac{b-y}{1+by} \text{ and } \frac{a+y}{1-ay}=\frac{b+x}{1-bx}.$$

37. Simplify $\left\{\dfrac{\sqrt{x+a}}{\sqrt{x-a}}-\dfrac{\sqrt{x-a}}{\sqrt{x+a}}\right\} \dfrac{\sqrt{x^3-a^3}}{\sqrt{(x+a)^2-ax}}$.

38. Prove that $\frac{3}{2}(\sqrt{3}+1)^2-2(\sqrt{2}+1)^2=\sqrt{69-24\sqrt{6}}$

39. Solve $\dfrac{1}{3x-1}+\dfrac{2(x+1)}{x-1}=1+\dfrac{3x^2+1}{3x^2-4x+1}$

40. Solve $x+y+z=0$, $ax+by+cz=0$, $\dfrac{x}{b-c}+\dfrac{y}{c-a}+\dfrac{z}{a-b}=3$

41. If $a-b=x-y$, shew that $a^2+b^2-\dfrac{a^3}{a+b}=x^2+y^2-\dfrac{x^3}{x+y}$.

42. The crew of a ship consisted of her complement of sailors and a number of soldiers, there were 22 soldiers to every 3 guns and 10 over; also the whole number of hands was 5 times the number of sailors and guns together. After an engagement in which the slain were $\frac{1}{4}$ of the survivors, there wanted 5 to be 13 men to every 2 guns. Required the number of guns, soldiers and sailors.

43. Find the value of $\dfrac{a}{b}-\sqrt{\dfrac{1+a}{1-b}}+\sqrt[5]{\dfrac{5a^2}{2b^4}}$, when $a=\dfrac{1}{4}$ and $b=\dfrac{1}{5}$.

44. Simplify $4\left\{a-\dfrac{3}{2}\left(b-\dfrac{4c}{3}\right)\right\} \left\{\dfrac{1}{2}(2a-b)+2(b-c)\right\}$.

45. Find the four factors of $(1+y)^2-2(1+y^2)x^2+(1-y)^2x^4$.

46. If $s=a+b+c$, prove that

$$(s-3a)^2+(s-3b)^2+(s-3c)^2=3\{(a-b)^2+(b-c)^2+(c-a)^2\}.$$

47. Shew that $\dfrac{a(x^2+ab)}{b(a^2+b^2)}-\dfrac{x^2}{ab}+\dfrac{b(x^2-ab)}{a(a^2+b^2)}$ is independent of x

48 If $x=\dfrac{a+b}{c-d}$, shew that $(a-cx)^2+(x^2-1)(b^2-d^2)$ is a complete square.

49. If $\dfrac{1}{s}=\dfrac{1}{a}+\dfrac{1}{b}+\dfrac{1}{c}+\dots$ to n terms, shew that

$$\frac{s-a}{a}+\frac{s-b}{b}+\frac{s-c}{c}+\dots \text{ to } n \text{ terms}=1-n$$

50. If $t=\dfrac{2}{2-w}$, $w=\dfrac{2}{2-z}$, $z=\dfrac{2}{2-y}$, $y=\dfrac{2}{2-x}$, then $t=x$.

51 Solve $\dfrac{1}{1-\dfrac{1}{x}}-\dfrac{\dfrac{1}{x}}{x-1}=6$

52 Solve $x+y+z=1$,

$$ax+by+cz=d,$$
$$a^2x+b^2y+c^2z=d^2.$$

53 If $a : b=c : d$, prove that

$$(la^3+mb^3)\left(\frac{l}{a^3}+\frac{m}{b^3}\right)=(ld^3+mc^3)\left(\frac{l}{d^3}+\frac{m}{c^3}\right).$$

54 The express leaves Bristol at 3 P M and reaches London at 6 P M , the ordinary train leaves London at 1 30 P M and arrives at Bristol at 6 P M If both trains travel uniformly, find the time when they will meet

55 Simplify $3a-[b+\{2a-(b-c)\}]+\frac{1}{2}+\dfrac{2c^2-\frac{1}{2}}{2c+1}$.

56 Resolve $\{x^2-(b+c)x+bc\}^2-(x-c)^2(x-a)^2$

57 If $s=a+b+c$, prove that

$$(as+bc)(bs+ca)(cs+ab)=(b+c)^2(c+a)^2(a+b)^2$$

58 Prove that

$$\left(x+\frac{a}{b}y\right)\left(x+\frac{b}{c}y\right)\left(x+\frac{c}{a}y\right)-\left(x+\frac{b}{a}y\right)\left(x+\frac{c}{b}y\right)\left(x+\frac{a}{c}y\right)$$
$$=\frac{xy(x-y)(a-b)(b-c)(c-a)}{abc}.$$

59. Find the value of $\dfrac{3^{\frac{2}{3}}+3^{\frac{1}{3}}+1}{3^{\frac{1}{3}}+1}+\dfrac{3^{\frac{2}{3}}-3^{\frac{1}{3}}+1}{3^{\frac{1}{3}}-1}$.

60 Simplify $\dfrac{\dfrac{x}{y}+1+\dfrac{y}{x}}{\dfrac{x}{y}-1+\dfrac{y}{x}}\times\dfrac{1+\dfrac{y^3}{x^3}}{1-\dfrac{y^3}{x^3}}-\dfrac{(x+y)^2}{x^2-y^2}$.

61 Find the value of $(x^a)^{b-c}(x^b)^{c-a}(x^c)^{a-b}$

62 The coefficient of x in the expansion of

$$(x-a)(x-b)(x-c)(x-d)(x-f) \text{ is } \frac{1}{A}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}+\frac{1}{f}\right),$$

find the value of A.

63. If $a+b+c=0$, prove that $\dfrac{a^2+b^2+c^2}{a^3+b^3+c^3}+\dfrac{2}{3}\left(\dfrac{1}{a}+\dfrac{1}{b}+\dfrac{1}{c}\right)=0$

64. Solve $\dfrac{2x}{3}-\dfrac{1-\frac{x}{2}}{4x}=\dfrac{8x+1}{12}$.

65. Solve $x+y+z=a+b+c$,

$$bx+cy+az=cx+ay+bz=bc+ca+ab$$

66 If a, b, c, are in continued proportion, prove that

$$\left(\frac{b}{c}+\frac{c}{a}\right) \quad \left(\frac{c}{a}+\frac{a}{b}\right)$$

is a ratio of equality

67 At an election, where each elector may give 2 votes to different candidates, but only one to the same, it is found on casting up the poll that of 3 candidates A, B and C, A has 158 votes, B 132, and C 118 Now 26 electors voted for A alone, 30 for B alone, and 28 for C alone How many voted for A and B jointly, how many for A and C, and how many for B and C?

68 Find the value of $\sqrt{\dfrac{2xz(x-z)}{y^3}}$, when $x=1$, $y=\cdot 2$, $z=\cdot 02$.

69 Resolve $ab(x^2+y^2)+(a^2+b^2)xy+(a-b)(x-y)-1$

70 Simplify

$$(a+x)^4+4(a+x)^3(a-x)+6(a^2-x^2)^2+4(a+x)(a-x)^3+(a-x)^4$$

71 If $3s=a+b+c$, prove that $(s-a)^4+(s-b)^4+(s-c)^4$

$$=2\{(s-a)^2(s-b)^2+(s-b)^2(s-c)^2+(s-c)^2(s-a)^2\}$$

72 Simplify $\dfrac{x-2}{x-2-\dfrac{x}{x-\dfrac{x-1}{x-2}}}$ 73. Reduce $\dfrac{a^6+2a^4x+a^3x^2}{a^3x+2a^2x^2+2ax^3+x^4}$.

74 Find the value of c which will make $x^4+5x^3+7x^2+cx-2c$ divisible without remainder by x^2+3x+2.

75 Establish the identity

$$\frac{x^2-xy+y^2}{x^2y^2}\left\{\frac{1}{x}+\frac{1}{y}\right\}=\frac{y^2-yz+z^2}{y^2z^2}\left\{\frac{1}{y}+\frac{1}{z}\right\}+\frac{x^2+xz+z^2}{x^2z^2}\left\{\frac{1}{x}-\frac{1}{z}\right\}.$$

76 Shew that $(2x+y^{-1})(2y+x^{-1})=\left(2x^{\frac{1}{2}}y^{\frac{1}{2}}+x^{-\frac{1}{2}}y^{-\frac{1}{2}}\right)^2$.

77. Simplify $\left\{\dfrac{a^{\frac{x}{x-y}}\sqrt[y]{b^{x-y}}}{\sqrt[y]{a^{x+y}b^{x}}\times a^{\frac{y}{x-y}}}\right\}^{-y}$.

78 Solve $3-\dfrac{2}{4+\dfrac{3}{1-x}}=\dfrac{11}{3}$.

79 Solve $\dfrac{5\sqrt{x+y}}{x}+\dfrac{5\sqrt{x+y}}{y}=10\frac{2}{3}$, $\dfrac{3\sqrt{x-y}}{y}-\dfrac{3\sqrt{x-y}}{x}=\dfrac{4}{5}$.

80 If $a : b=c : d$, shew that

$$la+mb : lc+md=\sqrt{(pa^2+qb^2)} : \sqrt{(pc^2+qd^2)}.$$

81. A letter carrier has to go daily from P to Q in a prescribed time If he goes a mile an hour faster than his ordinary rate, he arrives at Q half an hour before the time But if he goes a mile an hour slower, he arrives three-quarters of an hour too late Find his ordinary rate and the distance from P to Q

82. Find the value, when $a=2$, $b=3$, of

$$a^{a-1}b^{b+1}-(a+1)^b(b-1)^a+a^{b-1}b^{a+1}.$$

83. Simplify $(a+b+c)(x+y+z)+(b+c-a)(y+z-x)$
$+(c+a-b)(z+x-y)+(a+b-c)(x+y-z)$

84 Resolve into factors

$$(x-y)(x-2y)(x-3y)+9y(x-y)(x-2y)+18y^2(x-y)+6y^3$$

85 Expand $\{(x^2+x+1)(x^2-x+1)(x^2-1)\}^2$ in powers of x

86 Find the square root of $(x^2-3x+2)(x^2-4x+3)(x^2-5x+6)$

87 If $2s=a+b+c$, $2\sigma^2=a^2+b^2+c^2$, shew that

$$(\sigma^2-a^2)(\sigma^2-b^2)+(\sigma^2-b^2)(\sigma^2-c^2)+(\sigma^2-c^2)(\sigma^2-a^2)=4s(s-a)(s-b)(s-c).$$

88 Simplify $\left\{\dfrac{\sqrt{x}}{x^{\frac{3}{2}}-1}-\dfrac{1}{x^{\frac{3}{2}}+x+\sqrt{x}}\right\}-\left\{\dfrac{x^{\frac{3}{2}}+1}{\sqrt{x}}-\dfrac{x^{\frac{3}{2}}-1}{\sqrt{(x+1)}}\right\}$

89 If $x+y=a$ and $xy=b^2$, express x^3+y^3 and x^4+y^4 in terms of a and b

90 Prove the identity

$$\frac{1}{a+x}+\frac{2x}{a^2+x^2}+\frac{4x^3}{a^4+x^4}+\frac{8x^7}{a^8+x^8}=\frac{1}{a-x}-\frac{16x^{15}}{a^{16}-x^{16}}.$$

91. If $x=\dfrac{1}{\sqrt{a^2+b^2}}$, then $\dfrac{a^3}{1\mp bx}+\dfrac{b^3}{1\pm ax}=(a+b)(a^2+b^2)$.

92. Solve $\dfrac{x}{a^2+b^2}+\dfrac{x}{c^2+d^2}=\dfrac{a^2+b^2+c^2+d^2}{(ac+bd)^2+(bc-ad)^2}$

93 Solve $5^{x-2}+3^{y-2}=14$, $5^{x+1}-3^{y+2}+104=0$

94. If $a+b+c=0$, then

$$\frac{(b-c)^2(b+c-a)}{a(a-b)(a-c)}+\frac{(c-a)^2(c+a-b)}{b(b-c)(b-a)}+\frac{(a-b)^2(a+b-c)}{c(c-a)(c-b)}=6.$$

95 A criminal having escaped from prison, travelled 10 hours before his escape was known He was pursued so as to be gained upon 3 miles an hour After his pursuers had travelled 3 hours, they met an Express going at the same rate as themselves, who met the criminal 2 hours 24 minutes before In what time after the commencement of the pursuit will they overtake him ?

96 Find the value of $\sqrt[5]{\frac{3}{5}(3b+c)(b^2-2a-c)(2b-a)^2}$, when $a=5$, $b=4$, $c=3$

97. Multiply $x^2+(a-1)x+(a+1)$ by $(a-1)x-(a^2+a+1)$.

98. Shew that $(a+b+c)(ax^2+by^2+cz^2)-(ax+by+cz)^2$

$$=ab(x-y)^2+bc(y-z)^2+ca(z-x)^2$$

99 Resolve into factors $(a^2x^2-b^2y^2)^2+4abxy(bx+ay)^2-(bx+ay)^4$.

100. Shew that

$$\frac{a^2}{(a-b)(a-c)(1+ax)}+\frac{b^2}{(b-c)(b-a)(1+bx)}+\frac{c^2}{(c-a)(c-b)(1+c}$$

$$=\frac{1}{(1+ax)(1+bx)(1+cx)}.$$

101 If $2s=a+b+c$, $2\sigma^3=a^3+b^3+c^3$, prove that

$$(\sigma^3-a^3)(s-a)+(\sigma^3-b^3)(s-b)+(\sigma^3-c^3)(s-c)=a^4+b^4+c^4-s\sigma^3$$

102. Extract the square root of $\dfrac{a^2}{b^2c^2}+\dfrac{b^2}{c^2a^2}+\dfrac{c^2}{a^2b^2}+2\left(\dfrac{1}{a^2}+\dfrac{1}{b^2}+\dfrac{1}{c^2}\right)$.

103 Simplify $\dfrac{a^2}{(x-a)^n}+\dfrac{2a}{(x-a)^{n-1}}+\dfrac{1}{(x-a)^{n-2}}$.

104. Find the value of $\dfrac{(x^{a+b})^2(x^{b+c})^2(x^{c+a})^2}{(x^a x^b x^c)^4}$

105. If $x=\dfrac{2ac}{a+c}$, shew that the value of $\dfrac{(x-a)^2+(x-c)^2}{a^2+c^2}+\dfrac{4ac}{(a+c)^2}$ is the same for all values of a and c.

106. Simplify $\dfrac{\sqrt{a+x}-\sqrt{a-x}}{a+x+\sqrt{a^2+x^2}}\times\dfrac{\sqrt{a^2-x^2}}{\sqrt{a}+\sqrt{x}}\times\dfrac{x}{a-\sqrt{a^2-x^2}}$.

107. If $x+y=xy=1$, prove that $x^3=y^3=-1$.

108. Solve $\dfrac{1-\dfrac{1-\dfrac{a}{a-x}}{3-x}}{x}=\dfrac{1}{x-a}$

109. Solve $(a+b)x-(a-b)y=3ab$, $(a+b)y-(a-b)x=ab$.

110. If $a : b=x : y=p : q$, shew that $\left(\dfrac{a^3+x^3}{b^3+y^3}\right)^2=\dfrac{p^6}{q^6}$.

111. A, B, C, start at the same time from P to run to Q, their rates being such that B is always as much behind A as he is in advance of C. After A has reached Q he returns at once to P at the same rate, and meets B at a point whose distance from Q is equal to one-fourth of PQ. Shew that A meets C at a distance from P equal to one-third of PQ.

112. Find the value of $(\sqrt{x^2+y^2}-z)(\sqrt{x^2+y^2}+z)$, when $x=4$, $y=5$, $z=6$.

113. Simplify $\frac{1}{2}\{a-5(b-a)\}-\dfrac{3}{2}\left\{\dfrac{1}{3}\left(b-\dfrac{a}{3}\right)-\dfrac{2}{9}\left[a-\dfrac{3}{4}\left(b-\dfrac{4a}{5}\right)\right]\right\}$.

114. Resolve $(1+xz)^2(1+yz)^2-\{(1-xz)(1-yz)+2xyz\}^2$.

115. Find the square root of

$$x^4+2x^3(y+z)+x^2(y^2+z^2+4yz)+2xyz(y+z)+y^2z^2.$$

116. Find the value of $\sqrt{12}+\frac{1}{15}\sqrt{75}+6\sqrt{\frac{1}{12}}$.

117. Simplify $\dfrac{\left\{\dfrac{x}{1+x}+\dfrac{1-x}{x}\right\}\div\left\{\dfrac{x}{1+x}-\dfrac{1-x}{x}\right\}}{\left(1+\dfrac{x}{1-x}\right)\left(1-\dfrac{x}{1+x}\right)}$.

118. If $\dfrac{x}{b+c-a}=\dfrac{y}{c+a-b}=\dfrac{z}{a+b-c}$, prove that

$$(a+b+c)(yz+zx+xy)=(x+y+z)(ax+by+cz)$$

119. Show that

$$\frac{(ab-cd)(a^2-b^2+c^2-d^2)+(ac-bd)(a^2+b^2-c^2-d^2)}{(a^2-b^2+c^2-d^2)(a^2+b^2-c^2-d^2)+4(ab-cd)(ac-bd)}$$
$$=\frac{(a+d)(b+c)}{(a+d)^2+(b+c)^2}$$

120. If $c=a\sqrt{1-b^2}+b\sqrt{1-a^2}$, prove that

$(a+b+c)(b+c-a)(c+a-b)(a+b-c)=4a^2b^2c^2$

121. Solve $(x-a)^3+(x-b)^3+(x-c)^3=3(x-a)(x-b)(x-c)$.

122 Solve $ax+by+cz=a+b,\ bx+cy+az=b+c,\ cx+ay+bz=c+a.$

123. If $2a-b \quad a+2b=3c-d \quad c+2d$, then

$$3a+b \quad a-3b=4c+d \quad 2c-3d$$

124. A was following B, who after a time turned and without changing his pace walked in the opposite direction, A now approached B six times as fast as before. Compare their rates of walking.

125. Find the value of $x^4-2a(a-b)x^2+(a^2+b^2)(a-b)x-a^2b^2$, when $a=1,\ b=-2,\ x=3$

126. Divide $9a^3b^3-12a^4b+3b^6+2a^3b^3+4a^6-11ab^4$ by $3b^3+4a^3-2ab^2$

127 Resolve into linear factors

$$(x-a)^3(b-c)^3+(x-b)^3(c-a)^3+(x-c)^3(a-b)^3$$

128 Simplify $\dfrac{(yz-a^2)-(ca-y^2)(ab-z^2)}{(bc-1)(yz-a)-(z-by)(y-cz)}$

129. Reduce $\left(2-\dfrac{3x}{y}+\dfrac{9x^2-2y^2}{y^2+3xy}\right)\div\left\{\dfrac{1}{y}-\dfrac{1}{y-2x-\dfrac{4x^2}{x+y}}\right\}$

130 Find the fourth root of

$$x^8-\frac{3a^2x^7}{b}+\frac{27a^4x^6}{8b^2}-\frac{27a^6x^5}{16b^3}+\frac{81a^8x^4}{256b^4}$$

131 Simplify $1+\sqrt{8}+\sqrt{2}-\sqrt{27}-\sqrt{12}+\sqrt{75}-\sqrt{19+6\sqrt{2}}$

132 If $x^y=y^x$, prove that $\left(\dfrac{x}{y}\right)^{\frac{x}{y}}=x^{\frac{x}{y}-1}$, and that $y=2$, if also $x=2y$

133 If $x=\dfrac{1}{2}\left\{a+\sqrt{\left(\dfrac{4b^3-a^3}{3a}\right)}\right\}$ and $y=\dfrac{1}{2}\left\{a-\sqrt{\left(\dfrac{4b^3-a^3}{3a}\right)}\right\}$, find the value of x^3+y^3.

134. Solve $\sqrt{\dfrac{x}{a}}+\sqrt{\dfrac{(b-c)(ac-bx)}{abc}}=1$

135. Solve $(a+b)x-(a-b)y=m(x^2-y^2)$,

$$(a-b)x+(a+b)y=m(x^2-y^2)$$

136 If $\dfrac{x}{b-c+a}=\dfrac{y}{c-a+b}=\dfrac{z}{a-b+c}$, then each

$$=\frac{a(y+z)+b(z+x)+c(x+y)}{2(bc+ca+ab)}$$

137 A fruiterer sold for 19*s* 6*d.* a certain number of oranges and apples, of which the latter exceeded the former by 180 He sells the apples at the rate of 5 for $3d$, and 15 oranges bring him in $1\frac{1}{2}d.$ more than 35 apples How many are there of each sort, and what are the oranges worth a-piece ?

138. From $\{m(2m-3p)-2n(4n-3p)\}x+\{m(p-m)-p(2n+p)\}y$ take $3\{p\left(2n-\frac{3p}{2}\right)-\frac{p}{2}(2m-3p)\}x-\{p(p-m)+2n(2n+p)\}y$,

and find the value of this difference when $x=n=\frac{1}{2}$, $y=m=-2$

139 Reduce $\dfrac{12x^4-4ax^3-23a^2x^2+9a^3x-9a^4}{8x^4-14a^2x^2-9a^4}$

140 Multiply $\dfrac{x^3}{2}-5\dfrac{x^2}{3}+\dfrac{x}{12}+3$ by $\dfrac{x^2}{2}-x+3$

141. Resolve into linear factors $2x^2-7xy-22y^2-5x+35y-3$

142. Simplify $\dfrac{\left(\frac{x}{y}+1\right)^2}{\frac{x}{y}-\frac{y}{x}}\times\dfrac{\frac{x^3}{y^3}-1}{\frac{x^2}{y^2}+\frac{x}{y}+1}\div\dfrac{\frac{x^3}{y^3}+1}{\frac{x}{y}+\frac{y}{x}-1}$

143 If x be the L C M and y the H C F of a and b, and if $x+y=ma+\dfrac{b}{m}$, prove that $x^3+y^3=m^3a^3+\dfrac{b^3}{m^3}$

144 Find the value of $\left\{\dfrac{c^{-n}}{d^{-3q}}+\dfrac{a^{-\frac{m}{2}}}{b^{-p}}\right\}\left\{\dfrac{a^{-\frac{m}{2}}}{b^{-2p}}-\dfrac{c^{-n}}{d^{-3q}}\right\}$.

145 If $x(y-z)^2-z(y+z)^2=0$, then $\dfrac{\sqrt{x}+\sqrt{z}}{\sqrt{x}-\sqrt{z}}=\dfrac{y}{z}$.

146 Solve $\dfrac{3-2x}{1-2x}-\dfrac{2x-5}{2x-7}=1-\dfrac{4(x^2-1)}{7-16x+4x^2}$.

147. Solve $x+y+z=(a-b)(a-c)(b-c)$,

$$ax+by+cz=a^2x+b^2y+c^2z=0$$

148. If a, b, c, d be in continued proportion, shew that

$$\left(\frac{a-b}{c}+\frac{a-c}{b}\right)^2-\left(\frac{d-b}{c}+\frac{d-c}{b}\right)^2=(a-d)^2\left(\frac{1}{c^2}-\frac{1}{b^2}\right).$$

149 The sides of a right-angled triangle are as 5 to 12, and the hypotenuse is 130, find the sides

150. Find the value of $a^4+3a^3b+4a^2b^2+3ab^3+b^4$,

when $a=\dfrac{5+\sqrt{13}}{2}$ and $b=\dfrac{5-\sqrt{13}}{2}$

151 Multiply $x^6+a^6-ax(x^3+a^3)$ by $x^3+a^3-ax(x+a)$

152 Find $(x^2-2xy+4y^2)^{\frac{1}{2}}$ in terms of a and b, where $x=9a^2+12ab$ and $y=2b^2+6ab$

153 Multiply together $\sqrt{x}-3\sqrt{y}$, $\sqrt{x}-2\sqrt{y}$, $\sqrt{x}-\sqrt{y}$, $\sqrt{x}+\sqrt{y}$, $\sqrt{x}+2\sqrt{y}$, and $\sqrt{x}+3\sqrt{y}$

154 Extract the square root of

$$9\frac{x}{y}-24\sqrt{\frac{x}{y}}+34-24\sqrt{\frac{y}{x}}+9\frac{y}{x}$$

155 Shew that $\dfrac{x^2y^{-2}-xy^{-1}-x^{-1}y+x^{-2}y^2}{x^2y^{-2}-2+x^{-2}y^2}=1-\dfrac{xy}{(x+y)^2}$.

156 If $\sqrt{a^2+b^2}+a=bx$, find $x-x^{-1}$ and $x+x^{-1}$ in terms of a and b

157 Shew that $\sqrt[3]{(\sqrt{5}+2)}-\sqrt[3]{(\sqrt{5}-2)}=1$

158 Solve $\dfrac{x+\dfrac{1}{x}}{x^2+1}-\dfrac{1}{x+1}=\dfrac{3}{x^4+2x+1}$

159 Solve $x^3+y^3+z^3-3xyz=0$, $3a-x+z=3b-y+x=3c-z+y$

160 If $\dfrac{x}{2a-b+2c}=\dfrac{y}{2b-c+2a}=\dfrac{z}{2c-a+2b}$,

then $\dfrac{a}{2x+2y-z}=\dfrac{b}{2y+2z-x}=\dfrac{c}{2z+2x-y}$.

161. A has twice as many pennies as shillings B, who has 1s [illegible]d more than A, has twice as many shillings as pennies together they have one more penny than they have shillings How much has each?

162 Prove that $(ax-by+cz-du)^2+(ay+bx-cu-dz)^2$

$$+(az-bu-cx+dy)^2+(au+bz+cy+dx)^2$$
$$=(a^2+b^2+c^2+d^2)(x^2+y^2+z^2+u^2)$$

163 Divide $x^8+\dfrac{x^6}{y^2}+\dfrac{x^4}{y^4}+\dfrac{x^2}{y^6}+\dfrac{1}{y^8}$ by $x^4-\dfrac{x^3}{y}+\dfrac{x^2}{y^2}-\dfrac{x}{y^3}+\dfrac{1}{y^4}$

164 Find a value of x which will make $x^4+6x^3+11x^2+3x+31$ a perfect square.

165 Shew that the difference of $(e^x+e^{-x})^2$ and $(e^x-e^{-x})^2$ is independent of x, and that the sum of $\left(\dfrac{1}{x}+x\right)\left(\dfrac{1}{x}-x\right)$ and $\left(\dfrac{1}{x}+x\right)^2$ diminishes as x increases

166 Find the square root of $(3x+1)(3x+4)(3x+7)(3x+10)+81$

167. Simplify $\dfrac{a^2\left(\frac{1}{b}-\frac{1}{c}\right)+b^2\left(\frac{1}{c}-\frac{1}{a}\right)+c^2\left(\frac{1}{a}-\frac{1}{b}\right)}{a\left(\frac{1}{b}-\frac{1}{c}\right)+b\left(\frac{1}{c}-\frac{1}{a}\right)+c\left(\frac{1}{a}-\frac{1}{b}\right)}$

168. Find the value of $\dfrac{1+x}{1+\sqrt{1+x}}+\dfrac{1-x}{1+\sqrt{1-x}}$, when $x=\dfrac{\sqrt{3}}{2}$.

169. If $m=a^x$, $n=a^y$, $a^2=(m^y n^x)^z$, shew that $xyz=1$.

170. Prove that $\sqrt{y+\sqrt{2xy-x^2}}+\sqrt{y-\sqrt{2xy-x^2}}=\sqrt{2x}$.

171. Solve $\dfrac{a}{x+b-c}+\dfrac{b}{x+a-c}=\dfrac{b-c}{x+a}+\dfrac{a-c}{x+b}$.

172. Solve $\dfrac{x+y-z}{b+c}=\dfrac{y+z-x}{c+a}=\dfrac{z+x-y}{a+b}=a+b+c$.

173. If $\dfrac{x}{a+2b+c}=\dfrac{y}{2a+b-c}=\dfrac{z}{4a-4b+c}$,

then $\dfrac{a}{x+2y+z}=\dfrac{b}{2x+y-z}=\dfrac{c}{4x-4y+z}$.

174. A waterman finds that he can row with the tide from A to B, a distance of 18 miles, in an hour and a half, and that to return from B to A, against the same tide, though he rows back along the shore where the stream is only three-fifths as strong as in the middle, takes him just 2 hours and a quarter. Find the rate at which the tide runs in the middle where it is strongest.

175. Multiply $(2a-3c)y^2-(a-c)y+(2a+c)$

by $(2a+3c)y^2+(a+c)y-(2a-c)$.

176. Find the co-efficient of x^4 in $\left(1-\dfrac{x}{2}+\dfrac{x^2}{4}-\dfrac{x^3}{8}+\dfrac{x^4}{16}\right)^2$.

177. Resolve

$$(b-c)(b+c-2a)^2+(c-a)(c+a-2b)^2+(a-b)(a+b-2c)^2.$$

178. Simplify

$$\frac{a^4}{(a-b)(a-c)}+\frac{b^4}{(b-a)(b-c)}+\frac{c^4}{(c-a)(c-b)}.$$

179. Reduce $\dfrac{e^{x-y}+xy^{-1}+yx^{-1}+e^{y-x}}{xy^{-1}e^{x-y}+2+yx^{-1}e^{y-x}}$.

180. If $y=x+\dfrac{1}{x}$, express $x^5+\dfrac{1}{x^5}$ in terms of y.

181. If $x=\left(\frac{a+b}{a-b}\right)^{\frac{2mn}{m-n}}$, then $\frac{1}{2}\left(\frac{a^2-b^2}{a^2+b^2}\right)(\sqrt[m]{x}+\sqrt[n]{x})=\left(\frac{a+b}{a-b}\right)^{\frac{m+n}{m-n}}$.

182 If $x=\left\{-\frac{a}{2}+\sqrt{\left(\frac{a^2}{4}-\frac{b^3}{27}\right)}\right\}^{\frac{1}{3}}+\left\{-\frac{a}{2}-\sqrt{\left(\frac{a^2}{4}-\frac{b^3}{27}\right)}\right\}^{\frac{1}{3}}$,

prove that $$x^3-bx+a=0$$

183 Solve

$$(x-9)(x-7)(x-5)(x-1)=(x-2)(x-4)(x-6)(x-10)$$

184 Solve $\sqrt{y}-\sqrt{a-x}=\sqrt{y-x}$,

$$\sqrt{y-x}+\sqrt{a-x} \quad \sqrt{a-x} \quad 5 . 2$$

185 Two persons A and B run round a field, starting from the same point in opposite directions, A reaches the starting-point 4 minutes and B 9 minutes after they meet, if they continue to run at the same rate, in what time will they meet at the starting-point?

186 Find the product of $\frac{1}{1-ax}$, $\frac{1}{1-bx}$ and $\frac{1}{1-cx}$, and keep it in an *integral form*

187 Reduce $\frac{px^3-(p-1)qx^2+(p-q^2)x-pq}{pqx^3+(p^2+q^2)x^2+2pqx+p^2}$

188 Shew that $\left(\frac{x}{y}+\frac{y}{z}+\frac{z}{x}\right)\left(\frac{y}{x}+\frac{z}{y}+\frac{x}{z}\right)-1=\left(\frac{y}{z}+\frac{z}{x}\right)\left(\frac{z}{x}+\frac{x}{y}\right)\left(\frac{x}{y}+\frac{y}{z}\right)$.

189 If $\frac{x}{a}=\frac{y}{b}$ and $\frac{x(1-a)}{a(1-x)}+\frac{y(1-b)}{b(1-y)}=1$,

then $\frac{x}{a}=\frac{y}{b}=\frac{1}{1-\sqrt{(1-a)(1-b)}}$

190 If $x=\left(\frac{1-e}{1+e}\right)^{\frac{1}{2}}$, then shall $\frac{1-x}{1+x}=\frac{e}{1+\sqrt{1-e^2}}$.

191 Find the value of

$$\left\{\left(\frac{m^{-1}-n^{-1}}{a^{-2}-a^{-1}b^{-1}+b^{-2}}\right)^{-3}\right\}^{-4}\left\{\left(\frac{m^{-2}-n^{-2}}{a^{-3}+b^{-3}}\right)^{2}\right\}^{-6}.$$

192. Prove that $\dfrac{\frac{x}{\sqrt{1+x^2}}-\frac{x}{\sqrt{1-x^2}}}{\sqrt{1+x^2}+\sqrt{1-x^2}}=\frac{1}{x}\left\{1-\frac{1}{\sqrt{1-x^4}}\right\}$.

193. Solve $(x-2a)^3+(x-2b)^3=2(x-a-b)^3$

194 Solve

$$x-\frac{y+12}{5} : y-\frac{24-5x}{8} \quad 1 . 5, \quad \frac{x-2}{2}-\frac{19-y}{3}=\frac{\frac{x}{2}+8}{6}-\frac{y-9}{4}.$$

195 Fifteen guineas should weigh 4 oz. but a parcel of light gold having been weighed and counted, was found to contain 9 more guineas than was supposed from the weight and a part of the whole, exceeding the half by four guineas and a half, was found to be $1\frac{1}{2}$ oz. deficient in weight. What was the number of guineas ?

196 If $$A=c^2(ax+b)^2\{3a(cx+d)-c(ax+b)\},$$ $$B=a^2(cx+d)^2\{3c(ax+b)-a(cx+d)\},$$

the simplest value of $(A-B)^{\frac{1}{3}}$.

197. Multiply $$(a+b)x^2-abxy+(a-b)y^2,$$
by $$(a-b)x^2+abxy+(a+b)y^2$$

198 Find the co-efficient of x^6 in the product of
$$1+x+x^2+\ldots \text{ and } 1-x+x^2-\ldots.$$

199 Resolve
$$(y-z)^2(y+z-2x)+(z-x)^2(z+x-2y)+(x-y)^2(x+y-2z)$$

200 Simplify $$\frac{x^4(y-z)+y^4(z-x)+z^4(x-y)}{x^2(y-z)+y^2(z-x)+z^2(x-y)}$$

201. Reduce to its simplest form
$$\frac{1-x}{1+x}+\frac{x-y}{x+y}+\frac{y-1}{y+1}+\frac{(1-x)(x-y)(y-1)}{(1+x)(x+y)(y+1)}$$

202. Find the value of
$$\frac{\sqrt{x^2+a^2}+\sqrt{x^2-a^2}}{\sqrt{x^2+a^2}-\sqrt{x^2-a^2}}, \text{ when } x=\sqrt{\frac{a^4+1}{2}}.$$

203. If $$u=x\sqrt{1+y^2}+y\sqrt{1+x^2},$$
prove that $$\sqrt{1+u^2}=xy+\sqrt{(1+x^2)(1+y^2)}$$

204. Solve (1) $2ax^n-(b-1)x^{n+1}=(a-1)x^n+bx^{n+1}$

(2) $\sqrt{2}+1-\dfrac{1}{2^{\frac{1}{x}}-1}=0.$

205 Solve $$x^m y^n=a^{2m+n}b^n, \; x^n y^m=a^n b^{2m+n}.$$

206. If $$x(b-c)+y(c-a)+z(a-b)=0,$$
shew that $$\frac{bz-cy}{b-c}=\frac{cx-az}{c-a}=\frac{ay-bx}{a-b}.$$

207. Two loaded waggons were weighed, and their weights were found to be in the ratio of 4 to 5, parts of their loads, which were in the ratio of 6 to 7, being taken out, their weights were then found to be as the numbers 2 and 3, and the sum of their weights was then 10 tons What were the weights at first?

208. Shew that $(x^2+y^2-1)^2+(x'^2+y'^2-1)^2+2(xx'+yy')^2$
$$=(x^2+x'^2-1)^2+(y^2+y'^2-1)^2+2(xy+x'y')^2.$$

209 Resolve into four factors $x^2+y^2+z^2-2xy-2yz-2xz$

210 Reduce $\dfrac{(1-xy)(1+xy)-(x-y)(x+y)}{(1+xy)^2+(x-y)^2}$ to its lowest terms

211 Simplify $\dfrac{a^4(b^2-c^2)+b^4(c^2-a^2)+c^4(a^2-b^2)}{a^3(b^2-c^2)+b^3(c^2-a^2)+c^3(a^2-b^2)}$

212 Find the square root of $\dfrac{9x^{-\frac{4}{m}}}{4}-a^{-\frac{1}{n}}x^{-\frac{2a+m}{am}}+\dfrac{x^{-\frac{2}{a}}}{9a^{\frac{2}{n}}}$

213. If $xyz=1$, shew that
$(1+x+y^{-1})^{-1}+(1+y+z^{-1})^{-1}+(1+z+x^{-1})^{-1}=1.$

214 Simplify $\left\{\sqrt[4]{\dfrac{(x+y)^{\frac{1}{2}}+(x-y)^{\frac{1}{2}}}{(z+y)^{\frac{1}{2}}+(z-y)^{\frac{1}{2}}}}\times\dfrac{\sqrt{x+y}-\sqrt{x-y}}{\sqrt{z+y}-\sqrt{z-y}}\right\}^2.$

215 Solve $\sqrt[4]{\dfrac{a+x}{a-x}}-\sqrt[4]{\dfrac{a-x}{a+x}}=\sqrt[4]{b}$

216 Solve $\left(\dfrac{x}{a}\right)^m\left(\dfrac{y}{b}\right)^n=c,\ \left(\dfrac{x}{b}\right)^n\left(\dfrac{y}{a}\right)^m=d$

217 If $x-z \quad y-z=x^2 \quad y^2$, shew that $x+z : y+z=\dfrac{x}{y}+2 \quad \dfrac{y}{x}+2$

218 A pedestrian finding that he could walk forwards 4 times as fast as he could backwards, undertook to walk a certain distance ($\frac{1}{4}$ of it backwards) in a given time But the ground being bad, he found that his rate per hour backwards was $\frac{1}{8}$th of a mile less than he had supposed, and that to have won his wager, he must have walked forwards 2 miles an hour faster than he did What was his rate per hour backwards?

219 If $x+y+a=0$, $xy-b=0$, then $(1+x^2)(1+y^2)=a^2+(1-b)^2$

220 Resolve $(a+b+c)^3abc-(bc+ca+ab)^3$

221. Reduce $\dfrac{(1-x^2)(1-y^2)(1-z^2)-(z+xy)(y+zx)(x+yz)}{1-x^2-y^2-z^2-2xyz}.$

222 Simplify

$$\frac{bc}{a(a^2-b^2)(a^2-c^2)}+\frac{ac}{b(b^2-a^2)(b^2-c^2)}+\frac{ab}{c(c^2-b^2)(c^2-a^2)}.$$

223 Find the square root of $\left(1-\dfrac{1}{1-x^2}\right)\left\{\dfrac{1}{1-\dfrac{1}{x^2}}-1\right\}$.

224 Prove that

$$\frac{x^3-3x+(x^2-1)\sqrt{x^2-4}-2}{x^3-3x+(x^2-1)\sqrt{x^2-4}+2}=\frac{x+1}{x-1}\sqrt{\frac{x-2}{x+2}}$$

225 If $x\sqrt{a^2-y^2}+y\sqrt{a^2-x^2}=a^2$, then $x^2+y^2=a^2$.

226 Solve $\sqrt{1+a}\left(\dfrac{1+x}{1-x}\right)^{\frac{1}{4}}+\sqrt{1-a}\left(\dfrac{1+x}{1+x}\right)^{\frac{1}{4}}=2\sqrt[4]{1-a^2}$.

227 The trinomial ax^2+bx+c becomes 8, 22 and 42 respectively, when x becomes 2, 3, 4; what does it become when $x=-\frac{1}{3}$?

228 If $\dfrac{x}{y}-\dfrac{y}{x}=a$, and $\dfrac{x^2}{y^2}-\dfrac{y^2}{x^2}=b$, then $a^4+4a^2=b^2$.

229 Reduce to its lowest terms $\dfrac{(x+y)^7-x^7-y^7}{(x+y)^5-x^5-y^5}$

230 If $y=\dfrac{ax-b}{a-bx}$, shew that $\dfrac{x-y}{1-xy}=\dfrac{b}{a}$

231 Solve $(\sqrt[4]{x+a}+\sqrt[4]{x-a})^3(\sqrt[4]{x+a}-\sqrt[4]{x-a})=2c$.

232. Solve $(x-a)^3(b-c)^3+(x-b)^3(c-a)^3+(x-c)^3(a-b)^3=0$

233. If $\dfrac{x}{a}=\dfrac{y}{b}=\dfrac{z}{c}$, prove that

$$\frac{ax-by}{(a-b)(x+y)}+\frac{by-cz}{(b-c)(y+z)}+\frac{cz-ax}{(c-a)(z+x)}=3$$

234 If $x+y : 3y=6x-y : 8x$, find the ratio $x : y$.

235 A person, sculling in a thick fog, meets one barge and over takes another which is going at the same rate as the former; if a be the greatest distance to which he can see, and b, b' the distances that he sculls between the times of his first seeing and passing the barges, shew that

$$\frac{2}{a}=\frac{1}{b}+\frac{1}{b'}$$

ANSWERS

I [*pp* 5—7.]

1 $20m$, $240m$, $4m$, $8m$ 2. $\frac{x}{20}$, $12x$; $\frac{x}{2}$. 3 $\frac{y}{240}$, $\frac{y}{12}$, $2y$.

4 $\frac{a}{192}$, $\frac{a}{12}$, $\frac{a}{96}$ 5 $3x$; $36x$, $\frac{x}{1760}$ 6 $12x$, $\frac{x}{3}$

7 $20m+n+\frac{p}{12}$, $m+\frac{n}{20}+\frac{p}{240}$ 8 $x+\frac{y}{16}+\frac{z}{192}$; $16x+y+\frac{z}{12}$.

9 $10x$, $100x$, $1000x$, $\frac{x}{1000}$ 10 $\frac{y}{10}$, $\frac{y}{1000}$, $\frac{y}{1000000}$.

25 $12-x=5$ 26 $m-15=2$ 30 $(x-10)$ years 38 $y-x=a$.

II [*p* 8]

1 15 2 23 3 12 4 22 5 4 6 2 7 19 8 2
9 0. 10 43 11. 15 12 55 13 530 14. 282 15. 210.
16 $2\frac{1}{5}$ 17 $\frac{1}{6}$ 18 $\frac{19}{20}$ 19 3 20 $1\frac{3}{5}$ 21 $2\frac{1}{5}$ 22 $\frac{4}{5}$.

III [*p* 9]

1 168. 2 108 3. 0 4 0 5 84 6 $\frac{1}{3}$ 7 $\frac{1}{-}$ 8 1
9 56 10 1120 11. 0 12 $4\frac{1}{2}$ 13 165 14. 0
15 10 16 64 17. 6 18 60 19 $\frac{8}{25}$ 20 0
21 $10\frac{19}{48}$ 22 6 23 88 24 39 25 98 26 42.
27 0, 0, 0 28 $\frac{17}{18}$ 29 $\frac{4}{11}$ 30. $3\frac{1}{8}$.

IV. [*pp* 10—11]

1. 15 2 1. 3. 40 4 16 5 23. 6 6. 7 2 8 8
9 8 10 16 11 1 12 14. 13 6 14. 8. 15 2.
16 2 17 26 18 11 19 1 20 48 21 24. 22 33
23 $4\frac{3}{4}$ 24 $3\frac{1}{4}$ 25 $21\frac{3}{4}$ 26 $2\frac{77}{82}$ 27 $3\frac{37}{205}$ 28 122.
29. $6\frac{5}{24}$ 30. $2\frac{31}{240}$ 31 10·55 32 ·035 33 3·4599.

V [*p* 12.]

1 125. 2 64 3. 784 4 1728 5 6250000 6 729
7 32 8 1536 9. 224 10 5332 11. 0 12 82944.

13 $115\frac{1}{5}$ 14 396 15 0 16 20480 17 196608.
18 0 19 25 20 64 21. 648 22 5 23 4096
24 1000000 25 1296000000000, 26 11025 27. 15552
28 $112\frac{1}{2}$ 29 180 30. 2688 31 291600000000
32 18144 33 24192 34 $7\frac{19}{32}$ 35 $\frac{15}{32}$ 36 $\frac{8}{27}$.
37 $\frac{9}{100}$ 38 $\frac{4}{27}$ 39 $3\frac{13}{81}$

VI [*pp* 13—14]

1 12 2 432 3 300 4. 108 5 4860 6. 6
7 120 8 810 9 27 10 24. 11 4. 12 $\frac{1}{2}$
13 6 14 $\frac{1}{12}$ 15. $\frac{1}{81}$ 16 $\frac{1}{90}$ 17 0 18 $2\frac{2}{3}$
19 $\frac{4}{9}$ 20 4 21 $\frac{1}{16}$ 22 $\frac{1}{8}$ 23 125 24. 109
25 23 26 13 27 17 28 37 29 73 30 3
31. 27 32 28 33 64 34 80

VII. [*p* 15]

1 25. 2 250 3 159 4 722 5. 204 6 24575.
7 29 8 230 9 20. 10 39 11 161 12. 723
13 65 14 15 15 4069 16 62 17 840 18 290
19. 243 20 232 21 $\frac{3}{16}$ 22 $1\frac{7}{35}$ 23 $\frac{23}{4}$. 24. 2
25 5 26 86 27 $3\frac{4}{15}$ 28 $2\frac{4}{15}$ 29 3, 9, 17, 27, 39, 53
30 35, 24, 15, 0, 0, 24, 80

VIII [*p* 20]

23 -10 24 -5 25 5 26 -2 27 -13 28 -3
29 $-x$ 30 $-13x$ 31 $-19a$ 32 $-11m$ 33 $-6y$
34 $8ab$ 35. $-16x^2$ 36 $-9x^3$. 37 0 38 0 39 $17x$
40 $-11a^2$ 41 Gains $-$R5, loses $+$R5 42 -2 miles
$+2$ miles 43 A, $+8m$, B, $-5m$, A, $-8m$, B, $+5m$
44 -1 min. *i e*, 1 min to 12 noon.

IX [*pp* 23—24]

1 $+1, -14, +2, -3$ 2 $-7, +1$ 3. -5 4 $+15, +7, -4$
5 $+6, -25, +18, -6$ 6. -6 7 -2 8 $+4$ 9 $+10$
10 $-4, +5$ 11 $+5, 0$ 12 $-2, -2$ 13 0 14 0 15. $+2$
16 -16 17. $+16$. 18 -24. 19 $+14$ 20 $+26$.

X. [p 25]

1 $+13, -13, -1, +1$ 2 $+5m, +11m, -11m, -5m$
3 $-2, -1, +1.$ 4 $-4, -1, -100$ 5 The latter by $+2$
6 The former. 7. Yes, by $+4$ 8 The former by $+3$

XI [p 26]

1 $16a$ 2 $17y$ 3 $56x$ 4. $26z$ 5 $24m$ 6 $37n$ 7 $33r.$
8 $46y$ 9 $-42c$ 10 $-25x$ 11 $-21y$ 12 $-25ab$
13 $-16xy^2$ 14 $-73max$ 15 $\frac{37}{5}ab$ 16 $\frac{24}{5}xyz$ 17 $-\frac{617}{28}mn.$
18 $\frac{83}{35}pq$ 19 $(a+b+c+d)x$ 20 $(a+p+9)x$

XII [pp 26—27]

1. $-3x$ 2 a 3 $-2x$ 4 $28m$ 5 $-27xy$ 6 $abc.$ 7 $0.$
8 $\frac{17}{10}axy$ 9 0 10 $\frac{17}{40}mn$ 11. $-8x$ 12 $-17z.$ 13. $-56by$
14 $-\frac{11}{4}ab$ 15 $(a-b+c-d+f)x^3$ 16 $(4a+4p-q)x.$
17 $(7+4a)x^2y$

XIV [pp 29—30]

1. y 2 $a-x$ 3 $ax+1$ 4 $2a+2c$ 5 $3a-2b-e.$
6 $-x+y+3z$ 7 $7a-13b$ 8 $4a-2b+2c+3d.$ 9 6
10 $2ax+2cy$ 11 y^2

XVI [pp 32—33]

1 $2x$ 2. 58 3 $5a$ 4 $5ax+by$ 5 $ab-\frac{11}{6}xy$
6 $38a+2b+4c$ 7 $-ab-2cd$ 8 $-a+15b-8c$ 9 $3y.$
10 $14x^3$ 11 $-9ax+6by-7cz$ 12 $15a+3b-6c+5d$
13 $7a^2+25ax+2y^2$ 14 $\frac{73}{6}x^2-\frac{53}{3}ab+\frac{37}{4}a^2$ 15 $12x^2+7y^2$
16 $6+8ab-4bc+15ac$ 17 $-9x^3+2ax^2-31a^2x+16a^3.$ 18 $a-e$
19 $3x-7z$ 20 $2b+d-f$ 21 $11b^2$ 22 $10x^3-6mn+11r^2.$
23 $-x^2+2xy+4x-y$ 24 $-2bx^2+6d-1$ 25 $7x^3-2xy-2y^2.$
26. $4r^2-2q^2$ 27 $5x^3+4y^3+2z^3-24xyz$

XVII [pp 34—35]

1 $a+b$ 2 $a+x$ 3 $2y$ 4 $11x-5y$
5 $12a+15$ 6 $15a+12$ 7 $2a+1$ 8 $\frac{1}{2}x+\frac{3}{2}y-2$
9 $-mn-mx+my+2n.$ 10 $-2ax+2by+5$ 11. $x+5y-7z.$

12 $-x-3y+4z$ 13 $5a-5b+c+1$ 14 $a+b+2c-5d+9$.
15 $pq+1$ 16 $a^2+7a+3b-b^2$ 17 x^3-x^2+2x-2
18 $2-m^2-m^3+m^4$ 19 $\frac{1}{3}av-xy+1$. 20 $-x^2+2y^2+3z^2$
21 $3abc-3ab-2ac-1$ 22 a^2-2ax. 23 $2ar-6r^2-2a^2$
24 $(a-p)x^3-(b-q)x^2+(1-r)x$ 25 x^3+2x^2+4x-2
26 $3x^2+2x+3$ 27 $-\frac{1}{8}xy+\frac{1}{2}y^2-\frac{1}{6}$ 28 $-\frac{5}{2}y-\frac{21}{4}a-\frac{1}{12}x+\frac{1}{3}b$.
29 $7a^4+2a^3b-2ab^3-7b^4$
30 $ax^2-gx^2+2hxy+2fxy+by^2-hy^2+c+d$.
31. $2x^4-5x^3-2x$ 32 $(a-1)x^5+2x^2y^3+(b+5)y^5$.

XVIII [p 36]

1 $6+x$ 2 $x-y$ 3 $3a-7b$ 4 $1-2x$ 5 $5x-3a$
6 $x-a$ 7 $2x$ 8 $2a-2b+y$ 9 y^2 10 $12a-b$
11. $10a-4b$ 12 $12a-3x$ 13 $2bce+2bef$ 14 $4c$
15 $21+3x$ 16 $4a-5b+c$ 17 $-25x+2y$. 18. $a-2b$
19 $a+8b-c$ 20 $m-11n$

XIX. [p 37]

1. (1) $a-b+(c-d-e+f)$, (2) $a-b-(d+e-c-f)$.
2 (1) $(a-b)+(c-d)-(e-f)-(g-h)$
(2) $a-(b-c)-(d+e)+(f-g)+h$. (3) $(a-b+c)-(d+e-f)-g+h$.
(4) $a-(b-c+d)-(e-f+g)+h$ (5) $a-b+c-(d+e-f+g-h)$
(6) $a-(b-c+d+e)+f-g+h$ (7) $a-b+c-d-(e-f+g-h)$
(8) $a-(b-c+d+e-f+g-h)$.

Examples for Revision (A) [pp 37—41]

2. $24\frac{2}{5}$ 3 1 4 $6ax^2+2a^2x-ax$, $5a+2a^2$ 5 a^2+5a-b, 8
6 $2x+3z$ 7 $-2y$ 8 $(x+15)$ years 9 $-x^2+2x-1$
10 2, 6 11 $b-a+c$ 12 $2x-3x^2$, -1. 13. $2p-q$.
14 $9x^2-2xy-y^2$, x^2-2y^2 15 $10x+y$
16. $R\left(60-\frac{m}{16}\right)$, $R\left(60-a-\frac{b}{16}\right)$ 17 2, 3, a, $b+c$
18 $-3x^2+4x-1$, -1 20 $4x-y+9$ 21. $3-8a+6b$
22 $3a-8b-2c$ 23 x^2+ax-1.
24 $R(a-x)$, $R(b+x)$, $R(a-x+y)$, $R(b+x-y)$ 25 50.
26 18 27 $b+6a$. 28. $x-2y+9z-4$. 29. $x-3y+3z+1$.

31 $\frac{5}{3}a^2-ab-\frac{3}{4}b^2$ 32 Assets of R100 33 -1

34 $3x-5$, 0 35 -3 , $a+b-c-3$ 36 $x+y+5z$

37 $-8a+3b+c$ 38 x^2+6x-6 39 $5xy-x^2-2y^2$.

40 $x=y+30$ 41 $3-5x$ 42 0 43 $-5a-3b+3c+2d$

44 $a^4-a^3-ab-b^3+2$ 45 $m-3n$ 46 $3a-ab+2b$

48 $x-ch$ miles , 10 miles 49 6.

50. (i) $+33°$, (ii) $+13°$, (iii) $-24°$ 51. $8x^2-3ax+a^2$

52 $-\frac{x}{4}-\frac{y}{3}+\frac{11z}{4}+1$ 53 $6a+3b-4c$ 55 $\frac{1}{2}x^2-y^2$

56 20, -2, -7, -5, 2, 28 57 $-3a-2b+x-4$

58 $2a^2-b^2+\frac{9}{2}c^2$ 59 The latter 60 $-b-2c$, $1-2c$

61 $15x-9y-2z$ 63 3 miles north 65 $1-3x^2+3xy-3y^2$ 66 6.

67 $\frac{5a}{2}-\frac{b}{2}+\frac{17c}{4}$, $-\frac{3a}{2}+\frac{b}{2}-\frac{17c}{4}$ 68 x^3-x^2-3

69. $2x-5y+10z$ 70 -1 , $-a-3b+1$

71 (i) 0 ; (ii) $-a^2+4ab+5b^2$ 72 $(2x+3)$ rupees

74 -1 , -1, 0 76 x^5-2x^3-5x+1 77 $-3x^2+6x-3y$

79 33, 23, 15, 5, 3, 5 and 9 80 $(3a-2)$ rupees

XX. [p 43]

1. -93 2 -128 3 16 4. 90 5 25

6 -64 7 -120 8 -216 9 16 10 648

11. 750 12 -200 13 -45 14 -540

15. 64 16 -1440 17 -192 18 -108 19 0

20. 152 21 -5 22 595 23. 0 24 27 25 -125.

26 247 27 -16 28 164 29 0 30 0

31. 8, 14, 22, 58 32 18, 2, -2, 0, -18, -52

33 $+1$, -1 , $+1$, -1

XXI. [p. 46]

7 $-a^{m+1}$ 8 x^{2n-1}. 9 a^{2m} . 10 $-a^m$ 11 $-a^{m+1}$.

12 $-6a^3x^{n+1}$.

XXII [pp 46—47]

1. $3ax^2$ 2 $-10xy^2$ 3 $-104apx^2$ 4. $-2a^2bcx$

5 $56ab^2c^2$. 6 pq^2r^2x 7 $-775abcxyz$ 8. $17472mnpq$.

9 $648abcdyz$ 10 $-36ax^2yz$ 11 $80a^2b^2pqxy$
12 $-340abr^2suz$ 13 $6a^6$ 14 $50x^{10}$ 15 $-60y^6$.
16 $-10a^5$ 17 $72m^{17}$ 18 $77z^{12}$.
19 $-3a^2x^3$ 20 $-6x^4y^4$ 21. $2a^3b^3c^4$ 22 $4m^4n^2r^4$
23 $-6x^4yz^3$ 24 $24a^3b^5$ 25 $-2a^4b^3c^4d^5$ 26 $12a^4r^4b^4y^5$
27 $-21a^6b^7m^3$ 28 $160x^3xy^6z^4$ 29 $16a^2, -8a^3b^3, -a^5b^5c^5, 81x^8$
30 a^2bcx^3 31 $30abcdef$ 32 $-120abcd$ 33 $360abcpx$
34 $-640abcxyz$ 35 $-2160abcrsxyz$ 36 $960a^3b^3c^3$
37 $19200a^3b^3c^3x^2y^3$ 38 $6x^8$ 39 $-224mc^7$ 40 $-80a^{14}$
41. $3a^8bx^3y^2z$ 42 $-216a^7b^2c^3x^5y^6$ 43 $-120acf^3gx^4y^3$.
44 $5a^4b^3c^2mx^4y^6z^2$

XXIII [pp 48—49]

1 $3a^2x-3ax^2$ 2 $-ax+ay$ 3 $ab+b^2$ 4 $2a^2xy-aby^2$.
5 $12a^3b^2c^3-3abc^4$ 6 $5ab^2c+5bc^2d$ 7 $-2ax+2ay-4a$
8 $-bx^3+3b^2x+ab^4x$ 9 $-ab^2c^2-a^2bc^2+a^2b^2c$
10 $-6x^2yz+3xy^2z+12xyz^2$ 11 $2a^4b^2c^2+2a^2b^4c^2+2a^2b^2c^4$
12 $3a^2d+2abd-4acd+ad^2$ 13 $-2a^5b^3c^3+3a^3b^7c^2+4a^3b^2c^5$
14 $\frac34x^2-\frac{3}{10}xy+\frac38xz$ 15 $-\frac53a^3bc+\frac54a^2b^2c+\frac56a^2bc$
16 $\frac27a^3b+\frac37ab^3+\frac67ab$ 17 $\frac13a^3x^5+\frac58a^4x^4$ 18 $-\frac25a^6b^4+\frac{9}{10}a^5b^5+\frac45a^3b^6$
19 $\frac{14}{3}a^3b^2cx^4+\frac43b^3cx^3y^2-\frac{14}{15}b^2c^3x^3z^3$ 20 $10a^5b^3c+15a^3b^6c+20a^2b^3c^4$
21 $6a^3b^4c^3-24ab^6c^4-15a^2b^3c^5$ 22 $-4a^4x^8+8a^3x^{10}+12a^5x^6$
23 $16x^4y^4z^7-14x^3y^3z^6+6x^3y^5z^4$. 24 $8x^4y-24x^3y^3+24x^2y^3-8xy^4$
25. $-9a^4b^3c^2d+3ab^4c^3d^2+6a^2bc^4d^3-12a^3b^2cd^4$
26 $-20a^3b^4cxy+12a^2bx^4y^3+4a^2b^3x^2y-4abcx^3y^5$. 27 $-4a^4$
28 x^3+x^2-2 29 $4a^3x-4ax^3$ 30 $21ax^3+21bx^3$ 31 $18a^2-24b^2$
32 $32x^3+12y^3$ 33 $2x^2y^2$ 34 $4a^3-a^2b+2a^2c$ 35, 36, 37 0

XXIV. [p 52]

1 x^2+6x+5 2 $x^2+2x-15$ 3 x^2+x-56 4 $x^2-20x+99$
5 $-a^2+8a-12$ 6 $x^2+(a+b)x+ab$ 7. $x^2+(a-b)x-ab$
8 $x^2-(a-b)x-ab$ 9 $x^2-(a+b)x+ab$ 10 $x^2+16x+63$
11. $a^2+2ab+b^2$ 12 $a^2-2ab+b^2$ 13 a^2-b^2
14 $4a^2-13ab+3b^2$ 15 $1+4x-96x^2$ 16 $2-10x^2y+12x^4y^2$
17 $15a^4+3a^2b-10a^2b^2-2b^3$ 18 $24a^3b^2+18a^2bc-4ab^2c-3c^2$
19. $6a^3b^2-12a^2bc-10abc^2+20c^3$ 20 $35a^3b^2-5a^2c^3-21ab^2c^2+3c^5$

21. $8x^4-26x^2y+15y^2$ 22. $24a^6-10a^3b^2c-21b^4c^2$

23 $ac+bc-ad-bd$ 24 $ac-bc-ad+bd$ 25 $ac-bc+ad-bd$

26 $2m^2+17m-117$ 27 $6x^2-13xy+6y^2$. 28 $a^2x^2-b^2y^2$

29 $\frac{9}{10}x^2-\frac{19}{20}ax+\frac{1}{2}a^2$ 30 $\frac{1}{4}abx^2+(\frac{5}{8}a-\frac{8}{10}b)x-1$. 31. $2x-4x^2$

32 $-11qr$ 33 x^2-5x+4 34 $7a-a^2$

XXV [p 53]

1 $a^3+7a^2+17a+35$ 2 x^3+8 3 x^3-27 4. x^4-1.

5 $1-5x-5x^2+13x^3+60x^4$ 6 $6y^3+5y^2-19y-20$

7 $p^3+6p^2q+12pq^2+8q^3$. 8 $6x^3-19x^2y+29xy^2-21y^3$

9 $125x^3-150x^2y+60xy^2-8y^3$ 10 a^5+3a^3-4a+6

11. $-x^3-x^2y+5xy^2-3y^3$ 12 $-4c^3-4c^2d+5cd^2+3d^3$

13 $x^7-4x^3y^2-x^2y^3+2y^4$ 14 $35a^5-49a^4+19a^3-24a+64$.

15 $125a^3-8b^3$ 16 $125x^3+512y^3$ 17. a^5+b^5.

18 $ax^3+(a^2-b)x^2-2abx+b^2$ 19 $40x^4-30x^3+52ax^2+6ax-12a^2$.

20 $c^3-9cn^2+3c^2n-27c^3$ 21 x^4-a^4 22 $8x^2-13x+18$

23. $4a+3b$

XXVI [p 55]

1. x^2+6x+9 2 $a^2+8a+16$ 3 $1+4x+4x^2$ 4 $4x^2+28x+49$

5 $a^2+4ab+4b^2$ 6 $9x^2+6ax+a^2$ 7 $4x^2+12ax+9a^2$

8 $a^2x^2+2ax+1$ 9 $9+12xy+4x^2y^2$ 10 $x^2-2xy+y^2$

11 $9x^2-24x+16$ 12 $1-10x+25x^2$ 13 $4x^2-12xy+9y^2$

14 $9x^2-48ax+64a^2$ 15 $x^2-10xy+25y^2$ 16 $9a^2-24ab+16b^2$

17 $64-48ax+9a^2y^2$. 18 $1-2abc+a^2b^2c^2$ 19 $a^2b^2-2abxy+x^2y^2$.

20 $l^4+2l^2mn+m^2n^2$ 21 $4p^2q^2+4pqr^2+r^4$ 22 $a^2b^2+4abc^2+4c^4$

23 $c^4-4abc^2+4a^2b^2$ 24 $9p^2q^2-24pqr^3+16r^6$ 25. $1+2a+a^2$

26 $9a^2+6a+1$ 27 $a^4+2a^2x^2+x^4$ 28 $4a^2x^2+12abxy+9b^2y^2$

29 $x^6-2a^3x^3+a^6$ 30 $a^8-2a^4x^4+x^8$ 31 $a^4x^2+2a^3x^3+a^2x^4$

32 $a^6-6a^5b+9a^4b^2$ 33. $m^4n^4-2l^4m^2n^2+l^8$ 34. $4a^6+4a^3b^3+b^6$

35 $x^6+6x^3yz+9y^2z^2$ 36 $9m^2x^4-18m^3x^3+9m^4x^2$

XXVII [p 56]

1 x^2-4 2 x^2-25 3 $1-x^2$ 4 a^2-9 5 $4x^2-49$

6 $9-16x^2$ 7 $49-100x^2$ 8 $121-25x^2$ 9 $4x^2-y^2$.

10 $a^2-b^2x^2$. 11 $x^2-a^2y^2$ 12 x^2-m^2 13 $4a^2-y^2$

14 $1-a^2x^4$ 15 a^2x^2-9 16 $9x^2-16y^2$ 17 $4a^2x^2-9y^2$
18 x^4-4y^2 19 $4a^4-9b^2c^4$ 20 a^4-x^4 21 a^4-4b^4
22 $4a^2b^2-c^4$ 23 $25x^4-9a^2b^2$ 24 $16c^4-49a^2b^2$ 25 x^6-1.
26 $4a^6-9b^4$ 27 $9x^8-25a^6$

XXVIII [p 57]

1 x^2+3x+2 2 x^2+2x-3 3 $a^2-13a+36$ 4 $a^2-3a-40$.
5 $a^2-a-110$ 6 x^2-5x+6 7 $x^2+10x+21$ 8 x^2-x-20.
9 $z^2-15z+50$ 10 $x^2+19x+88$ 11 $m^2-3m-54$
12 $y^2-5y-24$ 13 $y^2-11ay+28a^2$ 14 $x^2+8ax+15a^2$
15 $4m^2+4m-3$ 16 $4a^2-20a+21$ 17 $49a^2+7a-12$
18 $9x^2+18x+8$ 19 $a^2m^2-6am-55$ 20 $25x^2-85x+66$
21 $16x^2+4ax-4bx-ab$ 22 $16x^2+44x+10$
23 $m^2x^2+7mx+12$ 24 $m^2x^2-18mx+80$.
25 $4x^2-2ax+2bx-ab$ 26 $x^4+(c+d)x^2+cd$.
27 $x^4+a^2x^2-b^2x^2-a^2b^2$ 28 $4x^6+24x^3+27$
29 $4x^6-24x^3+27$ 30 $9p^6-9p^3-4$

XXIX. [p 59]

1 byz 2 $-8ad$ 3 $-17mx$ 4 $-\frac{5cx}{2}$ 5 $6y$.

6 $-\frac{2bq}{3px}$ 7 $-\frac{c}{a}$ 8. $\frac{2nx}{3az}$

XXX [p 60]

1 $-a^2x$ 2 $3ab$ 3 $-4ax^2y$ 4 $-4a^6bcd$ 5 $43xz^2$.

6 $-\frac{3xz}{2a}$ 7 $\frac{3}{2}ax^3y$ 8 $-4xyz^9$ 9 $\frac{2}{3}ab^2m^3$ 10 $\frac{4}{5}a^3x^2y$.

XXXI [p 61]

1 $a+c$ 2 $-2a+b$ 3 $3a-2b$ 4 $3x^2-x$ 5 $-4b+2ab$
6 $10x+3$ 7 $5b-c+d$ 8 x^2-ax+a^2 9 $-2x^3+x^2-3x$.
10. $\frac{3}{2}x^2+2xy-3y^2$ 11 $-a+3b+2c$ 12 $8a^2x+a-2xy$
13 $12a^2+9ab-8b^2$ 14 $-5abx^2+3x-2a^2yz^2$ 15. $a-5x^2+4x$
16 $3a^2-4ab+5b^2$ 17 $4ay-\frac{7}{2}x+x^2$ 18 $5b^2-4c^2+2cx-1$
19 $-p^3+3p^2q-pq^2+\frac{2}{3}q^3$ 20 $4x^3y^3-\frac{4}{3}x^2y^2+2xy+1$.

XXXII [pp 64—65]

1 $x+3$ 2 $x+3$ 3 $x-7$ 4 $y+5$ 5 $2x+1$ 6 $x+5$
7 $2m+3$ 8 $5-4a$ 9 $3-2x$ 10 $5-4a$ 11. $2x+1$ 12 $x-3y$
13 $2x+3$ 14 $9x^2+3x+1$ 15 $4a^2+4a+1$ 16 $2x^2-2x+1$
17 $2x^2-3x+4$ 18 m^2-4m+3 19 $2x^2-x+1$.
20 x^2-ax+a^2 21 $x-4$ 22 $ax-b$ 23 $x+5$.
24 $2ab-3b^2$ 25. $27-18x+12x^2-8x^3$ 26 x^2-ax+1
27 x^2+ax-a 28 $ax^2+bx-ab$ 29 x^2+2x-1
30. $3x^2-x-4$ 31 $x-5$ 32 $9x-36$ 33 $4y-8a$
34 $5x-4$ 35 8 36 $x-2y$

Examples for Revision (B) [pp 65—68]

1 $3a^2-2xy+3c^2$ 2 $bx-ay$ 3 $x^3-a(a+1)x+a^2$ 4 $x+3a$.
7 $5x+50$ 8. 0 9 $2bx+2(c-d)y$ 10 a^2+3a+5
11 $-6x^3+131x^2-364x+264$ 12 x^2-2x+1
14 $(3x-24)$ years. 16 $9x^2-y^2$. 17 $3(a+x)-20+6(x+y)$
18 -52 19 ax^2-bx-a 21 $8(a+b)$, $a+9b$.
22 3, 1, 5, 2, 3, 5 23 $27x-15y+34z$ 24 $x-3$.
25 x^2-a^2 26 ax^2+3x-1 28 $\frac{1}{2}x$.
29. Hom, second, not hom, fourth, hom, third
30 $3x-1$, x^2+1. 31 x^4+2x^3-2x-1 32 $x+3$.
33 $(x-y)-xy$, $\frac{1}{y}-\frac{1}{x}$ 34 x 35. $\frac{ax}{192}+\frac{y}{4}+2$
37 $-a^3+6a^2x-4ax^2$ 38 $-23x^2+14xy$
39 $D=dQ$ 40. 0 42 $12a-x$ 43 $a^2+6ac+3b^2$, 79
44 $4x-10y$ 45 0, 0 46 x^2+2x+1 48 $(4x+3a)$ rupees
49 28, 6, 1, 3, 36 50 99 51. $-2x^4+3x^3y-3xy^3+2y^4$.
52 $a^3+2a^2x+2ax^2+x^3$. 53 $2x^2+x+3$, $2x^2-7x+9$
55 $29ax-10x^2-10a^2$ 57 -72 58 x^4-a^4
59. $2xy^2$, $-4by$, -1 60 $25x^2-50x+25$, x^2+2x+1.
61 $4x^2-38x+85$ 62 15 63 $-73, -13, -4, -1, 32, 71$.
64 $6ax-x^2-2a^2$ 65 $y^3-9x^2y+3xy^2-27x^3$.
66 $x-1$, -2 67 $4x^2-ax-5a^2$.
68 x^2-xy+y^2, $(a+b)^2-(a+b)c+c^2=a^2+2ab+b^2-ac-bc+c^2$.
69. $p+2q$ 70 $-55, -17, -3, -1, 15, 53$

XXXIII [pp 69—70]

1 $9x^2+4y^2+9z^2+12yz-18zx-12xy$

2 $4x^2+36y^2+z^2+12yz+4zx+24xy$

3 $9a^2+25b^2+16c^2+40bc+24ca+30ab$

4 $a^2x^2+b^2y^2+c^2z^2-2bcyz+2cazx-2abxy$

5 $a^2b^2+x^2y^2+c^4+2abxy-2c^2xy-2abc^2$

6 $a^4+2a^3b+3a^2b^2+2ab^3+b^4.$

7 $1-4x+10x^2-12x^3+9x^4$ 8 $a^4+b^4+c^4+2b^2c^2+2c^2a^2+2a^2b^2$

9 $a^2+b^2+c^2+d^2-2ab-2ac+2ad+2bc-2bd-2cd$

10 $16a^2+25b^2+c^2+4d^2+40ab-8ac-16ad-10bc-20bd+4cd$

11 $2x(3y+4z)+(3y+4z)^2=6xy+8xz+9y^2+16z^2+24yz$

12 $9x^6+a^6-6a^3x^3-12ax^5+4a^4x^2$ 13 $y^4+2xy^3+2x^2y^2+x^3y$

14 $20ax^3-37a^2x^2+30a^3x-9a^4$ 15 $p^2+q^2+2p+2q+2$

16 $4p^2-9q^2-4p+6q$ 17 625 18 0 19 0 20 $25a^2.$

21 $4x^2$ 22. 4 23 x^2 24 $4x^2$ 25 $4b^2-8b+4.$

XXXIV [pp 70—71]

1 $(a+b)^2-c^2=a^2+2ab+b^2-c^2$ 2 $a^2-b^2-c^2+2bc$

3 $a^2-2ac+c^2-b^2$ 4 $a^2+2ac+c^2-b^2.$

5 $x^2-4y^2-9z^2-12yz$ 6 $x^2-4xy+4y^2-9z^2.$

7 $x^2-4y^2-9z^2+12yz$ 8 $a^4-2a^2b^2+b^4$

9 $x^4+a^2x^2+a^4$ 10 $x^4+a^2b^2+2abx^2-b^4$

11 $(a-b)^2-(c-d)^2=\&c$ 12. $(x+1)^2-(y-z)^2=\&c$

13. a^4-x^4 14 x^8-y^8 15 x^8+x^4+1 16 $a^2-d^2.$

17 $4a^2$ 18 $4b^2+8bc+4c^2.$ 19 $8ab-4a^2b^2-3$

20 $a^2b^2+3c^2d^2$

XXXV [pp 72—73]

1 $(x-2)^3=x^3-6x^2+12x-8$ 2 $8x^3+36x^2y+54xy^2+27y^3$

3 $125m^3+150m^2n+60mn^2+8n^3$

4 $125x^3-225x^2y+135xy^2-27y^3$ 5 $27a^3+54a^2b+36ab^2+8b^3$

6. x^3-3x^2+3x-1 7 $1+6x+12x^2+8x^3$

8 $8x^3-36x^2y+54xy^2-27y^3$ 9 $8x^3-48x^2+96x-64$

10 $1-12x+48x^2-64x^3$ 11 $1+3x^2+3x^4+x^6.$

12 $8-12x^2+6x^4-x^6$ 13. $a^3x^3-3a^2x^2y^2+3axy^4-y^6$

14 $\{(a+b)-(a-b)\}^3=(2b)^3$ 15 $8x^3$ 16. -64 17. $27a^3$
18. $27(x+1)^3=27x^3+81x^2+81x+27$ 19 $8a^3$ 20. $(6c-2b)^3=$&c.
21. -144 22 0 23 -740 24 43 25. $0.$
26. 269 31 64 32 c^3 33. $1.$

XXXVI [pp. 73—74]

14 $1+x^6$ 15 $8x^3-27$ 16 $27x^6-8y^3z^3$.
17 $64a^6+125x^3$ 18 $8m^6-729n^6$ 19. $x+y$
20 x^3-8y^3 21 $2a^3+2ax^2-x^3$

XXXVII [p 74]

1. $(x+a)^2+3(x+a)+2=$&c 2 $(a+b)^2+3(a+b)-10=$&c
3 $(x+y)^2-5(x+y)-6=$&c 4 $(x+2y)^2-4(x+2y)+3=$&c
5 $(ax-b)^2-4(ax-b)-21=$&c 6 $(x^2-xy)^2+10(x^2-xy)+24=$&c
7 $(2a-b)^2-11(2a-b)+28=$&c 8 $(2x-3y)^2-(2x-3y)-30=$&c
9 $(3x^2-4x)^2-14(3x^2-4x)+48=$&c
10 $(2a^2-ab)^2+5(2a^2-ab)-24=$&c
11 $(4x-5y)^2-3(4x-5y)-54=$&c.
12 $(3x-8)^2+y(3x-8)-20y^2=$&c
13 $(a^2+3ab)^2-(c^2+d^2)(a^2+3ab)+c^2d^2=$&c
14 $(x^2-4z^2)^2-5y^2(x^2-4z^2)+6y^4=$&c

XXXVIII [p 75]

1 $a(b+c)$ 2 $m(1+x)$ 3 $x(2-x)$ 4. $a(a-c)$
5 $m(m+2x)$ 6 $3r(r^2-5s)$ 7 $ab(a+b)$ 8 $b(a-c+ac)$
9 $2a(4a-3b-2c)$ 10 $5p(4q-3pr+11r)$ 11 $3ab(a^2-b^3)$.
12 $ab(2ab+3)$ 13 $3pq(4p^2-q)$ 14 $2x^2(4-5mx^3)$.
15 $6m^5(3m^3-n)$ 16 $xy(3x+2y+1)$ 17. $4ab(a^2-4a+5b)$
18 $xy(2x^2y^2-3rx+2ry)$ 19 $3x^2y(2x-3y+4xy)$
20 $9x^4y^4(9x^2+7y^4)$ 21 $6m^2x^2(4m^2x^2y^3-7y^5)$ 22 $7a^2xy^2(a+2x-3y)$

XXXIX [p 76]

1 $(a+c)(b+d)$ 2. $(m+n)(m+1)$ 3 $(a+1)(x+1)$
4 $(2a-b)(x-2y)$ 5 $(1-x)(1-y)$ 6 $(a-b)(a-c)$
7 $(x-m)(x+y)$ 8 $(a+1)(a^2+1)$ 9 $(1-x)(1+x^2)$
10 $(f+h)(fg-ch)$ 11 $(a+bc)(b+ac)$ 12 $(ax-b)(cx-d)$

13 $(a-cq)(bq-a)$ 14 $(yz-x^2)(zx-y^2)$ $(a+b)(x-y+z)$
16 $5m(p-q)(p+2q-5m)$. 17 $x(y+b)(z+c)$ 18 $a(a+3)(b-4)$
19 $m(my-x)(mx-y)$ 20 $8mn(m-2n)(n-3p)$
21 $2a(ag-2bf)(2ah-cf)$

XL. [pp 77—78]

1 $(1+x)(1-x)$ 2 $(m+4)(m-4)$ 3 $(8+q)(8-q)$
4 $(1+9z)(1-9z)$ 5 $(5y+1)(5y-1)$ 6 $(4a+3b)(4a-3b)$
7 $(5ax+7b)(5ax-7b)$ 8 $(12x+13a)(12x-13a)$
9 $(9q+8r)(9q-8r)$ 10 $(25ax^2+11)(25ax^2-11)$
11 $(9p+10q)(9p-10q)$ 12 $(12x+11y)(12x-11y)$
13 $(7ac+9d^2)(7ac-9d^2)$. 14 $(lm+nq)(lm-nq)$
15 $(3z^2+4xy)(3z^2-4xy)$ 16 $(5az+2cy)(5az-2cy)$
17 $(a-x)(a+x)(a^2+x^2)$ 18. $(4x^2+5a^2)(4x^2-5a^2)$
19 $(2a^2+b^2)(2a^2-b^2)$ 20 $(6a^3+x^2)(6a^3-x^2)$
21 $(5a^4+3b^5)(5a^4-3b^5)$ 22 $(m^2+4)(m+2)(m-2)$
23 $(4a^2+1)(2a+1)(2a-1)$ 24 $(1+x^4)(1+x^2)(1+x)(1-x)$
25 $(a^4+x^4)(a^2+x^2)(a+x)(a-x)$ 26 $(x^8+a^8)(x^4+a^4)(x^2+a^2)(x+a)$
$\times(x-a)$ 27 $x(x+2y)(x-2y)$ 28 $2a(5a+1)(5a-1)$,
29 $3(1-4ax)(1+4ax)$ 30 $2(4mn+1)(4mn-1)$
31 $3a(a+x)(a-x)$ 32. $5ax(2a+b)(2a-b)$
33 $2x(5x+4y)(5x-4y)$ 34 $3x(1+2x)(1-2x)$
35 $2a^2x^2(5ab+2xy)(5ab-2xy)$ 36 $x(9x^2+4y^2)(3x+2y)(3x-2y)$.
37 $3x(x+\frac{1}{3}y)(x-\frac{1}{3}y)$ 38 $2a(a^2+\frac{1}{4}x^2)(a+\frac{1}{2}x)(a-\frac{1}{2}x)$
39 $2a^3b^2(9a+4b^2)(9a-4b^2)$ 40 $(x-y+z)(x-y-z)$
41 $(m+2n+q)(m-2n-q)$ 42 $(3y-4b+6x)(3y+4b-6x)$
43 $(6x-8y+1)(6x-8y-1)$ 44 $(2a+3b-9c)(2a-3b+9c)$
45 $(4+15z+18u)(4-15z-18u)$ 46. $(5p+6q-2)(5p-6q+2)$
47 $(5x-2y+3z)(5x-2y-3z)$
48 $(2lx-2my+9xy)(2lx-2my-9xy)$ 49 $25(r-s+t)(r-s-t)$
50 $(a+b+c+d)(a+b-c-d)$ 51 $(3x-2y+2z+1)(3x-2y-2z-1)$.
52 $96x$ 53 $-60ab$ 54 $13(a-b)(a+b)$. 55 $4(2ax-by)(ax+2by)$.
56 $-4ax(a^2+x^2)$ 57. $(3a-2b-c)(a+4b-5c)$
58 $-4(2a+c)(a+2b+5c)$ 59 $(1+x)(1-x)(1+y)(1-y)$
60 $y(y+x)(y^2-xy+2x^2)$ 61. $(m+1)(m-1)(x+y)(x-y)$

62 $(p+q)(p-q)(r+s)(r-s)$ 63 $8ax(a^2+x^2)$
64 60800 65 758000 66 81933 67 19455
68 3393600 69 4096 70 7225 71 21025
72 37249 73 173056 74 978121 75 99980001

XLI. [p 79]

1 $(a+4)(a^2-4a+16)$ 2 $(x-2y)(x^2+2xy+4y^2)$
3 $(3a-b)(9a^2+3ab+b^2)$ 4 $(3x+a)(9x^2-3ax+a^2)$
5 $(2a+1)(4a^2-2a+1)$ 6 $(4l-1)(16l^2+4l+1)$
7 $(1+3l)(1-3l+9l^2)$ 8 $(8x-3)(64x^2+24x+9)$
9 $(5a+3x)(25a^2-15ax+9x^2)$ 10 $2(3m-1)(9m^2+3m+1)$
11 $(x+4y)(x^2-4xy+16y^2)$ 12 $(ax-4y)(a^2x^2+4axy+16y^2)$
13 $(4a+5b)(16a^2-20ab+25b^2)$ 14 $(2ac-b)(4a^2c^2+2abc+b^2)$
15 $(7x+2)(49x^2-14x+4)$ 16. $(x^2+y)(x^4-x^2y+y^2)$
17 $8(3a-x^2)(9a^2+3ax^2+x^4)$ 18 $27(3+2c)(9-6c+4c^2)$.
19 $(x+\frac{1}{2}y)(x^2-\frac{1}{2}xy+\frac{1}{4}y^2)$ 20. $(a-6x^2)(\frac{1}{4}a^2+\frac{3}{2}ax^2+9x^4)$
21 $xy(x+2y)(x^2-2xy+4y^2)$ 22 $3p^2q\{(4p)^3-(5q)^3\}=$&c.
23 $\{x-2(y+z)\}\{x^2-2x(y+z)+4(y+z)^2\}=$&c
24 $\{4(a+b)+3c\}\{16(a+b)^2-12(a+b)c+9c^2\}=$&c
25 $\{(a+b)x-2cy\}\{(a+b)^2x^2+2x(a+b)cy+4c^2y^2\}=$&c
26 $(x-y)(x^2-5xy+7y^2)$ 27 $(a^2+bc)(a^4-4a^2bc+7b^2c^2)$.
28 $2(x+a)(x-a)(x^4-2a^2x^2+4a^4)$
29 $-(x-1)^2(x^4+2x^3+6x^2+2x+1)$
30 $9x^2-30x+100$ 31 $1-4c$ 32. $9a+5b$.

XLII [p 80]

1 $(1+a)(1-a)(1+ab)$ 2 $(a+b)(a-b)(1-ab)$
3 $(a-1)(a+1)(a^2-a+1)$ 4 $(x-1)^2(x^2+x+1)$
5 $(x+1)(x-1)(x-a)$ 6 $(1-ax)(a+x)(a-x)$
7 $(1+2x)(1+x)(1-x)$ 8 $(a+b)(a-b-ab)$
9 $(a+b)(a-b)(x+1)(x-1)$ 10 $(x-y)(x+y+a)$
11 $(x-y)(x^2+xy+y^2+ax+ay)$ 12 $(1+ax)(1+a-ax)$
13 $(a+x)(a+2x)(a-2x)$ 14 $(a+1)(a-1)^2(a^2+a+1)$.
15 $(a+1)(a-1)(x+1)(x^2-x+1)$ 16 $(a-1)(x-1)(x^2+x+1)$
17. $(a+1)(a-1)(x+1)(x-1)$ 18 $(a-1)(a^2x+ax+x+1)$

19 $(1+2x^2)(1+cx-2x^2)$ 20 $(a+b)^2(a^2-ab+b^2)$

21 $(a+x)(a-x)(a^2+ax+x^2)$ 22 $(a-b)(a+b)^2(a^2-ab+b^2)$

XLIII [pp 82—83]

1 $(x+3)(x+4)$	2 $(a+2)(a+5)$	3 $(x+3)(x+7)$
4 $(p+4)(p+20)$	5 $(z+14)(z+3)$	6 $(x+4)(x+5)$
7 $(a-1)(a-3)$	8 $(a-4)(a-5)$	9 $(x-2)(x-4)$
10 $(a-4)(a-3)$	11 $(x-2)(x-3)$	12 $(x-17)(x-10)$
13 $(x-4)(x+8)$	14. $(a+3)(a-7)$	15 $(a+9)(a-8)$
16 $(x+4)(x-8)$	17 $(x-4)(x+10)$	18 $(x-2)(x+21)$.
19 $(x-6)(x+7)$	20 $(x-9)(x+6)$	21 $(x+3)(x-18)$
22 $(m-8)(m+12)$	23 $(m+3)(m-32)$	24 $(m+4)(m-24)$
25 $(a-10)(a+12)$	26 $(a+5)(a-24)$.	27. $(p-9)(p+16)$
28 $(x-4)(x+12)$	29 $(x-6)(x+8)$	30 $(x+3)(x-16)$
31 $(l-1)(l+20)$	32 $(l+3)(l-26)$	33 $(l+6)(l-13)$
34 $(x-12)(x+13)$	35 $(x+6)(x-26)$	33 $(ab+3)(ab+12)$
37 $(ab-2)(ab-18)$	38 $(xy+3)(xy-6)$	39 $(pq-2)(pq+16)$
40 $(mn+2)(mn-8)$	41 $(ax-4)(ax+9)$	42 $(a+6b)(a-9b)$
43 $(x+5y)(x-20y)$	44 $(x-4y)(x+6y)$	45 $(x+2y)(x-12y)$
46 $(a-6b)(a+10b)$	47 $(l+3m)(l+15m)$	48 $(a-3b)(a+20b)$
49 $(m+16n)(m-8n)$	50 $(m+4n)(m-32n)$	51 $(p-19q)(p+20q)$
52 $(6+5x)(6-x)$	53 $(8+13a)(8-15a)$.	54 $(1-4m)(1+6m)$
55 $(k+20)(k-21)$	56 $(x+26)(x-25)$	57 $(q+25)(q+80)$
58 $(1-3x)(1+6x)$	59 $(1+2x)(1-9x)$	60 $(3-4x)(3+8x)$
61 $(3+2x)(3-16x)$	62 $(3-2x)(3-14x)$	63 $(5-12m)(5-6m)$
64 $(5-2x)(5+8x)$	65. $(5+x)(5-16x)$	66 $(m-20)(m+15)$
67 $(m+5n)(m-10n)$	68 $(x-8a)(x+10a)$	69 $(1-xy)(1+4xy)$
70 $(1+30xy)(1+25xy)$		71 $(k+16)(k-12)$
72. $(x+4y)(x-10y)$.	73 $(a-18)(a+27)$	74 $(x+18)(x-20)$
75 $(x-14)(x+40)$	76 $(x-8)(16-x)$	77 $(x+6)(12-x)$
78. $(x+20)(4-x)$	79 $(x+3\frac{1}{3})(x+\frac{1}{6})$	80 $(x-3)(x-\frac{1}{3})$
81 $(x+2)(x-\frac{5}{2})$	82. $(a+2)(a+z)$	83. $(x+y)(x+ay)$.

84. $(y+x)(y+x-1)$ 85. $(x-a+2)(x-a-3)$

86 $(x-y+1)(x-y+3)$ 87. $(x+2y-1)(x-y+2)$

88. $(x+a^2-2ab+b^2)(x-a^2-2ab-b^2)$. 89. $(x+a+1)(x+a-1)$

XLIV [p 84]

1 $(x+2)(4x+3)$ 2 $(a+4)(5a-1)$ 3 $(2x+3)(3x-4)$
4 $(2x-9)(3x-4)$ 5 $(x-5)(3x+5)$ 6 $(x+8)(4x-9)$
7 $(2y-5)(4y+7)$ 8 $(3x+8)(5x-12)$ 9 $(4x+1)(3x-5)$
10 $(2x-1)(5x+4)$ 11 $(3m+7)(3m-4)$ 12 $(x+8)(8x-3)$
13 $(5x+3)(3x-7)$ 14 $(4x-3)(2x+5)$ 15 $(3x-8)(4x+5)$
16 $(2x-1)(4x+7)$ 17 $(3x-2)(2x+5)$ 18. $(4x+3y)(x-4y)$
19 $(2+3x)(5-9x)$ 20 $(3x+2)(4x-3)$ 21 $(2x-5)(6x-1)$
22 $(4x+7)(x-3)$ 23 $(6p-5q)(4p-7q)$ 24 $(1-5x)(3+2x)$
25 $(9xy-2)(xy+6)$ 26 $(x+4a)(8x-3a)$ 27 $(3+2x)(2-5x)$
28 $(5a-3b)(2a-5b)$ 29 $(3x+4y)(4x+5y)$ 30 $(2a-5b)(6a+b)$
31. $(3m+5n)(6m-7n)$ 32 $(4x-3a)(3x-8a)$
33 $(3x-4)(14x+5)$ 34 $(7y+15z)(8y-3z)$
35. $(8x+9y)(3x-8y)$

XLV. [pp 88—89]

3 −28. 4 +6 5 +28 6 −8 7 +26 8. −25
9 20 yrs 10 R3 19. 1 84 ; 14 28 , 19 6 20 $5\frac{1}{2}$ ft.
21 23 4 ft 22 4 3 in 23 2 in 24 $22\frac{1}{2}$ ft
25 42 ft. 26 15 ft. 27 28 yd 28. 15 miles
29 27 mi 30 26 mi nearly 31 2×30 ft

XLVII. [pp. 94]

1 8 2 8 3 $\frac{1}{4}$ 4. 3 5. −1 6. 5 7. 12. 8 1
9 5 10 6 11 $5\frac{1}{3}$ 12 4 13 3 14 $\frac{5}{6}$ 15. 3 16 10.
17 15 18 $4\frac{1}{3}$ 19 $-\frac{4}{13}$. 20 29 21 4
22 $-3\frac{1}{2}$ 23 $3\frac{13}{16}$. 24 $\frac{23}{101}$

XLVIII [pp 95—97]

1 12 2 6 3 8 4 10 5 $\frac{3}{5}$ 6 120 7 10 8 30
9. −16. 10 $2\frac{1}{4}$ 11 $7\frac{2}{3}$ 12 $-\frac{5}{16}$ 13 $6\frac{1}{2}$ 14. 8
15 $3\frac{5}{21}$ 16 2 17 3 18 $1\frac{1}{6}$ 19 2. 20 16.
21 13. 22 5 23 $\frac{2}{3}$ 24 7. 25 2. 26 9.
27 2 28 2 29 −3 30 13 31. 0 32 3
33 7 34. 10. 35 $3\frac{2}{5}$ 36 −2 37 9. 38. 5. 39. 3

40. -8 41 3 42 9 43 8 44. 5 45. 8. 46 8
47. $\frac{89}{67}$ 48 3 49 35 50 72 51 51 52 7 53 8

XLIX [p 97]

1. 1 2 2 3 6 4 5 5 2 6 5 32 7 $13\frac{6}{7}$ 8 20
9 $-3\,9$ 10 1 11 4 12 11

LI [pp 101—103]

1 $3x-a$ 2 $31-x$ 3 $45+y$ 4 $20-x$ 5 $\dfrac{63}{x}$

6 $54x$ 7 $(x+34)$ yrs 8 $(50-3x)$ miles

9 A, $(6x+3)$ miles, B, $(6x-6)$ miles 10 $(5x+20)s$

11 $5x$ 12 $(x-6)$ yrs, $(x+6)$ yrs 13 $\frac{1}{4}W$ 14 $\dfrac{15m}{22}$.

15 $\frac{1}{3}ab$ 16 $\dfrac{16a}{x}$, $\dfrac{mx}{192}$ 17 $12x$ miles 18 $\dfrac{25}{24x}$. 19 $\dfrac{30}{x}$ hrs

20. $\dfrac{3x}{a}$ miles 21 $(y-x+3)$ Rs. 22 $(x+y-20)$ Rs 23. $5(x+3)$yrs

24 $\dfrac{a}{x-\frac{1}{4}}$ miles per hr 25 A has $(x-y)$Rs, B has $(x+y)$ Rs

26 $2m$, $8(m-3)$ 27 $10x+y$, $10y+$ 28 $100y+10z+x$,
$100z+10x+y$ 29 $10x+y$ 30. $x=a+15$ 31 $y=5-x$

32. $x-3=y+4$ 33 $\dfrac{x-3}{x}=\frac{2}{3}$, 9 34 $45-x=8x$, 5

35 $3x+3=54$, 17. 36 $3x=81$ 37 $40-x=7(10-x)$, 5
38 $x+15=3(x-15)$, 30 39 $x=4(x-m)$ 40 $18=(a-x)h$

LII [pp 103—107]

I (1) $162\frac{3}{4}$ sq ft II (1) 600 sq ft. III (1) $20\frac{4}{9}$ sq ft
(2) 11 ft 3 in (2) 72 ft (2) 5 ft 3 in
(3) 12 ft. (3) 32 ft 8 in.
(4) 10 st
(5) 18 ft

IV. (1) 5 ft 2 in V (1) 34 04 sq cm VI (1) 11 ft, 50 cm. nearly.
(2) $36\frac{1}{4}$ sq ft (2) 25 cm. (2) $1\frac{1}{4}$ yd
(3) 12 75 cm (3) $5\frac{1}{11}$ in

VII	(1) 707 14 sq cm., 50 29 sq in.	VIII	(1) 176 sq in	X.	(1) $3541\frac{2}{3}$ c. ft.
	(2) 6 cm		(2) 8 cm		(2) 8 cm.
			(3) 11 4 in		
			(4) 20 cm , 15 cm		

XI	(1) 14142 9 c cm	XII	(1) 439	XIII	(1) 37
	(2) 84 cm		(2) 17		(2) 5
			(3) 12		(3) 16, 30
			(4) 14		(4) 7, 24

XIV	(1) 15 miles	XV	(1) 49 ft	XVI	(1) R152 4*a*.
	(2) 5 hours		(2) 4 6 sec nearly		(2) R615
	(3) 45				(3) $3\frac{1}{2}$ years
					(4) 4

LIII [*pp* 108—116]

1 10 2. 120 3 114 4 71 5 28 6 31 7 4

8 810 9 24. 10 9 11 18 12 98, 85 13 38, 53

14 16, 10 15 60, 65 16 35, 29 17 22, 14

18 539, 540, 541 19 381, 383, 385 20 256, 257

21. 15, 21 22 Rs 875, Rs 125 23 153, 54.

24. *A* 695, *B* 570. 25 276, 245 26 Rs 53, Rs 19.

27. 117, 102, 130 28 Rs 114, Rs 95 29 *A*, 30*s*, *B*, 21*s*

30. £1 10*s*, £3 15*s* 31 9 years 32. In 15 years

33 3 years ago 34 45 yrs 35 40 yrs, 35 yrs

36. 38 yrs, 13 yrs 37 42 yrs, 12 yrs

38 *A*'s 35 *B*'s 30, *C*'s 27. 39 50 yrs

40 *A*'s 12, *B*'s 30, *C*'s 6 41 Rs 48 42 20, 15

43 *A* Rs 30, *B* Rs 15. 44 Rs 50 45 Rs 250

46 *A* Rs 510; *B* Rs 460. 47 30 ft 48 42

49. 1512000 sq miles 50 72 51. 30*d* 52 22 days

53 12 h-cr, 40 h-sov 54 3 sov, 12*s*, 60 six-d

55 pence 6, shil 27, h-cr 9 56 $3\frac{1}{2}$ mi. per hr

57, $14\frac{2}{3}$ mi from *A*'s starting pt

58. 4 mi per hr 59 7 mi per hr

60 $3\frac{4}{7}$ hr. after the second pipe is opened

LIV [*pp* 117—119]

1 39   2 23064 sq ft   3 289   4 $\frac{35}{37}$ 5 $\frac{15}{36}$.   6. 25
7 128   8 1680   9 672 sq ft.   10 72 lb   11 144
12 275   13 420   14 56   15 6 mi   16 110, 90.
17 $12\frac{53}{56}$ 18 $20\frac{1}{2}$ 19 36 yrs
20 *A* Rs 20 8*a*, *B* Rs 10, *C* Rs. 18 8*a*, *D* Rs 13
21. *A* Rs 30, *B* Rs 27, *C* Rs 23   22 *A*, 320, *B*, 280, *C*, 200
23. £2 10*s*.   24. 12 gals   25 44, 45   26 51   27 13*s*.
28. 10, 14   29. Rs 690   30 Rs 100   31 30, 40
32. 21 tolahs   33 20   34 *A* £60, *B* £50, *C* £30
35 $2\frac{1}{2}$ and $3\frac{1}{4}$ yds   36 162, 108   37 654

LV [*p* 121]

7 7   8 −2   9 2   10 3   11 $-5\frac{1}{2}$ 12 $\frac{1}{2}$

LVI [*pp* 123—124]

N B—The values of x *and* y *are given in order*

1 44, 15   2 $7\frac{1}{2}$, $7\frac{1}{2}$ 3 8, 3   4 7, 6   5 9, 11.
6 4, −3   7 3, −1   8 1, 3   9 12, 9   10 1, 5
11 3, 2.   12 3, 3   13 4, 10   14 2, 3   15 4, 7
16 −2, $2\frac{1}{2}$ 17 8, 6   18 15, 20.   19 18, 15   20 12, 3
21 9, 8   22 12, 20   23 5, 10   24 24, 12   25 4, 2
26 12, 5   27 12, 9   28 7, 8

LVII [*pp* 127—129]

N B—The values of x *and* y *are given in order*

1 25, 1   2 1, −1   3 10, 11   4 10, 4   5 8, 10
6 9; 8   7 8, 10   8 5, 12   9 3, 2   10 5, 2   11 2, 3
12 6, 8   13 5, 4   14 $2\frac{2}{15}$, $1\frac{1}{3}$ 15 144, 216   16 24, 12
17. 40, 60   18 5, 6   19 7, 10   20 $\frac{1}{7}$, 1.   21 $\frac{1}{2}$; −1.
22 4, 3   23 4, 10   24 $\frac{1}{2}$, $\frac{1}{3}$ 25 $3\frac{1}{2}$, $2\frac{1}{4}$ 26 $1\frac{2}{5}$, $2\frac{1}{2}$
27 2, 1   28 3, 2   29 3, −4   30 $\frac{1}{15}$, 18   31 3, $\frac{1}{5}$.
32 −5, 4   33 2, 3   34 1, 1   35 $\frac{1}{3}$, $\frac{1}{4}$ 36 $\frac{1}{2}$, 1
37 5, 6   38. $13\frac{1}{2}$, $1\frac{1}{2}$ 39 $-2\frac{1}{5}$, $\frac{1}{7}$ 40 11, 8
41 $\frac{1}{5}$, −17.   42. ·02, 29   43 $\frac{17}{37}$, $\frac{17}{11}$.   44 10, 5.

LVIII [pp 130—131]

N B—The values of x, y *and* z *are given in order*

1 5, 2, 4 2 2, 1, 3 3 16, $15\frac{1}{2}$, $16\frac{1}{2}$ 4 8, 2, 6 5 1, 5, 2.
6 3, −2, 1 7 2, −3, −4 8 11, 15, 17 9 6, −1, −4
10 4, 2, −1 11. 2, 4, 6 12 5, −3, 8 13 5, 8, 12
14 5, 12, 4 15 15, 12, 20 16 14, −18, −8 17 2, 2, 4
18 2, 1, −3 19. 5, 11, 7 20 8, 6, 12 21. 64, 80, 100
22 15, 13, 18 23 5, 7, 6 24 5, 7, −3

LIX. [p 132]

N B—The values of x, y *and* z *are given in order*

1 $\frac{1}{2}$, $\frac{1}{6}$, $\frac{1}{5}$ 2 6, $\frac{1}{8}$, $\frac{1}{4}$ 3 $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$ 4 3, 4, 6 5 2, 3, 6
6 4, 6, 8 7 1, $\frac{1}{3}$, $\frac{1}{2}$ 8 Each=2 9 $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$ 10 1, 2, 3
11 3, 5, 8

LX. [pp 133—138]

1 98, 85 2 49, 16 3 32, 15 4 63, 21 5 10, 16
6 24, 36 7 52, 18 8 232 9 $\frac{2}{15}$ 10 $\frac{3}{4}$
11 $\frac{15}{40}$. 12 $\frac{16}{24}$ 13 $\frac{3}{5}$ 14 $\frac{4}{5}$ 15 $\frac{5}{12}$ 16 $\frac{8}{13}$
17 8*p*, 1*a* 18 ₨15 8*a*, ₨8 4*a* 19 8 20. 24 bales or 72 casks.
21 69 fl, 36 half-cr. 22 667, 533 23 43, 34 24 75, 100, 125
25 35 and 14 yrs 26 40, 16, 14 27 18, 11 28 6 oranges, 4 apples 29 96 apples, 108 oranges 30 20, 18 31. 22, 26
32 40, 21 33 60 34 16 and 12 ft 35 Wine 40, water 15
36 36, 24

LXI [p 140]

1 58 or 85 2 97 3 54 4 37 5 91 6 48 7 17
8 63 9 38 10 17 11 631

LXII [pp 144—150]

1 (*a*) *Q* (*b*) *C* (*c*) *P* (*d*) *M* (*e*) *B* (*f*) *A* (*g*) *R* (*h*) *E* (*i*) *F* (*j*) *L*. (*k*) *S* (*l*) *G* (*m*) *N* (*n*) *H* (*p*) *K* (*q*) *D* (*r*) *T* (*s*) *W* (*t*) *V* (*u*) *Z*.

2. $A\,(9, 7)$ $B\,(12, 15)$ $C\,(-3, 6)$ $D\,(-8, -4)$ $E\,(11, -14)$
$F\,(8, 0)$ $G\,(-12, 11)$ $H\,(-5, -13)$ $K\,(-10, 0)$ $L\,(-5, 13)$
$M\,(0, -8)$ $N\,(13, 13)$ $O\,(0, 0)$ $P\,(-6, -10)$ $Q\,(14, -12)$
$R\,(0, 12)$ $S\,(-7, 9)$ $T\,(7, -11)$ $V\,(-3, -9)$ $W\,(6, 10)$
$Z\,(10, -12)$

3 (*r*) to (*v*), take 3, 4, 5, 3 and 4 divisions respectively as unit

4 (*a*) 5, 7 (*b*) −6, 8 (*c*) −7, −10 (*d*) 9, −9 (*e*) 0, 0
(*f*) 0, 0 (*g*) 0, 4 (*h*) −6, 1 (*i*) −1, −6 (*j*) 6, −2

6 (*a*) 10 (*b*) 13 (*c*) 17 (*d*) 5 (*e*) 10 (*f*) 13 (*g*) 13
(*h*) 17 (*i*) 15 (*j*) 13

7 20 8 25 9 7 81 10 30 sqs 11 144 sqs
12 81 sqs 13 35 sqs 14 50 sqs 15 48 sqs 16 48.
17 52 36 23 8, −5 24 6, 0, −1, 1 27. 17

LXIII [p 152*c*]

1 −17, −12, −7, −2, 3, 8, 13, 18, 23
2 8, 7, 6, 5, 4, 3, 2, 1, 0 3 13, 11, 9, 7, 5, 3, 1, −1, −3.
4. $-\frac{5}{3}, -\frac{4}{3}, -1, -\frac{2}{3}, -\frac{1}{3}, 0, \frac{1}{3}, \frac{2}{3}, 1$
5 $-\frac{5}{4}, -\frac{3}{4}, -\frac{1}{4}, \frac{1}{4}, \frac{3}{4}, \frac{5}{4}, \frac{7}{4}, \frac{9}{4}, \frac{11}{4}$
6 $\frac{13}{8}, \frac{10}{8}, \frac{7}{8}, \frac{4}{8}, \frac{1}{8}, -\frac{2}{8}, -\frac{5}{8}, -1, -\frac{11}{8}$.
7 12, 10, 6, 5, −6, −5, −12, −10.
8 14, −5, 9, −3, 4, −1, −1, 1, −6, 5, −11, 5
9 −11, −2, 1, −1, 13, 0

LXIV [p 154]

3 220 sqs, (i) 40 sqs, (ii) 48 sqs, (iii) 72 sqs, (iv) 60 sqs.

LXVI [p 159]

The required graph may be obtained by drawing a line through the points

7 (0, 2), (3, 3). 8 (2, 2), (6, 3) 9 $(0, \frac{1}{3}), (-\frac{2}{3}, 0)$, take 3 divisions of squared paper as unit 10 (−1, 2), (3, −1)
11 (7, 4), (15, 9) 12 $(0, \frac{2}{3}), (\frac{4}{3}, 0)$, take 3 divisions as unit
16. (0, 3), (6, 0) 17 (0, 6), (2, 0) 18 (−4, 0), (0, 2)
19 (0, −5), (2, 1) 20 (2, −2), (−1, 3) 21 (3, 0), (0, −5)

22 (−4, 0), (0, 5)   23 (3, 0), (0, 4)   24 (5, 0), (0, −3)

25 (2, −2), (7, 1)   26 (4, −5), (5, 2)   27 (10, 0), (−6, −2).

LXVII [*pp* 161—162]

1 −2, 3   2 1, 4   3 5, 2   4 3, −3

5 2, 6   6 −4, −3   7 3, 0   8 0, 5

9 −4, 0   10 −3, 4   11 −2, −3   12 1, 3

13 4, 5   14 $x=6, y=-2$ 15 4, −5   16 −3, −4

17 At the points (3, 0), (0, −2)   18 $11y=5x$

Miscellaneous Examples LXVIII [*pp* 163—164]

1. 32 units of area   2 98 units of area

−1, 2, 5, −4, −7, 0, $-\frac{11}{5}$ 6 (0, 0)   8 (5, 0), (−5, 6), (0, −4).

9 5, −2   11 17, 3, 5, 2, −7, 1, −19, 0   12 (0, 2)

13 No   16 $x=4, y=6$, 12, 9 and 2 8, −14   17 9

18 $3y=4x$ 19 $2x-9y=37$ 20 $7x+26y=33$ 21 (−3, 4).

LXIX [*pp* 166—167]

1 24   2. 1 4, 6   3 8, 3 5   4 1

5 12, 6 4   6 11 2   7. 1 25   8 1 7, 2

LXX. [*pp* 175—176]

1 149 m, 201 kilos   2 30 fr, 19s   3 6 6 cent, 3 1 in

4 81 7, 22   5 R3 11*a*, 27½ sr   6 R18 5*a*, 18 days

7. 39, 54   8 20*a*, R3, R14, 22*a* 6*p*, R7 8*a*, R25 15*a*

9 10*a*, R2 13*a*, 6*a* 3*p*   10 R474, 1150 copies

11 2 hr 24 min, 84 or 96 mi from start, 1 hr 48 min after start.

12 7 hr 25 min.   13. At 4 48 *p m*, $14\frac{2}{5}$ mi from *A*'s starting place, at 2 *p m*   14 At 6 *p m*, at 3 36 *p m*, at 8 *p m*.

15 (1) At 4 *p m*, (2) at 5 *p m*, (3) at 6 *p m*, (4) *A* 3 miles ahead of *B*, *B* 7 miles ahead of *C*.   16 3 4 days   17 12 days.

18 18 sec   19 13m 20 sec   20 1 hr

LXXII [*pp.* 178—180]

3 3 10, 4 11.   4. 31 1, 32 3.   5 14 5 lb, 9th month.

6 R23, R27, 37 yrs 7 6 56, 10 8 8 58 lb, 55 lb., 81 lb, 83 lb, both 104 lb 9 1s $9\frac{1}{2}d$, 1s $8d$, 1s. $6\frac{3}{4}d$, 1s $6\frac{1}{4}d$
10 R2 $4a$, R5 $12a$ 11 1828 and 1845, 1858
12 50, 95, 13

Examples for Revision (C) [pp 180—184]

1 $2d-2b-2c$ 2 $\frac{3}{4}$ 3 $2bx+2cy$ 5 (i) $\frac{1}{4}$, (ii) 19, 3
6 34 7 R17 8 $2a-6b$ 9 002
10 $2ax-2by$, $(a-1)x-y$ 12 (i) 10, (ii) 3, 2 13 18.
14 $\frac{2ac}{b}$ 15 $6x$ 16 $x^4+32x^2-320x-1024$ 17. 2, x^2
19 (i) 9, (ii) -3, 5 20 855300 6. 21. 15, 10 22 $(a^2-ab-b^2)x+(a^2+ab+b^2)y-(a^2+ab-b^2)z$ 23 $3x+8a$ 24 $x^3+5x^2y+7xy^2+8y^3$, $-x^3+8x^2y+7xy^2+7y^3$ 25 $(a+b-1)(a-b-1)$
26 (i) $2\frac{1}{2}$, (ii) 3, 2 27 695 mi from A, 663 mi from B.
28 $14d$, $10d$ 29 $3c$ 30 Diff $=(x-y)^2$ 31 $(1+2x)(1+4x^2)$, 0.
32 (i) $-2\frac{3}{5}$, (ii) 33, 3 33 $\sqrt{3}$ sec 34 In 20 days 35 $3x-4y=24$
36. $a+4b$, 0 38 $(a+b-c)x+(b+c-a)y+(c+a-b)z$
40 (i) $7\frac{1}{2}$, (ii) 4, 5 41 $ax+by$, $\frac{1}{2}(2ax+by)$ 43 59
44 $(a+b)(a+b+2c)$. 45 x^2-xy 46 (i) $2\frac{18}{19}$, (ii) 2, 6
48 156, 54 49 $x=3$ 50 $8a-3b$, 0 51. $8a^2+6ab-2b^2$, $8a-2b$ 52 0 53 4 9 54. 13 55 14, 7, 24
56 $25d$, $28d$ $29d$, $32d$

LXXIII [pp 186]

1 $x^4+a^2x^2+a^4$ 2 $a^4-2a^2b^2+b^4$ 3 $10a^4+a^3b+18a^2b^2-72ab^3-27b^4$ 4 $x^4-7x^2y^2+9y^4$ 5 $x^4+2abx^2+a^2b^2-b^4$ 6. $x^3+y^3+3xy-1$ 7 $a^3+b^3+c^3-3abc$ 8 $apx^3+2(aq-bp)x^2+(cp-4bq)x+2cq$ 9 $x^6-5x^4+3x^3+6x^2-7x+2$. 10 $3x^4-5x^3y+6x^2y^2+5xy^3-3y^4$ 11 $4x^6-10x^5y+10x^4y^2-21x^3y^3-5x^2y^4+5xy^5+y^6$ 12 $6a^5b-7a^4b^2-11a^3b^3+9a^2b^4-5ab^5$
13 $a^4-2a^2b^2+4abc^2+b^4-c^4$ 14 $35a^2+11a^3-15a^4+18a^5-68a^6+28a^7$ 15 $abx^4-(bp-ac)x^3+(bq-cp-ar)x^2+(cq+rp)x-rq$
16 $a^3(b+c)+b^3(c+a)+c^3(a+b)+3abc$ 17 $x^5-px^4+qx^3-qx^2+px-1$ 18 $4+5x+8x^2+10x^3-8x^4+5x^5-4x^6$
19 $1-\frac{1}{6}x-\frac{1}{6}x^2+\frac{5}{18}x^3-\frac{1}{9}x^4$ 20 $1+\frac{1}{6}x-\frac{1}{6}x^2+\frac{25}{36}x^3-\frac{1}{3}x^4$

21 $x^6-\frac{5}{2}x^5+\frac{21}{4}x^4-\frac{85}{8}x^3+\frac{21}{4}x^2-\frac{5}{2}x+1.$ 22. $agx^3+(2gh+af)x^2y$ $+(bg+2fh)xy^2+bfy^3$ 23 $a^2x^8+(2ac-b^2)x^6+(2af-2bd+c^2)x^4$ $+(2cf-d^2)x^2+f^2$ 24 $lx^3+my^3+mx^2y+lxy^2+(2lg-1)x^2$ $+(2fm-1)y^2+2(lf+mg)xy+(l-2g)x+(m-2f)y-1$

LXXIV [*pp* 188—189]

1 $1+x^2-x^4-x^6$ 2 $x^5+151x-264$ 3 $x^6-4x^5-4x^4+22x^3-12x^2-12x+9$ 4 x^8-y^8 5 $x^8-6x^6+2x^5+9x^4-13x^3+x^2+21x-7$ 6 $a^7-2a^6b+3a^5b^2-8a^4b^3+5a^3b^4+3ab^6-2b^7$

LXXV [*p* 189]

1 $x^3+6x^2+11x+6$ 2 $x^3-(a+b+c)x^2+(bc+ca+ab)x-abc.$

3 $abc+bcx+acy+abz+cxy+bxz+ayz+xyz$

4 $1-(a+b+c)x+(ab+ca+bc)x^2-abcx^3$

5. $a^2b+ab^2+b^2c+bc^2+a^2c+ac^2+2abc$

6. $ab^2-a^2b+bc^2-b^2c+ca^2-c^2a.$

7 x^4-16a^4 8. $1-x^8$ 9 $x^4-4x^3-7x^2+22x+24$

10 $x^4+(a+b+c+d)x^3+(ab+ac+ad+bc+bd+cd)x^2+(abc+acd+bcd+abd)x+abcd$ 11. $x^4-(a+b+c+d)x^3+(ab+ac+ad+bc+bd+cd)x^2-(abc+acd+bcd+abd)x+abcd$

12 $1-(ax+by+cz)+abxy+acxz+bcyz-abcxyz$

13 $a^3x^3-a^2(b-c+d)x^2y-(abc-abd+acd)xy^2+bcdy^3$

14. $a^2(b+c)+b^2(c+a)+c^2(a+b)-a^3-b^3-c^3-2abc$

15 $9x^4-52x^2y^2+64y^4$ 16 a^6-x^6 17 $x^{16}-a^{16}$

18 $a^8+a^4x^4+x^8$ 19 $2b^2c^2+2c^2a^2+2a^2b^2-a^4-b^4-c^4$

LXXVI [*pp* 191—192]

1. $x+6$ 2 $-2x^2+8x+1$ 3 $3x^2-4x-2$ 4 $-8x^3+5ax+a^2.$

5 $1-a+b$ 6 $x^2+y^2+a^2$ 7 x^2-xy+y^2 8 $a^2+2ab+2b^2$

9 $2x^2y^2+2xy+1$ 10 $16x^2-4xy+y^2$ 11 $1-2x+4x^2$

12. $a^2+3ab+6b^2$ 13 $2x^3-3x^2+2x$ 14. $2x^3+2x^2-5x$

15 $a^3+2a^2b+3ab^2+4b^3$ 16 $2a^2+3ab+2b^2$ 17 $x^2-xy+y^2.$

18 $y-2x-a$ 19 $x^4-2x^3y+3x^2y^2-2xy^3+y^4.$

20. $5x^4-4x^3+3x^2-2x+1$ 21. $m^4+2m^3n+3m^2n^2+4mn^3+5n^4$

22 $x^2-2ax-bx+a^2+ab+b^2$ 23 $a^2-ab+b^2-a-b+1.$

24 $1-x+2y+x^2+2xy+4y^2$ 25 $-4x^2-y^2-z^2-yz+2zx-2xy$
26 $-a^2-b^2-9c^2+3bc-3ca-ab$ 27 $7x^2+4x-9$
28 $3a^3-4a^2x+6ax^2+5x^3$ 29 $x^4+3ax^3+8a^2x^2-8a^4$.
30 $4x^2-5xy+2y^2$ 31 $x^2+y^2+z^2+1$ 32 $a^2+b^2+c^2+d^2$
33 $a^2-2ab+b^2-c^2$ 34 $x^3+x^2y+xy^2+y^3$
35 $x^4-x^3y+x^2y^2-xy^3+y^4$ 36 $a+b-c$ 37 $4x^2-2xy^2+y^4$
38 $x^2+y^2+z^2+xy-xz-2yz$ 39 $9a^4+6a^2x^3+4x^6$
40 $x^2+(m+1)x+1$ 41 $a^4-a^3b+a^2b^2-ab^3+b^4$.
42 $a^3-3a^2x+x^3$ 43 $y^4-(m-1)y^3-(m-n-1)y^2-(m-1)y+1$
44. $x^2+a(1-p)x+a^2$ 45 $\frac{1}{3}+2a+3a^2$ 46 $1-\frac{1}{2}x+\frac{3}{2}x^3$
47 $a-b$ 48 $ac-bc+a^2-b^2$ 49 y^2-yz+z^2

LXXVII [pp 193—194]

1 5 2 $-y^2$ 3 $5x-3$ 4 $2y^3$ 5 $m^2+2mn+1$.
6 x^2-2x+3 7. -216 8 $x+3$ 9 x^2+2

LXXVIII [p 195]

1. x^2+6x-4 2 $3x^2+9x+14$, $r=63$
3 $8x^2+15x-72$, $r=272$ 4 $x^3-2x^2+14x-28$
5 x^2-3x-2 6. $16x^3-24x^2y+36xy^2-27y^3$; $r=y^4$.
7 $3x^2-4x+5$ 8. $x^2+xy+3y^2$, $r=x^2y^3+7xy^4-7y^5$.
9 $2x^2-3x+4$, $r=x^3-4x-1$ 10. x^2-2x+3, $r=x-1$.

LXXIX [p 196]

1 $1+2x+2x^2+2x^3$ 2 $1+ax+a^2x^2+a^3x^3$ 3. $1-\frac{3x}{a}+\frac{6x^2}{a^2}-\frac{12x^3}{a^3}$.
4 $1-(b+c)\frac{x}{a}+c(b+c)\frac{x^2}{a^2}-c^2(b+c)\frac{x^3}{a^3}$ 5. $1-x+2x^2-2x^3$.
6. $1+x^2+x^3+x^4$

LXXX [p 197]

1 3, -13, 19, -5. 2 5, 2, 1, 26 3 4, -2, 13, -1
4 2 12, 0, 42 5 0, 0, -60, 120. 6 6, 0, -6, -120
7. $6x-1$. 8. $2(a^2+x^2+2x-3)$. 9. 32.

LXXXI [p 199]

1 12 2 9 3 -3. 4. 5. 5. 0. 6. c^4-3c^2+1.

7. $-q^5+2q^3+8q+10$ 8 0 13 4 14. 16 15 $p=-10, q=8$.
16 $a=5, b=24$

LXXXII [p 200]

1 $a^3 \mp a^2b+ab^2 \mp b^3$ 2 $a^4-a^3b+a^2b^2-ab^3+b^4$.
3 $a^5 \mp a^4b+a^3b^2 \mp a^2b^3+ab^4 \mp b^5$
4 $x^6-x^5y+x^4y^2-x^3y^3+x^2y^4-xy^5+y^6$.
5 $x^7 \mp x^6y+x^5y^2 \mp x^4y^3+x^3y^4 \mp x^2y^5+xy^6 \mp y^7$
6 $x^8-x^7y+x^6y^2-x^5y^3+x^4y^4-x^3y^5+x^2y^6-xy^7+y^8$
7. $a^9 \mp a^8x+a^7x^2 \mp a^6x^3+a^5x^4 \mp a^4x^5+a^3x^6 \mp a^2x^7+ax^8 \mp x^9$
8 $a^{10}-a^9x+a^8x^2-a^7x^3+a^6x^4-a^5x^5+a^4x^6-a^3x^7+a^2x^8-ax^9+x^{10}$.
9. $x^{15} \mp x^{14}a+x^{13}a^2 \mp x^{12}a^3+x^{11}a^4 \mp x^{10}a^5+x^9a^6 \mp x^8a^7+x^7a^8 \mp x^6a^9$
$+x^5a^{10} \mp x^4a^{11}+x^3a^{12} \mp x^2a^{13}+xa^{14} \mp a^{15}$.

LXXXIII [p 202]

1 484 2 1. 3. 15 4 85, 97, 170 5 63001.
6 529282, 235359 7. $6a^2b^2+2(a^4+b^4)$ 8 19. 9 37.
10 $3a^2b^2+c^2d^2$, $a^2b^2+3c^2d^2$. 11 (1) $(x-5)^2-5^2$, (2) $(x+9)^2-9^2$;
(3) $(x+3a)^2-(2a)^2$, (4) $(x+a)^2-(2a)^2$, (5) $(x^2+5x+7)^2-(2x+5)^2$ or $(x^2+5x+\frac{11}{2})^2-(x+\frac{5}{2})^2$ or $(x^2+5x+5)^2-1$

LXXXIV [p 203]

1. 36 2 $\frac{5}{6}$ 3. 2, 40 4 26 5. -296 6. 100.
7 -9 8. 3

LXXXV [p 203]

1 $15x^2+22x+8$ 2. $6x^2+13x+6$ 3 $32x^2+28x-15$.
4 $15x^2+2x-24$ 5 $6x^2-25x+24$ 6 $18x^2+43x-5$
7. $12a^2-25a+12$ 8. $35x^2+33x-54$ 9 $16x^2-134x+105$

LXXXVI [p 204]

1. $m^3+6m^2+11m+6$ 2. $x^3-6x^2+11x-6$ 3. x^3+4x^2+x-6.
4 $x^3+15x^2+66x+80$ 5 $k^3+2k^2-11k-12$ 6 $p^3+2p^2-91p+88$.
7 $z^3-2z^2-11z+12$ 8 $h^3-h^2-30h+72$ 9 $x^3+6ax^2+11a^2x+6a^3$.
10 $x^3-7a^2x-6a^3$ 11 $x^3-9ax^2+23a^2x-15a^3$ 12 $y^3-4my^2+m^2y+6m^3$ 13 $m^3-16m^2+53m+70$ 14 $h^3-5nh^2-2n^2h+24n^3$.
15 $27x^3-135x^2+162x-40$ 16. $125x^3-125x^2-10x+24$.

LXXXVII [*pp* 205—206]

2. 256 3 $\frac{3}{2}$ 4 5 5 5 6 $(a+b+c)^2=$&c 7 $(a+b)^2=$&c.

LXXXVIII [*p* 206]

1—10 0

LXXXIX [*p* 207]

1. $(2x+3)(3x+2)$ 2 $(3x+5)(5x+3)$ 3 $(3x+4)(4x+3)$
4 $(3x-8)(8x-3)$ 5 $(x-12)(12x-1)$ 6 $(5x-8)(8x-5)$
7 $(x+8)(8x-1)$ 8 $(3m-4)(4m+3)$ 9 $(15x+1)(x-15)$
10 $3(4x+1)(x-4)$ 11 $(x+16)(16x-1)$ 12 $(x+11)(11x-1)$
13 $(x+9)(9x-1)$ 14 $(x-6)(6x+1)$ 15 $(5k-9)(9k+5)$.
16. $(m-8n)(8m-n)$ 17 $3(3x+4)(4x-3)$ 18 $(3x+2y)(2x-3y)$.
19 $7(x-2y)(2x-y)$ 20 $(x-7y)(7x+y)$ 21 $(5x-8y)(8x+5y)$.

XC [*pp* 208—209]

1 $(a+b)(a-b)(a-c)$ 2 $(ax+b)(cx^2+d)$ 3 $(x+a)(x^2-bx+1)$
4 $(x-2c)(x^2-2ax+3a^2)$ 5 $(x-a)(x-b-c)$
6 $(x-b)(x+2a+b)$, 7. $(1-ax)(1-ax-cx^2)$
8 $(3x-1)(x+5y+4)$ 9 $(a-x)(ay+a-x)$ 10 $(x-y)(x-5y+2z)$
11. $(a-b)(a+2b-3c)$ 12 $(a-4b)(a-9b+5c)$ 13 $(x-a)^2(x-b)$.
14 Put $c^2=m$, $(a-b^2)(a-4b^2)(a-c^2)$ 15 $(x^2+px-q)(x^2-ax-1)$.
16 $(x-2y)(x+3y-1)$ 17. $(x-y+1)(x+y-2)$
18 $(x-3a)(x+a-2)$ 19 $(a+2)(a+6b-5)$ 20 $(a+3b)(a-b+2c)$
21. $(x-3a)(x+5a+2)$ 22. $(x+a+ab)(x-b-ab)$.
23. $(x-3a-6b)(x-2a+6b)$ 24 $(x-ab-a^2)(x+ab-b^2)$
25 $(x-y+3)(x-3y+1)$ 26 $(a-3b+2c)(a-b-2c)$.
27 $(x-3a)(3x-2a+5b)$

XCI [*pp* 211—212]

1 $(x+\frac{5}{2})^2-(\frac{5}{2})^2$ 2 $(x-10)^2-10^2$. 3. $(x-9)^2-9^2$.
4. $(x+\frac{13}{2})^2-(\frac{13}{2})^2$ 5 $2\{(x-\frac{5}{4})^2-(\frac{5}{4})^2\}$ 6 $5\{(x+\frac{12}{10})^2-(\frac{12}{10})^2$
7. $\left(x+\frac{m}{2}\right)^2-\left(\frac{5m}{2}\right)^2$ 8 $(x+q-\frac{1}{2})^2-(\frac{1}{2})^2$ 9 $(x^2+3x+1)^2-1$.
10. $(x^2+ax-a^2)^2-a^4$. 11 $(x+12)(x+16)$ 12 $(k-6)(k-15)$.
13. $(a+18)(a-9)$ 14. $(x-8)(x-10)$ 15. $(a+19)(a-20)$.

16 $(3x+4)(x-8)$ 17 $(3a-2)(4a-5)$ 18 $(5-x)(6-5x)$
19 $(4+7x)(3-x)$ 20 $(5x+3y)(2x-5y)$ 21. $(3x-2a)(2x+9a)$.
22 $(4x-3a)(x-12a)$ 23 $(3-20x)(4+3x)$ 24 $(5x-31)(3x+25)$.
25 $(4z-35)(6z-49)$ 26 $(y+94)(y-95)$ 27 $(x+81)(8x-135)$.
28 $(x+y+3)(x-y-1)$ 29. $(x+y-4)(x-y-2)$
30 $(a-b-c)(a-5b+c)$ 31. $(x+y+z)(x+9y-z)$
32 $(a-c)(a-2b-3c)$ 33 $(a-b)(a+3b-2c)$
34. $(x-a+c)(3x+a-3c)$ 35 $(x+2y+3)(y+1)$
36 $(2x+3y+4)(y-3)$. 37. $(x+y+z)(x-2y-z)$
38 $(x+5y+4)(3x-1)$ 39 $(2y-x+a)(y-2x-a)$.
40 $(x+3y+3a)(x+y+a)$

XCII. [pp 214—215]

1. 0 2 0 3 $(b-c)(c-a)(a-b)$ 4 $-(b-c)(c-a)(a-b)$.
5. $(y-z)(z-x)(x-y)$ 6 $-(y-z)(z-x)(x-y)$.
7. $-(b-c)(c-a)(a-b)x^2$. 8 $-(b-c)(c-a)(a-b)$.
9. $-(b-c)(c-a)(a-b)$ 10 $(b-c)(c-a)(a-b)(a+b+c)$
11. $-(b-c)(c-a)(a-b)(a+b+c)$ 12 $(b-c)(c-a)(a-b)$
13 $-(b-c)(c-a)(a-b)x$ 14. $-4(b-c)(c-a)(a-b)$

XCIII [pp 215—218]

1. $a^3+b^3-c^3+3abc$ 2. $8x^3-y^3+z^3+6xyz$ 3 $27x^3-8y^3-64z^3$
$-72xyz$ 4 $1-x^3+8y^3+6xy$ 5 $-x^3-8y^3-18xy+27$.
6 $27a^3-64b^3-72ab-8$ 7 $x^3+y^3+3xy-1$ 8 $(a+b+1)$
$\times(a^2-ab+b^2-a-b+1)$. 9 $(a+2b-1)(a^2-2ab+4b^2+a+2b+1)$
10 $(x+y-a)(x^2-xy+y^2+ax+ay+a^2)$ 11. $3(y-z)(z-x)(x-y)$
12 $3abc(b-c)(c-a)(a-b)$ 13 $3(2x-y)(x+y)(x-2y)$ 19 0
20 $2x^3=\frac{137842}{421875}$

XCIV. [p 220]

1. $(x+1)(x+2)(x+4)$ 2 $(x+1)(x+4)(x+5)$
3. $(x-1)(x+2)(x+8)$ 4. $(x-1)(x-3)(x+5)$.
5 $(x+1)(x^2+3x+8)$ 6 $(x-2)(x+3)(x-4)$
7. $(2x+1)(x+3)(3x-2)$. 8 $2(x+2)(x-2)^2$
9. $(a-1)^2(a+2)$ 10. $(x+1)(x-2)^2$. 11. $(x+1)(x+5)(x-6)$.

12. $(2a-b)(2a^2+ab+b^2)$.
13 $(x+1)(x+3)(4x-3)$
14. $(2x-3)(4x^2-2x-3)$
15 $(a-3b)(a^2+2ab+6b^2)$.
16. $(a+b)(a^2+8ab-8b^2)$
17 $(x-2y)(x-4y)(x+6y)$
18. $(a-3b)(a^2+ab+3b^2)$.
19 $(a-b)(2a-b)(3a+b)$
20 $(x+1)(5x-3)(5x-2)$
21. $(x+2)(x-3)(x^2+x+7)$.
22. $(x-2)(x+4)(x^2-2x+12)$
23 $(2x+1)(2x-1)(3x^2+1)$.
24 $(x-3)(x^3-2x^2-6x-18)$
25. $(x+1)(x-2)(3x^2-2x+4)$
26 $(a-b)(a+2b)(3a^2+2ab+4b^2)$
27. $(x-1)(x+2)(x-3)(x+4)$
28. $(x-5)(x+1)(3x+2)(4x-3)$.

XCV. [pp 222—225]

1 $(1-x)(1-x-a)$
2 $(2m+2q+1)^2$
3 $(ax-y)(ax-y-1)$
4 $ax(x+y)(3x+3y-1)$
5 $(a+b)(a+b+2c)$
6 $(1-y+z)(1-y-z)$
7. $(1+ax+x)(1+ax-x)$
8. $(1+a+b)(1-a-b)$
9 $(a+b+c)(a-b+c)(a+b-c)(a-b-c)$.
10. $(a+b+c)(b+c-a)(c+a-b)(a+b-c)$
11 $(x+y+z-u)(x+y-z+u)$
12 $(b+c+d-a)(c+d+a-b)(d+a+b-c)(a+b+c-d)$.
13 $(x^2+2x+2)(x^2-2x+2)$
14 $(2a^2+2ab+b^2)(2a^2-2ab+b^2)$
15 $(x^2+x+1)(x^2-x+1)$
16 $(a^2+ab+b^2)(a^2-ab+b^2)$.
17 $(x^2+2x+3)(x^2-2x+3)$
18 $(x^2+pxy-y^2)(x^2-pxy-y^2)$
19 $(x+1)(x-1)(x+5)(x-5)$
20 $(a^4-a^2x^2+x^4)(a^2+ax+x^2)(a^2-ax+x^2)$
21 $(x+a)(x-a)(3x+5a)(3x-5a)$
22 $(5x^2-2x+1)(x^2-2x+5)$.
23 $(x+y)(x-y)^2$.
24. $(a+3b)(a+b-3)$
25. $(x-y-z)(x-y+z+2)$
26 $(x+y-z)(x-y+z+1)$
27. $(a-b-c)(a+b+c+1)$
28 $(x+2)^2(x-2)^2$.
29 $(1-x)^2(1+x)(1-4x+x^2)$
30. $(x+y)(x-y)(x^2+xy+y^2)(x^2-xy+y^2)$
31 $\{x-2(y+z)\}\{x^2+2x(y+z)+4(y+z)^2\}=$&c
32 $-(x-1)^2(x^4+2x^3+6x^2+2x+1)$.
33. $2(x+a)(x-a)(x^4-2a^2x^2+4a^4)$
34 $(a^2+bc)(a^4-4a^2bc+7b^2c^2)$
35 $(x-y-z)(x^2-2xy+xz+y^2-yz+z^2)$
36 $(x-y+1)(x^2+xy+y^2-x-2y+1)$
37 $(a+2b)(a^2+ab+b^2)$
38 $(a+b-1)(a^2+2ab+b^2+a+b+2)$

39 $(x-2a)(2x^2-5ax+17a^2)$. 40. $(a+b+c)(a^2+b^2+2bc+c^2)$.

41 $(x-1)(x+1)(x^2+12)$ 42. $(x^2-1)(3x^2+4)$

43 $(x-y)(x+y)(2x^2+3y^2)$

44 $(x+1)(2x-1)(x^2-x+1)(4x^2+2x+1)$

45 $(x-y)(x^2+xy+y^2)(3x^3+2y^3)$.

46 $(a+2)(a-2)(a^2+4)(2a^2+2a+1)(2a^2-2a+1)$

47. $(a+x)(a-x)(a^2+x^2)(a^2+2ax+2x^2)(a^2-2ax+2x^2)$

48 $(x+1)(x+2)(x+3)(x+4)$ 49 $(x+1)(x+2)(x^2+3x-3)$

50. $(x-1)(x+5)(x+1)(x+3)$ 51. $(x-4)(x-6)(x+1)(x-11)$

52 $(x+1)^2(x-1)^2(x^4-2x^2+3)$ 53 $(x-3)(x-4)(3x^2-21x-8)$

54 $(x-2)(3x+4)(3x^2-2x-6)$ 55 $5(x+y)(8x+7y)$

56. $(5x^2-3y^2)(13y^2-11x^2)$ 57 $(2a+7x)(a+10x)$

58 $(a^2+2ab-b^2)(a^2-3ab+4b^2)$ 59. $(x+y)^2(x+4y)(10x-11y)$

60 $(a-1)(a+1)(a+5)(a+7)$ 61 $(x+2)(x+7)(x^2+9x+4)$

62. $(x+5)(x+6)(x^2+11x+8)$ 63 $(x+1)(x-6)(x^2-5x+16)$

64 $(2x+a)(x-3a)(2x^2-5ax+8a^2)$.

XCVII [*pp* 229—230]

1. $-(b-c)(c-a)(a-b)(bc+ca+ab)$

2 $(b-c)(c-a)(a-b)(bc+ca+ab)$ 3 $-(b^2-c^2)(c^2-a^2)(a^2-b^2)=$&c.

4 $-(b^2-c^2)(c^2-a^2)(a^2-b^2)=$&c 5. $3(b+c)(c+a)(a+b)$

6 $(a+b+c)(bc+ca+ab)$ 7 $(b+c)(c+a)(a+b)$

8 $(a+b+c)(bc+ca+ab)$ 9. Expanding we get $a(b^3-c^3)$
$+b(c^3-a^3)+c(a^3-b^3)-3abc\{(b-c)+(c-a)+(a-b)\}$,
$(b-c)(c-a)(a-b)(a+b+c)$

10, 11 $-(b-c)(c-a)(a-b)(a^2+b^2+c^2+bc+ca+ab)$

12. Ex. 10 with sign changed 13. Expanding we get
$abc\{(b-c)+(c-a)+(a-b)\}-\{a^3(b-c)-b^3(c-a)+c^3(a-b)\}$;
$(b-c)(c-a)(a-b)(a+b+c)$ 14 Expanding we get $bc(b-c)$
$+ca(c-a)+ab(a-b)-abc\{a^2(b-c)+b^2(c-a)+c^2(a-b)\}$,
$(b-c)(c-a)(a-b)(abc-1)$

15 We have $(b^2-c^2)(b+c)^2+(c^2-a^2)(c+a)^2+(a^2-b^2)(a+b)^2$
$=(b^4-c^4)+2bc(b^2-c^2)+(c^4-a^4)+2ca(c^2-a^2)+(a^4-b^4)+$
$2ab(a^2-b^2)$, $-2(b-c)(c-a)(a-b)(a+b+c)$

16. Proceed as in Ex. 15; $2(b-c)(c-a)(a-b)(a+b+c)$,

XCVIII [p. 231]

1. $x^3+(a+b-c)x^2+(ab-ac-bc)x-abc$ 2 ax^3+bx^2-bx-a
3 $a^3+2a^2c-a(b^2-bc-c^2)-bc(b-c)$ 4 $x^3-2bx^2-(a^2-ab-b^2)x+ab(a-b)$ 5 $(a^3+1)x^3+(a-2)x^2+2x-1$ 6 $(x^3-1)a^3-(x^3+x^2-2)a^2+(4x^2+3x+2)a-3(x+1)$ 7 $a^3-b^3+c^3+3abc$.
8. $x^3-3x^2+3x+y^3-1$ 9 $a^3(b-c)-a(b^3-c^3)+b^3c-bc^3$.
10 $(m+1)a^5-(m^2-2)a^4+(m^2+2)a^3+(m^2-2m-1)a^2-(m^2+m+2)a+2m$ 11 $(a^3-b^3)x^4+(a^2+2ab+3b^2)x^3-(a^2b-ab^2+2a+2b+a^2+ab+b^2)x^2+(a+b-2ab)x+ab$ 12 $2(a-b)x^5-(a^2-2ab-b^2)x^4+2(a+b-ab^2)x^3-(2a^2+2b^2-a^2b^2)x^2+2ab^2x-(a^2-b^2)$
13 $a^3(b^2-c^2)-a^2(b^3-c^3)+b^2c^2(b-c)$.

XCIX. [p 232]

1 $a^2-ab+b^2-2a+b+1$ 2 $a^2+b^2+c^2+bc+ca+ab$
3 $x-a+b$ 4 $a^2+b^2+c^2$ 5 $x-c$ 6 $a+b+c$.
7 $x^2+x+(a-b)$ 8 *Sym of ref* is a, $a^2(b-c)-a(b^2-c^2)+bc(b-c)$ 9. *Sym of ref* is a, a^2-xa+z 10 $x-a-b$.
11 $(a-b)x+(a^2-b^2)$ 12 $x^2+(a-2b)x+a^2+3b^2$
13 $(a^2+a+1)x^2+(a+1)x+1$ 14 $a^2+b^2+c^2-bc-ca-ab$.
15 $1-x+2y+x^2+2xy+4y^2$ 16 $(a+b)(x+y)+1$
17 $x^2-a^2+(n+1)ax$ 18 *Sym of ref* is a, $(x-1)a+x^2$
19 x^2-xy+y^2 20. *Sym of ref* is a; $(2x-y)a^2+(x+y)xa-x^3$.

C [pp. 233—235]

1 a^2-ab+b^2 2 $x-y$ 3. $a-b+c-d$ 4 $2(x-y)$
5 $x^2-xy+y^2-3x-3y+2$. 6 $x^2+y^2+z^2+1$ 7. $a+b$
8 $a^2-ab+b^2-2a+b+1$ 9 x^2-ax+a^2 10 $(1+a)^2+x^2$
11. $8ab$ 12 $x^2-xy+y^2+x+y+1$ 13 x^4-x^2-1
14 $3(b+c)$ 15 $x^2-2xy+4y^2+x+2y+1$. 16 $a-b+c$
17. $(x^2+ax+a^2)(x^4-a^2x^2-a^4)=$&c. 18 x^2+3x-1.
21 The other factor$=x^2+y^2+z^2+2xyz-1$ 22. $x+y+z+xyz$.
23. $a^2+2ab+b^2-5a-5b+4$ 24 $4(a^2+b^2+c^2)$ 25 $8(ac+bd)$
26 $x^4+8x^3+24x^2+32x+16$ 27. 13 28 $4a^2b^2$.
29 $(b+c)(c+a)(a+b)$ 30 0. 31 1 32 0 34 0
35. $\frac{1}{3}$ 36 $\frac{203}{72}$ 37 $b-ap+q$ 38 $\frac{1}{16}$.

40 $(ab+a-b+1)(ab-a+b+1)$ 41. $(a-b)(a+b+c)(a^2+ab+b^2)$ $\times(a^2+b^2+c^2-bc-ca-ab)$ 42 $(2x+a)^2(x-4a)$ 43 $(a-b)^2$ $\times(a+2b)$ 44 $(x^2+x+a+1)(x^2-x-a+1)$ 45 $\{(a+b)x+(a-b)y\}\{(a-b)x+(a+b)y\}$ 66 $(x-a)(x-b)(x^2-ab)$

CI. [pp 236—237]

1. ab 2 ax 3 a^2 4 a^2b^2 5 ab^2c^3.
6. $15a$ 7 $21ax$ 8 ab 9 4 10 $16m^2$
11. $7a^2b^3x^3y^3$. 12. $3xyz$ 13 a^2c^2 14 $4m^3x^3$
15 $2mp^2$. 16 ax. 17 $18m^2n^3$ 18 $18a^2mx$.

CII [pp 237—239]

1. $a+x$ 2 a^2-ax 3 $2ax$ 4 $x+2c$ 5. x
6 $a-1$ 7 $x-2y$ 8 $y(x+y-z)$ 9 $(a+b)^2x^3$.
10 $a(a-b)^2$ 11. $a(p+q)x$ 12 $4(a-1)^2(x-a)^2$
13. $6b(x-a)^2(x^2-y^2)$ 14 $8(a+x)$ 15 $3(a+b)$
16 $8(x^2-x+1)$ 17. $3a(a-b)(a+b)^2$ 18 $5(x+1)(x+2)$
19. $4a(a-x)$ 20. $3(a+b)(a-b)^3$. 21. $4(x+1)(x+2)$
22 $x+2$. 23 $2x-1$ 24 $x+2$. 25 $1+x$ 26 $x+3$
27 $a-c$. 28 $1+x$ 29 $x-2$ 30 $x-2$
31. $x-1$. 32. $5x^3-1$. 33 x^3-1 34 $xy+ab$
35. $1-ax$ 36 $2a+3b+c$

CIII [pp 241—244]

1. $x-3$ 2 $x-1$ 3 $3x+1$ 4 $2x+3$ 5 $x+5$
6 $3x-2$ 7 x^2+x+1. 8 x^2+2x-1 9 $x+6$ 10 $x-7$.
11. $2x-1$ 12 $x-3$ 13 $x+2$ 14 $3x-7$ 15 x^2+x+1.
16 $2x^2-3x+1$ 17 $x+4$ 18 $2x^2+1$ 19 x^2+7x+5.
20 x^2-x+1. 21 $3(2x+3)$ 22 $2(x^2+ax-2a^2)$ 23 $x(x-2)$.
24 $2x(2x-1)$ 25 $x(x+2)$ 26 $2ab(2a-3b)$ 27. $x-2$
28 $2x-1$. 29 $x+5$ 30 $2x^2-xy+y^2$ 31 $a^2+2ab+3b^2$.
32 $x-3a$ 33 $3x^2+y^2$ 34 $a-2b$ 35 $x-3$ 36 x^2+3x+5.
37 $2x^2-6x$ 38 $2x-y$ 39. a^2x-3ay^2 40 x^3-2x^2+x.
41. $x^2+7xy-y^2$ 42 x^2-1 43 x^2-a 44 $x^2-(a-b)x+b^2$.
45 $(x+1)^3$. 46 $2x^3-4x^2+x-1$ 47 x^2-2x+3 48 x^2+2x+3.
29 $a-2b$

CIV. [p 245]

1. $x+7$. 2 $a-2$. 3 $a-b$. 4 $x-2a$ 5. $1+2x$
6 $x-1$. 7 $2x-9$.

CV. [p 248.]

1. $10a^2b^2$. 2 $24xy^2$. 3 $48a^2b$ 4 $63a^2x^4$ 5. $168x^3y^5$.
6 xyz. 7. $2a^3x^3$ 8 $60a^3b^3x^2$ 9 $50m^3n^3$ 10 $570a^2bx^2y^3$.
11. $24a^2x^2$ 12 $60a^3x^4$. 13. $960a^2x^3y^3z^2$

CVI. [pp 248—249]

1 $3ax^2(2+x)$ 2 $axy(x-y)$ 3. $xy(x+y)$. 4 $6a(x^2-y^2)$.
5. $12x(x^2-1)$. 6 $60xy(a^2-x^2)$ 7. $210(a^4+a^3b-ab^3-b^4)$.
8 $(2x-1)(8x^3+1)$ 9. $10(x^2-y^2)(x+y)$ 10 $(a^3+x^3)(a^2+ax+x^2)$
11 $42x(x^3-y^3)$ 12 $2ax(a^2-x^2)$ 13 $(x-y)(x^3+y^3)$.
14 $(x^2+1)(x^3+1)$ 15 $(1+a)(1-a)^2$ 16 $12x^2y^2(x^2-y^2)$
17. $(y-z)(z-x)(x-y)$ 18 $-ax^2(a^2-x^2)$. 19 $x^2(x+1)(x+2)$
20 $120xy(x^2-y^2)$ 21 $(x+2)^2(x-2)^3$. 22 $75(x+y)(x+2y)(x+3y)$
23 $36xy^3(x^2-y^2)$ 24 $12xy(2-a^2)$. 25 $a^3x^3(a^2-y^2)$ 26. x^6-1.
27 $60(x+y)(x-y)^2(x^3+x^3)$ 28 $x(x^2+x+1)(x^4-1)$
29. $x^3y^3(x^2-y^2)(x^6-y^6)$

CVII [p 250]

1 $(3x-1)(x-3)(x-6)$ 2 $(x-3)(2x-7)(3x-14)$. 3. $(x-1)^3(x+1)$
4. $(x-a)(x-b)(x-c)$ 5 $(3x-2y)(x-y)(4x^2-y^2)$ 6 $(x-1)(x^2-4)$
$\times(x^2-9)$ 7 $x(x-2)(x+3)(x-4)$ 8 $mn(x-1)(x-5)(x+6)$
9 $(a-x)(2a+3x)(3a^2+2ax+x^2)$ 10 $ab(a^2-b^2)(ax-by)(x^2-y^2)$
11. $(x^2-1)(x^4-4)(x+3)$ 12. x^6-a^6. 13. $a^2x(2a-3x)(a+3x)$
$\times(3a-x)(a-x)$ 14 $(x+3a)(x-a)\{x^2+(2a-3)x-6a\}$
15 $(x^2-2x+3)(x^2-2x-4)(x^2-2x-5)$
16 $(2x^2-1)(x-6)(x+8)(3x-1)(x+20)$.

CVIII [p 251]

1. $(x+2)(2x-1)(3x+1)$ 2 $(x-3)(2x-1)(4x+5)$ 3 $(x+9)(x-8)^2$
$\times(3x+10)(3x-10)$ 4 $(1+2x)(1-2x)(1+2x+4x^2)(1+2x-4x^2)$.
5 $(9x^2-1)(9x^4-1)(x^2-3)^2$.

Miscellaneous Examples CIX. [pp 253—254.]

1. $x+2a$ 2 x^2-2x+3 3 $x-3$ 4. x^2+2x+3 5 $4a^2-3ab+b^2$.

6 $(x^2+2x-3)(x^2+3)(x^2+2x+3)$ 7. $6(x-2a)(x^2-9a^2)(x^2-16a^2)$.

8. $(x+6)(x-7)(x^2-25)$. 9 $x(x-1)(3x+1)(4x^2+2x-1)$.

10 $(x-1)^2(7x-5)(2x^2+3x-5)$ 11 $bx+a$; $(bx+a)(a^2x^2-b^2)$

12 $x+y-1$; $(x+y+1)(x^3+y^3+3xy-1)$ 13 $8(x^2+y^2)$; $48(x^4-y^4)$.

14 $x-8$, $(x+9)(x-8)^2(9x^2-100)$ 15 $x-y$ 16 $(x-1)^2$.

17. $x^3-3ax+2a^3$. 18 $3x-1$, $\frac{1}{2}$ 19 2. 20 8

CXI. [pp. 257—259]

1. $\frac{3a}{bc}$ 2 $\frac{4y}{5x}$ 3 $\frac{2a^2bx}{3y}$ 4. $\frac{7n^2}{45m^2}$.

5. $\frac{2x-3y}{y^2(4x+5y)}$. 6 $\frac{ay}{a-y}$. 7. $\frac{2m-n}{2a}$. 8 $\frac{3m}{p-q}$.

9. $\frac{c}{d}$. 10 $\frac{x}{a^2(c-x)}$ 11. $-\frac{x}{y}$. 12 $-\frac{4x^2}{3a(2x+3a)}$

13 $\frac{b(a-b)}{4a(a^2-ab+b^2)}$. 14 $\frac{2a^2(x-a)}{3x^2(x^2-ax+a^2)}$ 15. $a(m+n)$.

16. $\frac{1-a-b}{1-b}$. 17. $\frac{a+b+c}{a-b+c}$. 18. $\frac{2x-3a}{4x^2+6ax+9a^2}$. 19. $\frac{ax+by}{ax-by}$.

20 $\frac{1-a}{1-b}$. 21. $\frac{x-1}{x+1}$. 22. $\frac{x+4}{x-4}$ 23 $\frac{x-1}{x+2}$ 24 $\frac{3a-2x}{3a+2x}$.

25 $\frac{2a-3x}{2a+3x}$. 26. $\frac{1+4x}{3+2x}$ 27 $\frac{3y+4}{5y-4}$. 28 $\frac{x+(a-b)y}{x-(a-b)y}$.

29. $\frac{ax-by-cz}{ax+by+cz}$ 30 $\frac{(a+b)x-ab}{(a-b)x-ab}$ 31. $\frac{a+b}{a-b}$. 32. $\frac{2y^2+3y-5}{7y-5}$.

33 $\frac{x^2-x+1}{x^3-3x+1}$ 34 $\frac{x^2-x+1}{x^2+x+1}$ 35. $\frac{x^2-x+1}{x-4}$ 36 $\frac{3x^2+4x+2}{4x^3+x+2}$.

37 $\frac{2x^2-6x+5}{3x^3-5}$ 38 $\frac{x(x+5)}{9x^2-x-3}$ 39 $\frac{6x^2-4ax-a^2}{6(x-a)}$.

40 $\frac{x^2-xy+y^2}{x^2+xy+y^2}$ 41. $\frac{a-bx}{a+cx}$. 42 $\frac{a^2-ab+b^2}{a^2+ab+b^2}$ 43 $\frac{3a^2+2b^2}{5a(2a+3b)}$

44. $\frac{2x^2+3x+5}{2x^2+3x-5}$ 45. $\frac{x^2+x-12}{x^2-x-12}$ 46 $\frac{3a^2+ax+2x^2}{2a^2+ax+3x^2}$

47. $\frac{3a(a^2-7ab+12b^2)}{2b(a^2+7ab+12b^2)}$ 48 $\frac{2(x^2-2ax+3a^2)}{3a(2x^2+5ax-3a^2)}$.

CXII [pp 260—261]

1 $\frac{20a}{180}$, $\frac{75a}{180}$, $\frac{84a}{180}$ 2 $\frac{24x}{36}$, $\frac{27y}{36}$, $\frac{10z}{36}$. 3 $\frac{a^3}{abc}$, $\frac{b^3}{abc}$, $\frac{c^3}{abc}$

4 $\frac{5x^2}{xyz}$, $\frac{4y^2}{xyz}$, $\frac{6z^2}{xyz}$, $\frac{x^2+2y^2}{xyz}$. 5 $\frac{4x(a+x)}{36ax}$, $\frac{3a(a-x)}{36ax}$

6 $\frac{5(4x-5)}{50}$, $\frac{20x}{50}$, $\frac{2(7x+6)}{50}$ 7. $\frac{6+6a}{30}$, $\frac{15-5a}{30}$, $\frac{3a-24}{30}$.

8 $\frac{cx^2-abc}{abc}$, $\frac{ay^2-abc}{abc}$, $\frac{bz^2-abc}{abc}$ 9. $\frac{a(a-b)}{a^2-b^2}$, $\frac{b(a+b)}{a^2-b^2}$, $\frac{c(a-b)}{a^2-b^2}$.

10 $\frac{4x(1-x)}{12(1-x)}$, $\frac{15a(1-x)}{12(1-x)}$, $\frac{6(1+x)}{12(1-x)}$. 11 $\frac{1+x}{1-x^2}$, $\frac{1-x}{1-x^2}$, $\frac{1}{1-x^2}$

12 $\frac{2(x^2-4)}{(x-1)(x^2-4)}$, $\frac{-3(x+2)(x-1)}{(x-1)(x^2-4)}$, $\frac{4(x-1)}{(x-1)(x^2-4)}$ 13 $\frac{x(x-y)}{2y(x-y)}$, $\frac{2xy}{2y(x-y)}$, $\frac{2(1-y)}{2y(x-y)}$ 14 $\frac{x^2(a-x)^2}{a^2x^2(a^2-x^2)}$, $\frac{a^2(a+x)^2}{a^2x^2(a^2-x^2)}$, $\frac{ax(1-x^2)}{a^2x^2(a^2-x^2)}$.

15 $\frac{ab(b-c)(c-a)}{(a-b)(b-c)(c-a)}$, $\frac{bc(c-a)(a-b)}{(a-b)(b-c)(c-a)}$, $\frac{ca(a-b)(b-c)}{(a-b)(b-c)(c-a)}$

16 $\frac{a^2(a+b)}{ab(a^2-b^2)}$, $\frac{b^2(a-b)}{ab(a^2-b^2)}$, $\frac{ab(a+b)}{ab(a^2-b^2)}$

17. $\frac{3(4x^2-1),\ 5(2x-1)^2,\ 3x(2x-1),\ 4x(2x+1)}{(2x+1)(2x-1)^2}$.

CXIII. [pp. 262—263]

1. $\frac{29x}{30}$ 2 $\frac{ax}{315}$ 3 $\frac{44a-7x}{75}$. 4. $\frac{41x}{24y}$ 5 $\frac{2axy^2+3bx^3-aby}{x^2y^2}$

6 $\frac{96x^3-9x^2y+10y^2}{24xy}$. 7 $\frac{a^2+b^2+c^2}{abc}$ 8 0.

9 $\frac{8a^3+6a^2x+ax^2+x^3}{30a^2x^2}$. 10 $\frac{x^2-2xy-y^2}{y(x-y)}$ 11 $\frac{2}{1-x^2}$ 12 $-\frac{1}{1+x}$

13 $\frac{2mq}{q^2-p^2}$ 14 $\frac{a^2+x^2}{a^4(a+x)}$ 15 $\frac{a+bx}{c+dx}$ 16. $\frac{2ax+3by}{6}$ 17 1

18 $\frac{x^2+a^2}{2a(x+a)}$ 19 $\frac{1}{2}$ 20 $\frac{29x}{50}$ 21 $\frac{2x^2+(m-n)x+8a^2}{2x^3}$.

22 $\frac{2(1+x^2)}{x(1-x^2)}$ 23 $a-x$ 24 1 25 $\frac{2a^2}{x(a^2-x^2)}$ 26 $\frac{x+3}{x^4-1}$.

27 $\frac{4}{(9x^2-4)(3x-1)}$. 28 $\frac{4(2-x)}{x^4-1}$ 29 $\frac{y}{x+y}$ 30 $\frac{1}{1-x^4}$ 31 $\frac{18}{x^2-9}$.

32. $\frac{x^2}{(x-1)(x-2)(x-3)}$ 33 $\frac{3n^2}{(3m+2n)(9m^2-n^2)}$. 34 $\frac{1+2x+3x^2}{4(1-x^4)}$.

35 $\frac{2}{x(x-2)}$ 36 $\frac{1}{(x-1)(x-3)(x-5)}$ 37. $\frac{12}{x^8-5x^4+4}$.

38 $\frac{4(x^4+a^4)}{x^4-a^4}$. 39 $\frac{1}{x^3-1}$ 40 $\frac{3x-4}{(x+1)(x^3-1)}$ 41. $\frac{a+bx}{b+ax}$.

42 $\frac{3+2x+x^2}{(1-x)(1-2x)(1-3x)}$ 43 $\frac{-2x+2x^2-x^3}{(1-x)^4}$ 44 $\frac{a^3}{a^4-b^4}$.

45 $\frac{x^2+1}{(x+1)^2(x+3)}$. 46 $\frac{x+c}{(x-a)(x-b)}$ 47. $\frac{23+16x-30x^2-3x^3}{6-11x-21x^2+x^3+3x^4}$.

48 $\frac{1}{(x+1)(x+3)}$.

CXIV [p 265]

1. $\frac{3ay}{10bx}$. 2 $\frac{3}{2}$. 3 $\frac{a^2(a+x)}{y^2(a-x)}$. 4 $\frac{8(c+d)}{cd(m-n)}$ 5 $\frac{ax+x^2}{ab-bx}$

6 $\frac{a^4-x^4}{a^2x}$ 7 $\frac{a-x}{a+2x}$ 8. $\frac{3a^2(a-b)}{b}$. 9 $\frac{x}{a}+\frac{bx^2}{a^2y}+\frac{ay^2}{b^2x}+\frac{y}{b}$

10 $\frac{1}{y}+\frac{x}{y^2}+\frac{x}{yz}+\frac{y}{xz}+\frac{1}{z}+\frac{y}{z^2}$ 11. $\frac{bx}{ay}-\frac{ay}{bx}$. 12 $\frac{8ab}{9x^2}+2+\frac{9x^2}{8ab}$.

13 $\frac{x}{x+a}$ 14 $\frac{a^4}{a^2+x^2}$ 15 $\frac{ax+ay}{cy-y^2}$ 16 $\frac{1}{(x-y)^2}$ 17 $\frac{a^4x(ax-1)}{a-b}$.

18 $rs+(qs+rt)x+qtx^2$ 19 $x^2+1+\frac{1}{x^2}$. 20 $1-x$

21. $a^2x^2+1+\frac{1}{a^2x^2}$ 22 $\frac{3a^4}{x^4}-\frac{19a^3b}{10x^3y}+\frac{21a^2b^2}{5x^2y^2}-\frac{9ab^3}{10xy^3}+\frac{b^4}{y^4}$.

CXV [pp 267—268]

1. $\frac{1}{4a}$ 2 $\frac{8x^2}{3ab}$. 3 $\frac{a^2c}{mb}$ 4. $a-x$ 5 $\frac{a}{a^2+b^2}$.

6 $\frac{cx-x^2}{c^2+cx+x^2}$ 7. $\frac{2ay^2(1-x)}{c}$. 8 $\frac{x-1}{x^2}$ 9. $\frac{x^3+x}{3x^2+3}$.

10 $-\frac{a}{b}$ 11. $\frac{1}{x^2+y^2}$ 12 1 13 $\frac{x^2}{y^2}$ 14 $\frac{(x-1)(x-2)}{x^2}$.

15 $\frac{1}{(1+x)(1-x)^2}$ 16 y^2+y+1. 17. $z-1+\frac{1}{z}$. 18 $\frac{3x}{4b}-\frac{2a}{x}$.

19 $x+\frac{1}{x}$ 20 $x+\frac{1}{x}$. 21. $\frac{acx^2}{bd}+\frac{bx}{cd}+\frac{a}{b}$.

CXVI. [pp. 269—270]

1 $\frac{3x-4}{6x}$ 2 $\frac{2a}{4-a}$. 3 $2m-1$ 4 $5x-3$ 5 $\frac{2x}{3x+2}$

6 $\frac{ay+bx}{ab}$ 7. $\frac{m-n}{x+y}$ 8 $\frac{x}{z}$. 9. $\frac{(a-b)x}{ab}$. 10 $\frac{2a}{a-x}$.

11 $-\frac{b}{a}$ 12 $\frac{xy-y^2}{x^2+xy}$ 13. $x-1$. 14 $\frac{y^2}{xy-x^2}$. 15 1

16 $\frac{1}{m^2+n^2}$. 17 y. 18 $\frac{a-b}{a+b}$ 19 $\frac{3(x^2+1)}{x+2}$. 20. $\frac{a^2+x^2}{2ax}$

21. x. 22 $\frac{y+z-x}{y+z+x}$. 23 $\frac{y-x}{y}$ 24. 1. 25 $\frac{1}{x+1}$.

26 $\frac{a^2-b^2}{a^3b}$. 27 $\frac{2bc}{(b+c-a)^2}$ 28 $\frac{3x+1}{2x+1}$. 29. $\frac{ab-1}{a^2b-ab^2-a+2b}$.

30 $\frac{4}{3x}$. 31. $\frac{61-42x}{11-12x}$.

CXVII [pp 271—272]

1. 0. 2 0 3. 1. 4 0 5. 0 6 $\frac{1}{xyz}$. 7. 1 8. 1.

9 x^2. 10 $\frac{1}{xyz}$. 11. 1 12 1 13 1. 14 1. 15. 1

16 -1 17 1. 18 $\frac{1}{(x+a)(x+b)(x+c)}$ 19. $\frac{x}{(x-a)(x-b)(x-c)}$.

20 $\frac{x^2}{(x+a)(x+b)(x+c)}$ 21 See examples 18, 19 and 20

CXVIII. [p 273]

1 0 2 0 3 $\frac{3x^2-(a^2+b^2+c^2)}{(x-a)(x-b)(x-c)}$ 4. 0 5 0. 6 0

7 0

CXIX. [pp 274—277]

1. 1 2 $b^2-a^2+\frac{b^4}{a^2}-\frac{a^4}{b^2}$ 3 $\frac{ax+by}{ax-by}$. 4 $\frac{2a^2}{(a-x)^2}$.

5. $\frac{(a-b+c)(a-b-c)}{2ab}$. 6 $-\frac{b}{a}$ 7 $\frac{a}{a+b}$ 8 $\frac{a+b}{c^2-a^2-b^2+2ab}$.

9 1 10 a^4+b^4. 11 1. 12 $\frac{2(b+c)}{a+b+c}$ 13 0 14 $\frac{x-y}{x+y}$

15 $a^2+2ab+b^2-c^2$. 16 1 17 $\frac{1}{1-x^2}$, $\frac{(m+n)^2}{4mn}$ 18 $\frac{x-3y}{x+3y}$

19 $(a+b+c-d)(a+b-c+d)(c+d+a-b)(c+d-a+b)-4(ab+cd)^2$.

20 $\frac{x^2-x+1}{x^2-1}$ 21. $3x-5y$. 22. $\frac{a^4-10a^3b-6ab^3-b^4}{a^4+10a^3b+6ab^3-b^4}$ 23 1.

24. $a^3+\frac{1}{a^3}$ 25. $\frac{9(a-b)(a-c)(b-c)}{(b+c-2a)(c+a-2b)(a+b-2c)}$.

26. $\frac{x^2-bx+b^2}{x^2+bx+b^2}$ 27. $2(a+b+c)$ 28 $\frac{x-z}{1+xz}$ 29 $a+b+c$.

30 $\frac{3x-a-2b}{(x-a)(x-b)}$ 31 $\frac{5x}{2(x^2-1)}$. 32 $\frac{(1-a)(1-b)}{(1+a)(1+b)}$

33 $\frac{(1-x)(3-x^2+x^5+x^6)}{(1+x)(1+x^2)(1+x^3)}$ 34 $\frac{ab(2-x^2)}{2(b^2-a^2)x}$ 35 1.

36 $\frac{4xy^3}{x^4-y^4}$ 37 $\frac{(a+b+c+d)(a+b-c-d)(a-b+c-d)(b+c-a-d)}{4(ab-cd)^2}$.

CXXI. [p. 287.]

1 a^6; a^{21}, x^6y^9, $-243a^{15}x^{10}$, $-a^{18}x^{36}y^{45}$ 2 $-27a^6x^3y^9$; $256x^8y^4z^{16}$; $32a^{15}b^{10}x^{20}$, $-x^7y^{21}z^{42}$ 3 $\frac{x^4}{y^8}$; $-\frac{27a^3}{x^3}$; $-\frac{a^{10}b^{15}c^5}{32}$, $\frac{x^{21}y^{32}}{a^{16}c^8z^{40}}$

4 $-256x^9y^8$, $-288a^{13}b^2x^{19}$, $-256a^5b^{22}c^{10}$

5 $-81a^{14}b^3c^4$, $x^{2mp}y^{3np}z^{4p}$, $(-1)^na^nx^{mn}y^{np}$

6 a^{2m}, $(-1)^{m+1}a^{m+1}$, $-\frac{a^{2m-1}}{b^{2m-1}}$, $\frac{a^{2m+2}}{b^{2m+2}}$

CXXII [p 288]

1 $\frac{x^2}{4}+\frac{xy}{3}+\frac{y^2}{9}$. 2 $\frac{a^2}{b^2}-10+\frac{25b^2}{a^2}$ 3 $\frac{x^2}{4}-\frac{2}{3x}+\frac{4}{9x^4}$ 4 $1+x-\frac{5x^2}{12}-\frac{x^3}{3}+\frac{x^4}{9}$. 5 $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}+\frac{2xy}{ab}-\frac{2xz}{ac}-\frac{2yz}{bc}$ 6 $\frac{a^4}{x^2}-\frac{4a^2b^2}{xy}+\frac{6a^2c^2}{xz}+\frac{4b^4}{y^2}-\frac{12b^2c^2}{yz}+\frac{9c^4}{z^2}$. 7 $2a^2x^2+8b^2y^2-24bcyz+18c^2z^2$. 8 $a^{2m}+2a^mb^n+b^{2n}$ 9 $a^{2m}+4b^{2n}+c^2-4a^mb^n+2a^mc-4b^nc$ 10 $e^{4x}+2e^{3x}+3e^{2x}+2e^x+1$ 11 $(a^2+2ab+b^2)x^2-2(a^2-b^2)xy+(a^2-2ab+b^2)y^2$.

12 $\frac{x^4}{a^4}+\frac{2x^2y^2}{a^2b^2}+\frac{y^4}{b^4}-\frac{2x^2}{a^2}-\frac{2y^2}{b^2}+1$ 13 $\frac{a^2x^2}{m^4}+\frac{2abxy}{m^2n^2}+\frac{b^2y^2}{n^4}-\frac{2ax}{m^2}-\frac{2by}{n^2}+1$

14 $x^6-4x^5+6x^4-8x^3+9x^2-4x+4$

15 $a^2+2a(bx+cx^2+dx^3)+b^2x^2+2bx(cx^2+dx^3)+\&c$

CXXIII [p 288]

1 $\frac{8a^3}{b^3}-\frac{12a^2}{b^2}+\frac{6a}{b}-1$ 2 $\frac{64}{x^3}-\frac{144}{x^2y}+\frac{108}{xy^2}+\frac{27}{y^3}$ 3 $a^6-6a^4b^2+9a^4c^2$ $+12a^2b^4+27a^2c^4-36a^2b^2c^2-54b^2c^4+36b^4c^2-8b^6+27c^6$ 4 $1-6x+21x^2$ $-44x^3+63x^4-54x^5+27x^6$ 5 $a^3+3a^2bx+3(b^2+ac)ax^2+(b^2+6ac)bx^3$ $+3(b^2+ac)cx^4+3bc^2x^5+c^3x^6$ 5 $\frac{x^6}{a^6}+\frac{3x^4y^2}{a^4b^2}-\frac{3x^4}{a^4}+\frac{3x^2y^4}{a^2b^4}-\frac{6x^2y^2}{a^2b^2}+\frac{3x^2}{a^2}$ $+\frac{3y^2}{b^2}-\frac{3y^4}{b^4}+\frac{y^6}{b^6}-1$ 7 $\frac{27x^{12}}{8y^{12}}-\frac{9x^5}{2y^5}+\frac{2y^3}{x^4}-\frac{8y^9}{27x^9}$ 8 $\frac{x^{3m}}{a^{3n}}+\frac{3x^{2m}y^n}{a^{2n}b^m}$ $+\frac{3x^my^{2n}}{a^nb^{2m}}+\frac{y^{3n}}{b^{3m}}$ 9 $x^{12}-\frac{6ax^{10}}{c}+\frac{15a^2x^8}{c^2}-\frac{20a^3x^6}{c^3}+\frac{15a^4x^4}{c^4}-\frac{6a^5x^2}{c^5}+\frac{a^6}{c^6}$

CXXIV [p 291]

1 $a^5-5a^4b+10a^3b^2-10a^2b^3+5ab^4-b^5$

2 $a^8-8a^7x+28a^6x^2-56a^5x^3+70a^4x^4-56a^3x^5+28a^2x^6-8ax^7+x^8$

3 $1-12x+60x^2-160x^3+240x^4-192x^5+64x^6$

4 $2187x^7-5103x^6-5103x^5-2835x^4+945x^3-189x^2+21x-1.$

5 $\frac{x^5}{a^5}+5\frac{x^3}{a^3}+10\frac{x}{a}+10\frac{a}{x}+5\frac{a^3}{x^3}+\frac{a^5}{x^5}$

6 $64\frac{a^6}{b^6}-192\frac{a^5}{b^5}+240\frac{a^4}{b^4}-160\frac{a^3}{b^3}+60\frac{a^2}{b^2}-12\frac{a}{b}+1.$

CXXV. [p. 292]

1 a^2b 2 $5xy^3z^3$ 3. $7m^4n^3z$ 4. $12x^6y^8z^2$. 5 $8x^my^{2n}$.
6 $9a^{x+1}b^{\frac{3}{2}x+1}$. 7 $13x^{2m}y^{4n}z^{\frac{1}{2}m}$ 8 $ab^{\frac{3}{2}}$. 9. $\frac{5ax^2}{3yz^3}$

10 $\frac{6x^my}{7y^nz^3}$ 11 $\frac{5a^2x}{4yz^3}$. 12 $\frac{3a^{\frac{3}{2}}}{5b^2}$ 13 axy^3 14. $-2xy^4$.

15. $x^my^{\frac{1}{2}n}$. 16 $-ax^2y^3$ 17 $\frac{5x^2y}{2ab^{\frac{2}{3}}}$. 18 $\frac{2mn^3}{3p^3}$ 19 $5^{\frac{1}{m}}x^ay^c$

20. a^mx^{3+a}.

CXXVI [p 293]

1. $3a+b$ 2 $4x^2-3$ 3 $x(x-4a)$ 4 $8ax+3b^2$
5. $2ax(3ax-b)$. 6. $5x^2+3xy$ 7 $2ab-3c$ 8 $3x^2y+11z$
9 $3a-17bc$ 10 $x+\frac{y}{2}$ 11 $x-\frac{1}{2}$ 12 $3x-\frac{y}{3}$ 13 $2a-\frac{x}{4}$
14. $xy^2+\frac{z^3}{2}$ 15 $\frac{a}{c}+x$ 16 $\frac{x}{y}-\frac{y}{x}$. 17 $\frac{x}{a}-\frac{a}{2y}$ 18 $a^3+\frac{1}{a^3}$
19 $\frac{5}{24}-\frac{x}{5}$ 20 $3a^4-\frac{a^2b^2}{6}$

CXXVII [pp 294—295]

1. $a-b+c$ 2 $3x-2y+z$ 3 $x+y-a$ 4 $x+\frac{a}{3}-\frac{b}{2}$
5. $2x-a+1$ 6 $ax-bx^2-1$ 7. $x^2-a(x-b)$ 8 $\frac{a}{x}+\frac{b}{2y}-\frac{3c}{z}$.
9 $a-\frac{x}{2}-x^2$ 10 $a-b+c-d$ 11. $2x^2+2ax+4b^2$

CXXVIII [pp 298—299]

1. $3x^2+x+1$ 2 x^2+2x-3 3 x^2-x+1. 4 $2x^2-3x+4$.
5 $1-x+2x^2$. 6 $4x^2-3x-5$ 7 $5x^2-5x+6$ 8 $9x^2+6x-2$
9. $3-4a-14a^2$ 10 a^6-4a^3+1 11. $x^2+2xy+4y^2$ 12 x^3+4x-1
13 $2+3ax+x^2$. 14. $8p^2-3pq+2q^2$ 15 x^2-2x-2 16 $2x^2-4x-6$.
17. $2ax^2-3a^2x+a^3$ 18 $x^2+x-\frac{1}{2}$ 19 $x+1-\frac{1}{x}$ 20. $x-2-\frac{1}{x}$.
21. $x+4-\frac{8}{x}$ 22 $x-\frac{1}{x}+\frac{1}{x^2}$ 23 $x^2-\frac{3x}{2}+\frac{1}{4x}$ 24 $\frac{x^2}{2y^2}+\frac{2y^2}{x^2}+1$
25 $\frac{x}{y}-\frac{1}{2}-\frac{y}{2x}$ 26 $4ax-2ab-3by$ 27 x^3+2x^2+3x-1
28 $4a^3-3a^2+2a-1$ 22 $a^2-\frac{b}{2}+\frac{c^2}{3}-\frac{d^3}{4}$ 30 x^2+3x+1.
31 $\frac{2x}{x^2-1}$ 32 a^2+b^2 33 $(a+b)x-(a-b)y$.

CXXIX. [pp. 299—300]

1. $2x-a$ 2 $x+3y$ 3. $2a-5b$. 4. x^2-4a^2.

5 $x+\frac{a}{3}$ 6 $\frac{a^2}{3}-3b$ 7 x^2-x+1 8 $1+2a-a^2$.

9. $2a^2+ab-3b^2$ 10 $x-3y+2z$ 11 $\frac{2x}{y}-1+\frac{y}{2x}$

CXXX [*p* 300]

1 $x+1$ 2 $a-2$ 3 $a-b$ 4 $x-\frac{1}{x}$ 5 x^2-1 6 $2a-x^2$.

CXXXI [*pp* 303—304]

1 $1, 1, \frac{1}{x^3}, 1, 0$ 2 $x^{\frac{3}{2}}, x^{\frac{5}{3}}, x^{\frac{8}{3}}; a^{\frac{3}{4}}b^{\frac{1}{2}}; x^{\frac{10}{3}}$.

3 $\frac{x^{-2}}{2^{-1}}, \frac{1}{4^{-1}x^{-2}x^{-3}}; \frac{y^{-1}z^{-3}}{3^{-1}x^{-2}}, \frac{x^{-1}y^{-\frac{1}{2}}}{4^{-1}z^{-1}}, \frac{x^{-1}z^{-\frac{3}{2}}}{5^{-1}y^{-\frac{3}{2}}}$

4 $5\sqrt[5]{a^3}$ or $5(\sqrt[5]{a})^3$, $3\sqrt[3]{x^5y^2z^4}$ or $3(\sqrt[3]{x})^5(\sqrt[3]{y})^2(\sqrt[3]{z})^4$, $\frac{1}{3}\sqrt[6]{\frac{a^2x}{y^4}}$

5 $\frac{3a}{b}, \frac{x^3}{ay^3}, \frac{b^{\frac{1}{3}}c^4}{x^{\frac{2}{5}}}$ 6 216 7 $-\frac{4}{25}$ 8 -1152 9 -61.

10 $\frac{x^2}{cy}$ 11 $-\frac{1}{27x^6}$ 12 $\frac{b^{nx}c^x}{a^{mx}d^{2x}}$ 13 $\frac{b^{3x-3y}c^{3x}}{a^{2m}}$ 14 1 15 mx.

CXXXII [*pp* 306—307]

1 $3a^{\frac{2}{5}}-4a^{\frac{2}{6}}$ 2 $6x^{-\frac{2}{3}}-5a^{-\frac{1}{2}}x^{-\frac{1}{3}}+6\frac{1}{2}$ 3 $x+y$

4 $x^{\frac{5}{2}}-x^2-4x^{\frac{3}{2}}+6x-2x^{\frac{1}{2}}$ 5 $6x^{-8}b-9a^{-9}b^3-4a^{-7}b^{-1}+6a^{-8}$

6 $\frac{1}{2}a^{-2}x^{-6}-\frac{1}{30}a^{-1}b^{-2}x^{-3}y-\frac{2}{15}b^{-4}y^2$ 7 $a^m+a^{\frac{m}{2}}b^{\frac{n}{2}}+b^n$

8 $a-a^{\frac{1}{2}}b^{\frac{1}{2}}+b$ 9 $9+3a^{-\frac{1}{6}}+a^{-\frac{2}{6}}$ 10 $x^{\frac{2}{3}}+2x^{\frac{1}{2}}+3x^{\frac{1}{3}}+2x^{\frac{1}{6}}+1$

11 $x^2+x^{-2}-1$ 12 $x^{\frac{2}{3}}+y^{\frac{2}{3}}+z^{\frac{2}{3}}-x^{\frac{1}{3}}y^{\frac{1}{3}}+x^{\frac{1}{3}}z^{\frac{1}{3}}+y^{\frac{1}{3}}z^{\frac{1}{3}}$ 13 $-\frac{x}{y}$

14 $\left(\frac{y}{x}\right)^{\frac{1}{6}}$ 15 $\left(\frac{a}{b}\right)^{mn+qr}$ 16 x 17 $a^{\frac{1}{x}}x$ 18 $\frac{a^2x^{2m-1}}{y^{2m+1}}$

19 $\left(\frac{4y^{-2}}{b^{-4}}-\frac{9x^2}{a^2}\right)^{-2}$ 20 $\frac{b-a}{x-y}$ 21 $\frac{(3y)^{2m}}{(z-y)(x-a)^{2m-1}}$ 22 1

23 $(x^m-y^n)(x^m+y^n)$ 24 $(x^{-3}+y^{-1})(x^{-3}-y^{-1})$ 25 $a^{-1}x^{-1}-1+ax$

26 -17 27. $\dfrac{a^4b^4}{(b+a)^3(b-a)}$. 28 $x^{\frac{2}{3}}+x^{\frac{1}{3}}y^{\frac{1}{3}}+y^{\frac{2}{3}}$.

29 $\dfrac{c^x-1}{c^x+1}$. 30 $\dfrac{ab^{-1}+yx^{-1}}{ab^{-1}-yx^{-1}}$. 32 a^n.

CXXXIII. [p 308]

1. $\sqrt{50}$ 2 $\sqrt[3]{108}$ 3 $\sqrt{100ax^3}$ 4. $\sqrt[3]{a^{10}x^7}$.
5. $\sqrt[m]{q^{2m+3}r^{2m+2}}$.

CXXXIV. [p. 309]

1 $6\sqrt{2}$ 2 $6\sqrt[3]{5}$ 3 $72\sqrt{2}$ 4 $a^2\sqrt[5]{3b^3c^4}$ 5. $a(a+x)\sqrt{2x}$.

CXXXV [p. 310]

1 $3\sqrt{5}$ 2 $\sqrt{5}-4$. 3 $ab\sqrt[4]{b^3}$ 4 $\sqrt[12]{\dfrac{b^4c^4x}{a}}$ 5 x.

6 $(a+c)\sqrt{(ab)}$ 7 $\sqrt{bx(a^2-x^2)}$ 8 $3a^2$ 9 $\dfrac{\sqrt{a}}{x^2}$

10 $\dfrac{1}{a}\sqrt{x}-\sqrt{y}$. 11 $\dfrac{a}{b}$ 12 $\dfrac{9a}{x}\left(\dfrac{a^3}{2x^3}\right)^{\frac{1}{4}}$ 13 $3\sqrt{\dfrac{x+y}{x-y}}$

CXXXVI. [pp 312—313]

1 $3-\sqrt{3}$ 2 $2\sqrt{7}+3\sqrt{3}$ 3 $\frac{1}{4}(3+\sqrt{2}+6\sqrt{3}+2\sqrt{6})$ 4 $7-4\sqrt{3}$.
5 $\dfrac{x}{a-\sqrt{a^2-x^2}}$ 6 $5+2\sqrt{2}+\sqrt{3}$ 7 $\dfrac{2\sqrt{b}}{a^2-b}$ 8 $\dfrac{2(x+y)}{x-y}$

9 2 10 $\frac{1}{3}\sqrt{6}$ 11 $5\sqrt{6}$ 12 $\dfrac{2a}{\sqrt{a^2-x^2}}$ 13 $\dfrac{2}{x^2}$ or $\dfrac{2\sqrt{1-x^2}}{x^2}$.

14 $\dfrac{4x\sqrt{(x^2-a^2)}}{a^2}$. 15 $2x^2$ 16 $\dfrac{2(1-x)}{\sqrt{1-4x}}$ 17 $\dfrac{2a\sqrt{a^2-x^2}}{x^2}$, $1\frac{1}{3}$

18 9 19 $\dfrac{2a^3}{a^4-x^2}$ 20 0 21 0 22 0

24 Given expn $=(x^3+y^3+z^3-3xyz)+2xyz=2xyz$ [$x+y+z=0$],
$2(q-r)\sqrt{p}+2(r-p)\sqrt{q}+2(p-q)\sqrt{r}$

CXXXVII [p 314]

1 $3+\sqrt{2}$ 2 $3+\sqrt{5}$ 3 $\sqrt{3}+\sqrt{5}$ 4 $4-\sqrt{3}$ 5 $\sqrt{5}-2$

6. $2\sqrt{3}+3\sqrt{2}$. 7 $2\sqrt{2}-1$. 8 $2-\frac{1}{3}\sqrt{3}$. 9. $\dfrac{\sqrt{2}}{2}(\sqrt{3}+1)$

10. $\sqrt{a}+\sqrt{a+2b}+\sqrt{\frac{1}{2}(1-x)}$. 11. $\sqrt{x+1}-\sqrt{y-1}$. 12 $\sqrt{\frac{1}{2}(1+x)}$
13 $\sqrt{x}-\sqrt{y+z}$. 14 $a-\sqrt{ax-a^2}$.
15. $x+\sqrt{a^2-x^2}$. 16 4 17. $\frac{1}{2}\left(xy+\frac{1}{xy}\right)$.

Examples for Revision (D). [*pp.* 314—319]

1. $12x^4+180x$, $18\frac{3}{4}$ 2. 1. 3 $(b-a)^3=$etc 4 c^3.
5 $(a+1)x^2+(a^2+1)x+a^3$ 6 $\{(a+b)x-(a-b)y\}\{(a-b)x+(a+b)y\}$
7. $2x^2-x+2$ 8 $-16ab^{-\frac{1}{2}}$ 9 $4(a^4-a^3b+ab^3-b^4)$ 10. $4x^2y+2y^3$
11. $a+b+c-3x$. 12 $15x^3$ 14 $\frac{b^3-a^3}{3a}$. 15 $x^{12}-2x^6+2$
16 $5a=b$ 17. 1 18 $(a^2-b^2)x^2-4abxy-(a^2-b^2)y^2$.
19. $\sqrt{a^2-4b^2}$ 20. (i) $(a^2+b^2)(x^2+y^2)\{a(x-y)+b(x+y)\}$
$\times\{a(x+y)-b(x-y)\}$ (ii) $(x+y+z)(x+y-z)(x^2-2xy-y^2-z^2)$.
21 $(x^2+12ax+31a^2)^2-(4a^2)^2$ 22 3. 25 0 26 $3abc-a^3-b^3-c^3$.
27. a^2-b^2 28 $x-a$ 29. 5 30. (i) $(a^2+b^2)(x^2+y^2)$.
(ii) $(x+a)^3(x-a)^3$ 31 1 33 $3-29x-{\cdot}01x^2$, ${\cdot}2709$.
34 (i) $(x-y)(x+y)^3$ (ii) $(x-a)(x^2+ax+a^2)(ax+1)(a^2x^2-ax+1)$
35 $2+3x-4x^2$ 36 $2(ac+bd)$ 37 $(x-1)^2$. 38 $x^{-1}+y^{\frac{1}{3}}$.
39 $(a+b-c)(bc+ca+ab)(3abc-a^3-b^3-c^3)$ 41. 1.
42. $(a+1)(a-1)(b+1)(b-1)(a^2+1)(b^2+1)$ 43 $\frac{x}{y}-\frac{y}{x}+1$.
44 $a^2b^{-2}+1+b^2a^{-2}$ 45 $\frac{(x^2-x+1)(e^x-1)}{(x-1)(e^x+1)}$. 48 $b-a$.
49 $\left(\frac{1}{x}-\frac{1}{2}\right)^2$, $(a-b)\left(a+b-\frac{1}{ab}\right)$ 50 $a^{\frac{3}{2}}-b^{-1}$
51. $16x^4-8x^2(2y^2+a^2)+(4y^2-a^2)^2$ 53 $(a+2)x+(a-1)$
54 $x+\frac{1}{x}+2a$. 56 $x+1-\frac{1}{x}$ 57 $a^5+3a^3b^2+4ab^4$ 58 a^2+b^2
59 3 60 $b^n-a^{1-m}b$ 61 $\{x-a(b+2)\}\{x+b(a-2)\}$ 63. $3\sqrt{2}-\sqrt{3}$.
64. $x^{(mp+nr)(mn-pr)+mn}$ 65 $(2a-b)^3-(a+b)^3=7a^3-15a^2b+3ab^2-2b^3$.
66 (i) $(lx+my)(mx-ly)$, (ii) $2(x-y)(1-xy)$. 67. abc.
68 $\left(x+\frac{1}{x}\right)^3=$etc 69. $x^{\frac{1}{p^2+q^2}}$ 71. $x^{2n}+2$ 72. $b+\sqrt{a^2-b^2}$

73. 52. 74 $x+x^{-1}+1$. 75 Put $m=x+y$, $n=x-y$; thus expn. $=(am-n)(bm-n)$; $\{a(x+y)-(x-y)\}\{b(x+y)-(x-y)\}$.

76. $(x-y)^2$, $4a^2b^2$. 78 $\frac{2}{x^2}\sqrt{a^2-x^2}$ 79 The other factor is $4b^2+(a-c)^2$ 80 $19x-2$. 82. $\{x(a+1)+y(b+1)\}(x^2-xy+y^2)$

84. $1+x+x^2+x^3+x^4$; x^5. 85 $(mn-1)\left(\frac{x}{mn}-1\right)$ 86 15 87451.

87. $\frac{8}{41}$ 88. $8x-1$.

CXXXVIII [p 320]

1. $\frac{a-d}{c+3}$. 2 $\frac{c-d}{a-1}$. 3 $\frac{ma}{2b+1}$. 4 $\frac{c}{a-b}$. 5. $-\frac{3}{3}$. 6. 5.

7 $\frac{m^2}{n}$ 8 $\frac{1}{a}$ 9 $\frac{b^3}{a-c}$ 10 $\frac{acc}{cd-bc}$. 11. $\frac{1}{ab}$ 12 $\frac{d}{c}$.

13 $\frac{a^3+b^3}{a+b}$. 14. 1. 15 $\frac{b(h-g)}{a-bq}$. 16. $\frac{ab}{a-b}$. 17. $\frac{b-1}{a+2c}$

18. $\frac{d-c}{a-b}$ 19 $\frac{70ab-3ac}{320c}$. 20. $\frac{bf(h-q)}{af+2bc-bfq}$. 21. $\frac{3ac-b}{a-3b-c}$.

22 $a+b$. 23 $\frac{2npb-2mpc+6mn}{p(2na-2mb+nd)}$ 24 $\frac{ac}{b}$. 25 $\frac{4ab^2-10a}{4a-3b}$.

26 $\frac{b}{a}(a-b+c)$ 27. $-\frac{2a}{3}$ 28. $\frac{4a^2(a^2+ab-b^2)}{3a^3-6a^2b+ab^2+6b^3}$.

CXXXIX. [pp 321—326.]

1. 8. 2 20 3 1 4 4 5 7 6 8. 7. 4. 8. $\frac{3}{2}$.

9 7. 10. $\frac{ad-bc}{(a-c)n-(b-d)m}$. 11 $\frac{b}{5a}$. 12. $\frac{br-cq}{cp-ar}$ 13 $\frac{1}{3}$

14. 7. 15. $\frac{5}{a}$ 16 7. 17. $3\frac{1}{2}$ 18 $2\frac{1}{2}$ 19. $-3\frac{1}{3}$

20 $-2\frac{1}{2}$ 21 $6\frac{3}{4}$ 22 $36\frac{3}{11}$ 23. $\frac{a(m+n)-2ap}{p(m+n)-2mn}$.

24 5 25 $\frac{-ab}{a+b+c}$. 26 $\frac{a^3(1-b^2)+b^3(1-a^2)}{ab(a^2+b^2-2)}$

27. $\frac{(m-n)(n^2-m^2)}{m^2-mn+n^2}$ 28. 2 29 12 30. 3. 31. 3 32. 2

33 6 34 $3\frac{3}{8}$. 35. 4. 36 3. 37. 14. 38. 3. 39. 4.

40 10 41 1 42 $\frac{7}{8}$ 43 15 44 $\dfrac{ab(c+d)-cd(a+b)}{cd-ab}$

45 $\dfrac{1}{b-a-1}$ 46 $\dfrac{c^2-ab}{a+b-2c}$ 47 $\frac{1}{2}(a+b)$ 48 $4\frac{1}{3}$ 49 $\dfrac{a^2+b^2}{a+b}$

50 $-3a$ 51 $1-a$ 52 $a+3b+5c$ 53 $-(a+b+c)$

CXL [pp 328—329]

1 10 2 a 3 $\frac{1}{8}$ 4 19 5. $(2^m-3)^2$ 6 $\frac{1}{6}(a+1)$

7 $-11\frac{15}{16}$ 8 2 9 $\frac{9}{20}$ 10 16 11 25. 12 $\dfrac{\sqrt{a}}{\sqrt{a+2}}$.

13 a^{12-m} 14 $3\frac{6}{7}$ 15 $\frac{1}{2}a$ 16 $\frac{16}{25}$ 17 4 18. $\frac{4}{7}\sqrt{3}$

19 $\dfrac{(a-1)^2}{2a-1}$ 20 5 21 $6\frac{1}{4}$ 22 $\frac{25}{27}$ 23. $\dfrac{1}{a}\left(b-\dfrac{c^2}{c-1}\right)^2$.

24 $\dfrac{9a}{16}$ 25 $\frac{25}{16}$ 26 $2\sqrt{ab-b^2}$

27 $\dfrac{ac^{\frac{n}{n+1}}}{a^{\frac{n}{n+1}}-c^{\frac{n}{n+1}}}$ 28 $\left(\dfrac{2a}{1+a}\right)^2$. 29 $2a$ 30 49

31 $\dfrac{8a^3+15a^2b+6ab^2-b^3}{27b}$. 32 $\left\{a^2+\left(\dfrac{2a}{3c}-\dfrac{c^2}{3}\right)^3\right\}^{\frac{1}{2}}$. 33 37

CXLI [p 330]

1 $-\frac{1}{3}$ 2 $2a-1$ 3 $m-3n$ 4 8 5 $\frac{3}{3}m$ 6 7 7 3.

8. $\dfrac{a}{2}-2$. 9 2 10 $\sqrt{(\frac{3}{2}a)}$ 11 $\left(\dfrac{m}{2}\right)^{\frac{1}{m-2}}$

CXLIII [p 332]

1 Second 2 Third 3 Second 4 First if x, and second if x and y, are variables

CXLIII [p 332]

1 $4x+3=0$ 2 $(a-c)x+(b-d)=0$ 3 $\frac{1}{2}x+9=0$.

4 $\left(1-\dfrac{a}{c}\right)x+\dfrac{b^2}{c}=0$ 5 $\left(\dfrac{b}{c}-\dfrac{d}{e}\right)x+a=0$ 6 $\dfrac{9}{20}x-\dfrac{9}{10}=0$.

CXLIV [pp 334—337]

N B—*The values of x and y are given in order*

1 $\frac{1}{2}(a+b)$, $\frac{1}{2}(a-b)$ 2 $\frac{ac+b^2}{a^2+b}$, $\frac{ab-c}{a^2+b}$ 3 $1, 0$

4 $\frac{nc+bd}{na+mb}$, $\frac{mc-ad}{na+mb}$ 5 $\frac{bc'-b'c}{ab'-a'b}$, $\frac{ca'-c'a}{ab'-a'b}$

6 $\frac{(a+b)m+(a-b)n}{2(a^2+b^2)}$, $\frac{(a-b)m-(a+b)n}{2(a^2+b^2)}$ 7 $\frac{b^2}{2a}$, $\frac{2a^2+b^2}{2a}$

8 $\frac{a+b-c-d}{bc-ad}$, $\frac{a-b-c+d}{bc-ad}$ 9. $\frac{a^2bc}{a^2+b^2}$, $\frac{ab^2c}{a^2+b^2}$

10 $\frac{b}{a(b-a)}$, $\frac{a}{b(a-b)}$ 11 a, b 12 $x=y=\frac{ab}{a+b}$.

13 $3a, -2b$ 14 $\frac{a}{b}\frac{a^2+ab+b^2}{a+b}$, $-\frac{a^2}{a+b}$ 15 a, b

16 $\frac{ac(bm+dn)}{ad+bc}$, $\frac{bd(cn-am)}{ad+bc}$.

17 $5(m+n)$, $3(m-n)$ 18 $\frac{abc(ab+ac-bc)}{a^2b^2+a^2c^2-b^2c^2}$, $\frac{abc(ac-ab-bc)}{a^2b^2+a^2c^2-b^2c^2}$

19 $\frac{12mab}{a+b}$, $\frac{m(a-b)(7b-5a)}{a+b}$. 20 $\frac{b+c}{2a}$, $\frac{a+c}{2b}$ 21 $b+c, a+c$

22 $\frac{a}{a+b}$, $\frac{b}{a-b}$ 23. $\frac{2ab}{a+b}$, $\frac{2ab}{b-a}$ 24 $\frac{1-ab}{n-bm}$, $\frac{1-ab}{m-an}$

25 $\frac{bc-ad}{nb-md}$, $\frac{bc-ad}{mc-na}$ 26. $\frac{1}{a}$, b.

27 $\frac{(a^2-b^2)c+(c^2-d^2)b}{ac-bd}$, $\frac{(a^2-b^2)d+(c^2-d^2)a}{bd-ac}$.

28 $\frac{mp-nq}{pa}$, $\frac{mp-nq}{qa}$ 29 $\frac{(a^2+b^2)c}{a^2-b^2}$, $\frac{(a^2+b^2)c}{2ab}$.

30 $\frac{a^2-b^2-c^2}{n(a+c)-mb}$, $\frac{a^2-b^2-c^2}{m(a-c)-nb}$ 31. $a+b, a-b$

32 $a\,\frac{abc}{a^2+b^2}$, $b\,\frac{abc}{a^2+b^2}$ 33 $2b-a, 2a-b$ 34 $a+b, a-b$.

35. $\frac{a^2-b^2}{a^2+b^2}$, $\frac{2ab}{a^2+b^2}$ 36 m^2, n^2 37 $\frac{a^2+ab+b^2}{a+b}$, $\frac{a^2-ab+b^2}{a-b}$.

CXLV [p 338]

N B—*The values of* x, y *and* z *are given in order*

1 $\frac{1}{2}(3a-b-c)$, $\frac{1}{2}(3b-c-a)$, $\frac{1}{2}(3c-a-b)$.

2 $\frac{a}{6}$, $\frac{5a}{12}$, $\frac{2a}{3}$ 3 $\frac{5(a+2b)}{6}$, $\frac{11a-14b}{6}$, $\frac{a+2b}{6}$.

4. $\frac{1}{3}(2b+2c-a)$, $\frac{1}{3}(2c+2a-b)$, $\frac{1}{3}(2a+2b-c)$

5 $\frac{1}{2}(b+c)$, $\frac{1}{2}(c+a)$, $\frac{1}{2}(a+b)$ 6. $\frac{b^2+c^2-a^2}{2bc}$, $\frac{c^2+a^2-b^2}{2ca}$, $\frac{a^2+b^2-c^2}{2ab}$.

7 $\frac{2a}{m+p-n}$, $\frac{2b}{m+n-p}$, $\frac{2c}{n+p-m}$ 8. $\frac{a-b}{c}$, $\frac{b-c}{a}$, $\frac{c-a}{b}$

9 $\frac{1}{x}=\frac{1}{2}\left(\frac{m}{b}+\frac{n}{c}-\frac{l}{a}\right)$; $\frac{1}{y}=\frac{1}{2}\left(\frac{n}{c}+\frac{l}{a}-\frac{m}{b}\right)$, $\frac{1}{z}=\frac{1}{2}\left(\frac{l}{a}+\frac{m}{b}-\frac{n}{c}\right)$.

10 $\frac{2}{x}=\frac{1}{b}+\frac{1}{c}$, $\frac{2}{y}=\frac{1}{c}+\frac{1}{a}$, $\frac{2}{z}=\frac{1}{a}+\frac{1}{b}$

CXLVI [p 342]

N B—*The values of* x, y *and* z *are given in order*

1 $4, -20, 22$ 2 $4, 5, 2$ 3 $\frac{11}{25}, -\frac{32}{15}; -\frac{3}{4}$

4. $\frac{1}{(a-b)(a-c)}$, $\frac{1}{(b-a)(b-c)}$, $\frac{1}{(c-a)(c-b)}$.

5 $bc(b-c)$, $ca(c-a)$, $ab(a-b)$ 6 $(b^2-c^2)(c+a)(a+b)$,
$(b+c)(c^2-a^2)(a+b)$, $(b+c)(c+a)(a^2-b^2)$ 7. $b-c$, $c-a$, $a-b$.

8 $b+c-a$, $c+a-b$, $a+b-c$

9 $\lambda(a^2-bc)$, $\lambda(b^2-ca)$, $\lambda(c^2-ab)$, where $\lambda=\frac{2}{a+b+c}$

10 a, b, c 11. Each $=a^2+b^2+c^2-bc-ca-ab$ 12 $-a, -b, -c$.

13 $\lambda a(b^2-c^2)$, $\lambda b(c^2-a^2)$, $\lambda c(a^2-b^2)$,

$$\text{where } \lambda=\left\{\frac{a}{b-c}+\frac{b}{c-a}+\frac{c}{a-b}\right\}-2(bc+ca+ab)$$

CXLVII. [pp 345—346]

1 $x=15, y=3$ 2 $x=21, y=20$ 3 $x=5, y=6$.

4 $x=4, y=3$. 5. $x=3, y=-\frac{5}{2}$. 6 $x=2, y=4$

7 $x=y=2$ 8 $x=7, y=\frac{6}{7}$.

9 $x=1, y=3, z=4, u=2$ 10. $u=1\frac{1}{2}, x=1, y=4, z=\frac{1}{2}$.

11. $x=4, y=9, z=16, u=25$ 12. $x=5, y=2, z=3, u=4, v=1$.
13 $x=5, y=4, z=3, u=2, v=1$ 14 $x=abc, y=ab+bc+ca,$
$z=a+b+c$ 15 $x=0, y=z=\frac{1}{2}$ 16 $(b+c-a)x=(c+a-b)y$
$=(a+b-c)z=k$, where $k=(a+b+c)^2$ 17. $x=b-c, y=c-a,$
$z=a-b$ 18 $u=abcd, x=-(abc+abd+acd+bcd),$
$y=ab+ac+ad+bc+bd+cd, z=-(a+b+c+d)$.
19 $x=\frac{b+c-a}{b+c+a}, y=\frac{c+a-b}{c+a+b}, z=\frac{a+b-c}{a+b+c}$ 20 $x=y=1$
21 $x=3, y=1$ 22 $x=\frac{9}{4}, y=\frac{27}{8}$ 23 $x=y=3$ 24 $x=2, y=3$

CXLVIII [pp 348—373]

1. 52 2 86 or 68 3 73 4 57 5 46.
6 84 7 305 8 15 days 9 $1\frac{7}{8}$ hr 10 $7\frac{1}{5}$.
11. $17\frac{1}{7}$ hrs. 12 25 days 13 300 cub ft 14 In 30 hrs
15 In 25 and 100 hrs 16 *A* in 32 hrs ; *B* in $53\frac{1}{3}$ hrs
17 24 days 18 *B* in 15 min , *C* in 12 min 19 28 miles
20 14 miles 21 5 miles 22 60 miles , 6 miles
23 $3\frac{3}{4}$ miles per hour 24. 4 miles per hour 25 6 hrs 25 miles
26 $15\frac{1}{3}$ miles ; 4 hrs 8 m , and 3 hrs $52\frac{1}{2}$ m 27 $1\frac{1}{4}$ miles ;
$4\frac{1}{2}$ miles 28 5 miles 29 24 miles per hr 30 $3\frac{1}{2}$ min.
31 15 miles 32. 77 miles 33 5 miles per hr , 45 miles
34 $9\frac{1}{7}$ miles from Ely 35 6 miles per hr , 32 miles
36 Constable's speed at first $8\frac{1}{4}$ miles per hr , thief's $9\frac{3}{4}$ miles ,
$71\frac{1}{2}$ miles
37. 30 miles per hr ; $26\frac{4}{5}$ miles 38 $131\frac{1}{4}$ miles
39 60 miles 40 12 min past 4 41 30 hrs after they
first started ; 20 miles from the place whence the quicker
walker started
42 10 miles per hr 43 3 miles per hr
44 1 mile per hr 45 30 min.
46 91 miles , $4\frac{2}{3}$ hr ; $8\frac{3}{4}$ hr 47 2 miles per hr
48 $2\frac{2}{3}$ miles per hr , 70 miles 49 4 miles per hr
50 (1) $5\frac{5}{11}$ min past 1 , (2) $21\frac{9}{11}$ or $54\frac{6}{11}$ min past 1 51 (1) $33\frac{8}{11}$
min past 6 , (2) exactly at 6 , (3) $16\frac{4}{11}$ min or $49\frac{1}{11}$ min
past 6
52 (1) $16\frac{4}{11}$ min., (2) $49\frac{1}{11}$ min , (3) $32\frac{8}{11}$ min past 3

53 $54\frac{6}{11}$ min past 10, at 12 54 (1) 12 min past 2, (2) $3\frac{3}{11}$ min past 2
55 $17\frac{5}{11}$ min or 48 min past 6 56 36 min past 3
57 24 min past 8 58 $1\frac{1}{59}$ min past 12
59 $32\frac{4}{13}$ min past 5 60 $5\frac{5}{11}$ min after 2
61. 15 lbs 62 28 sr, 7 sr 63 1 pint
64 $\frac{8}{9}$ sr, $\frac{1}{9}$ sr 65 50 gals 66 100 gals 67 10 sr.
68 40 gals, 49 gals 69 10 gals from 1st, 5 gals from 2nd 70 9 gals, 6 gals 71 36 72 640
73 4550 74 1064 75 1008 76 1184 77 864, 576
78 530 79. 180000 80 6 cr, 4 half-cr. 81. 4*s* $4\frac{1}{2}d$., 4*s* $10\frac{1}{2}d$
82 4096 83 22. 84 183 85 1400
86 Rs 19200 87 Rs 5 88 £1480 89 *A*, £24, *B*, £36.
90 Rs 7 as 2 91 5, 10, 15 92 500 93 5000
94 Rs 46 8*a* 95 70, 35 and 14 respectively 96 180
97 $x=2$, £5 5*s* 98 1819 99 4 sov, 59*s*, 55 six-pences
100. 48 101. 2340 102 36 and 64 tolahs 103 40 and 24 days; Rs 6 4*a*, Rs 13 12*a* 104 6*d* 105. Rs. $3\frac{1}{2}$ per maund
106. 20 mds, 30 bighas 107 4500 108 Up to 14 lbs, 3*d*, every additional 7 lbs, 2*d*. 109 17 and 15 chataks
110 $3\frac{3}{4}$ cubits. 111 $c\left(1+\frac{ab}{a+b}\right)$. 112 $\frac{a(c-b)}{a-b}$, $\frac{b(a-c)}{a-b}$
113 $\frac{nb+m}{a+b}$ and $\frac{na-m}{a+b}$ days 114 $\frac{abc}{b+c}$ miles
115 $\frac{2c(a+b)}{a-b}$ miles, or $\frac{2c(a+b)}{b-a}$ miles. 116 $\frac{(100)^2c}{(b+d)(100+a)}$ rupees

CXLIX [*pp* 377—381]

1. *A*, Rs 55, *B*, Rs 20 2 *A*, 48, *B*, 28. 3. 31, 7.
4 £288, 5 per cent 5 Rs 22, Rs 24. 6 1*s* 4*d*, 2*s* 10*d*.
7 *AB*, 35 miles, *BC*, 29 miles, *AC*, 45 miles
8 315, 289 and 304 rupees. 9 12, 5*s* 10 30*d*, 15 11 63.
12 42 13 432 14 416. 15 3 16 39*s*., 21*s*, 12*s*.
17. $14\frac{34}{49}$, $17\frac{23}{41}$, $23\frac{7}{31}$ days 18. 90, 72, 60 min.
19. Each equal cock in 32 hrs, the other in 24 hrs.

20. 3 miles per hour 21 15 and 2 miles respectively.
22 In 10 min 23 At 10. 4 A M. 24. 12 and 4 miles per hour.
25. 1080 yds, $16\frac{1}{2}$ min 26 24 miles per hr, 96 miles.
27. 4 and 5 yds 28 100 lb, 2 cwt 3 cwt
29. 8 and $7\frac{1}{3}$ yds per sec 30 A in 5 min, B in 5 min 20 sec.
31 30 mi, 6 mi per hr
32 $AB=31\frac{1}{2}$ miles, $BC=63$ miles, 21 and 42 miles per hr
33 35. 34 72 weeks 35 $\dfrac{d(a-b)}{b-c}$ miles, $\dfrac{d(a-b)(a-c)}{2(b-c)}$ miles
36. Rate of faster train$=\dfrac{(a+b)(m+n)}{2mn}$ ft., and of slower train $=\dfrac{(a+b)(n-m)}{2mn}$ ft. per sec; 44 ft. and 36 ft per sec 37 Time of going$=\dfrac{ct}{b+c}$ hrs, time of returning$=\dfrac{bt}{b+c}$ hrs, stream's vel. $=\dfrac{a(b^2-c^2)}{2bct}$ miles per hr; 3 hrs, 13 hrs, 5 miles per hr

CL. [p 384]

1. 16 . 5 2 5 : 8 3 10 . 3 4 Impossible.
5 5 . 6 is greatest, 2 3 least 6. 3 2. 7 3 4.
8. $\dfrac{\sqrt{a}+1}{\sqrt{a}-1}$. 9. 13 14 10. $3n-m$. $2m+n$. 11 1 8
12. 3 2. 13 4 . 3 or -3 4. 14. $\frac{13}{11}$ 15 11.
16. 3 17. 1 2. 18 $\dfrac{c(b-a)m}{bm-an}$, $\dfrac{c(b-a)n}{bm-an}$.
19. 60, 75. 20 20, 28 21 18, 24.

CLI. [p 385]

1. 1 . 5 2 25 6 3. $3x : 4y$ 4 $1-y : x$.
5 16 25, 1 . 8; $5\sqrt{10}$. 19, 13 . 14. 6. $x=17\frac{1}{2}$. 7 $x=26$.

CLII [pp. 390—391]

1. $4\frac{2}{3}$ 2. $7\frac{1}{2}$ 3 (i) 15, (ii) 1; (iii) $3b^2$, (iv) $\dfrac{-a(x-1)^2}{x}$
4. 5 5. 3. 6. 5 2 7 $b(m+1)$ $a(m-1)$ 8 2 · 5.
9. 2 1, 1 . 11. 10 4 5, 77 · -13. 12 $x=4$ 13. $x=3$.

CLV [pp 397—398]

1. $\frac{4}{9}$. 2. $\frac{35}{34}$ 3. $\frac{5}{4}$. 4. $b-\frac{a}{c^3}\left(\frac{c-1}{c+1}\right)^2$. 5. $a\frac{25b^2-1}{25b^2+1}$.

6. $\frac{3ab}{1+b^2}$ 7. $\frac{1}{2}(b-a)$. 8 $\left(\frac{2a}{1+a}\right)^2$ 9. $\left\{\frac{2ac}{b(c^2+1)}\right\}^{\frac{1}{2}}$.

10 $\sqrt{ab}$ 11. $\frac{2}{\sqrt{4ab-b^2}}$ 12 $\frac{1}{a}\sqrt{\frac{2a}{b}-1}$. 13. 1 or 9

14 $a\left\{1-2\sqrt{\frac{b}{c}}\right\}$ 15. $\frac{a(1-\sqrt{b})^2}{1+b}$ 16 $\frac{a}{\sqrt{2b-b^2}}-a$.

17. $2a\sqrt{1-a^2}$ 18. 8 19 $\frac{a}{(\sqrt{b}-1)^2}$ 20 $\sqrt{1+4(a-1)^2}$.

21. $\sqrt[4]{\frac{1}{2}+\frac{1}{2}\sqrt{2}}$. 22 $\sqrt{\frac{a^2-1}{a^2+3}}$ 23 $\sqrt{\left(\frac{3a+1}{2}\right)^4-1}$.

24. $1-\sqrt{1-a^2}$ 25 $\frac{1}{2}\left(a+\frac{1}{a}\right)-1$ 26 a or $\frac{1}{a}$.

CLVI. [p. 399]

1. 17. 2 2 3 $\frac{b^2-c^2}{4a-b}$. 4. $\frac{ad-bc}{(a-c)n-(b-d)m}$. 5. $a-2b$.

6. $\frac{c^2(a-b)^2}{\{(b+c)\sqrt{b}+(a+c)\sqrt{a}\}^2}$ 7. $\frac{b-q}{a-p}$ 8 $\frac{b}{a}$.

9 $\frac{ab(c+d)-cd(a+b)}{cd-ab}$ 10 $\frac{ab(c+d)-cd(a+b)}{ab-cd}$.

CLVII. [pp 402—403]

1. 24; x^2y^2 2 $87\frac{1}{2}$ 3 $1\frac{1}{3}$ 4 3, −3 5. 6, 12 15. 1

CLVIII [pp 406—410]

8. 0 9. 0. 18 Put each of the given ratios $=l$, then each of the latter $=l^2$. 23 $\frac{a^2+bc}{ac}$ 26 $2ax=(a+b+c)(a^2+2b^2+c^2)$, $2by=(a+b+c)(a^2+b^2+2c^2)$, $2cz=(a+b+c)(2a^2+b^2+c^2)$.

27. $\frac{1}{a+b+c}=\frac{\frac{2a}{x}}{b+c-a}=\frac{\frac{2b}{y}}{c+a-b}=\frac{\frac{2c}{z}}{a+b-c}$.

28 $x=\frac{1}{2}k(b+c-a)$, where $k^2=1\div(a^2+b^2+c^2)$, &c.
32 $2 . 3 . 4$ 33 $2c$ 35 $, 12, 3$ 36 16 37 9 years.
38 $3 . 13$ 39 Length 30 and breadth 25 yds 40 300
41. 11 24 42 22 miles 43 $6\frac{2}{3}$ inches 44 $6 . 7$.
45. 6 inches. 46. Val of gold : val of silver$=20n^2q : m^2p$

CLIX. [pp 410—430]

3 $a=b=c$. 5 $x=a, y=b$ 6 $\frac{x}{a}=\frac{y}{b}=\frac{1}{a^2+b^2}$ 7 $3a^3$. 8 3.
9. $\frac{y+2}{y}$ 16 16 17 -3 18. $\frac{-d+q\{b-q-p(a-p)\}}{c-q(a-q)-p(b-q)+p^2(a-p)}$.
19 $p^2-4q=0$ 20 $a=0, b=3, c=2, d=0, e=5$
21. $p=2, q=6, r=2, s=-21$ 23 $a=2, b=0, c=-7, d=-2$
24 $l=-6, m=12, n=-8$ 25 $2x+3, -15$ 26 $p=2a, q=a^2$.
27. $p=-10, q=8$ 28. $a=-25, x^2-3x-4$.
29 $l=0, m=-1, n=-12$. 30 $a=-29, b=-3$ $c=27$
31 $a=14, b=20$
39. $(x-1)^2(x^2+5x+2)$ 40 $(x+1)(12x^2+x+3)$
41. $(x-1)^2(4x+3)(5x-1)$ 42 $(x^2-1)(x+1)(x+12)(3x-5)$
43 $(x+1)^3(3x^2-9x+8)$ 44 $(x^2-1)^2(x-5)(5x-2)$
45. $(a-b)(a+2b)(3a+b)$ 46 $(x+y)(2x+5y)(4x+3y)$
47. $(a^2-x^2)(a-2x)(3a+4x)$. 48 $(x+a)^4(x-3a)$
49 $(x-3)^2(2x^2+3x-4)$ 50. $(3x-y)(x-y)(2x^2+3xy+3y^2)$
51 $(a-x)(a-2x)(a-3x)(a-4x)$ 52 $(x-4)^2(x^2+1)$.
53 $n(nc+bd)+a(mc-ad)=m(mb+na)$ 54 $ab(c+d-e-f)$
$+cd(e+f-a-b)+ef(a+b-c-d)=0$ 55 $a^3+b^3+c^3-3abc=0$
56 $x^2+y^2+z^2+2xyz=1$ 59 $a^2+ab+b^2=3$ 60 $a^2=b^2-2c^2$.
61. $(p-q)^3=(p+q)^2(p^2+pq+q^2)$ 62 $a^3+b^3+c^3+abc=0$
63 $m^{\frac{2}{3}}-n^{\frac{2}{3}}=4$ 64 $(ab'-a'b)^2=(bc'-b'c)^2+(ca'-c'a)^2$
65. $(a+b)c^2=(ab)^{\frac{1}{4}}\{(ab)^{\frac{1}{2}}+2(1-c^2)(ab)^{\frac{1}{4}}+1\}$,

CLX. [pp 434—436]

1. 4. 2 $2\sqrt{6}$ 3 12, 16, 24 5. x^2-2x+1 8. $a=3, b=4$,
$c=12$ 10. $5 ; 5x+4$.

CLXI [*pp* 439—440]

1 ± 4 2. ± 3. 3 ± 5 4 ± 2 5 $\pm \frac{\sqrt{85}}{2}$.

6 ± 4 7 $0, 4$ 8 $\pm \frac{2}{5}\sqrt{10}$ 9. $\pm \sqrt{6}$ 10 $0, a+b$

11. $\pm \frac{7}{2}\sqrt{2}$ 12. $0, \frac{2ab}{a+b}$ 13 $0, 3$ 14 $\pm 2\sqrt{-1}$

15 ± 6. 16 $0, \pm 9$ 17 $\pm \sqrt{3}$ 18. $0, 3$ 19 $5, 5$.

20 $\pm \frac{1}{2}$ 21 $\pm \frac{24a^2}{25b}$ 22 $0, \pm \sqrt{3}$ 23 $\pm \frac{\sqrt{3}}{2}$

24 $0, \pm \frac{2}{a}\sqrt{\left(1-\frac{1}{a^2}\right)}$ 25 $\pm \sqrt{\left(\frac{a-2}{a+4}\right)}$ 26 ± 5.

CLXII [*pp* 442—447]

1. $-3, -8$ 2 $11, 39$ 3 $5, -16$ 4 $-25, -80$

5 $6, -\frac{27}{2}$. 6 $\frac{9}{2}$ $\frac{4}{3}$ 7. $4, -\frac{22}{3}$ 8 $\frac{5}{2}, -\frac{2}{3}$

9 $\frac{5}{2}, \frac{3}{5}$ 10 $2, \frac{7}{24}$ 11 $13, -\frac{13}{3}$, 12 $\frac{4}{3}, -\frac{5}{14}$.

13 $12, -\frac{105}{8}$. 14. $\frac{4}{3}, \frac{5}{4}$ 15 $2, -3$ 16. $\frac{1}{4}, -2$

17 $\frac{p}{4}, \frac{3p}{4}$ 18 $\frac{a}{3}, -\frac{3a}{4}$ 19 $\frac{4a}{3}, -\frac{5a}{2}$. 20 $\frac{5a}{3}, -\frac{7a}{4}$

21 $6, -13$ 2 $\frac{1}{7}, -1$. 23 $3, \frac{1}{2}$ 24 $2\frac{2}{5}, 2\frac{2}{3}$

25 $\frac{1}{2}(11 \pm \sqrt{233})$ 26 $6, -1$ 27. $2, 6$ 28 $3, 8\frac{1}{3}$

29 $2, -\frac{1}{2}$ 30 $2 \pm \sqrt{3}$ 31 $a \pm b$ 32 $\frac{-b \pm \sqrt{b^2-ac}}{a}$

33 $\left(\frac{a+b}{a-b}\right)^{\pm 1}$. 34. $-b, b-2a$ 35 $7, 16$ 36 $1\frac{1}{3}, -1\frac{1}{2}$

37. $a, m+n$ 38 $-a, -b$ 39 $5, -2$ 40 $5, 13$

41 $5a, -2a$ 42 $\frac{1}{2}(11 \pm \sqrt{33})$ 43 $4, 1\frac{2}{5}$ 44 $a-c, c-b$

45 $3, -\frac{1}{2}$ 46 $3, 1\frac{10}{11}$ 47 $3, -1\frac{11}{13}$ 48 $5, -\frac{1}{3}$.

49 $12, 11$ 50 $6, -\frac{1}{3}$ 51. $4, 2\frac{2}{3}$ 52 $5, -6$. 53 $4, -2$

54. $24, -6$ 55 $10, -\frac{2}{5}$ 56 $\frac{7}{4}, 1$ 57 $\frac{13}{3}, 1$ 58 $3, 1\frac{1}{5}$

59 $3, -\frac{4}{5}$ 60 $2 \pm \sqrt{6}$ 61 $1, -1\frac{2}{5}$ 62 $4, -2\frac{2}{3}$ 63 $6, 3\frac{4}{7}$.

64 $6 \pm \sqrt{601}$. 65 $3, 1\frac{1}{5}$ 66 $1, -2\frac{4}{7}$ 67 $\frac{1}{9}(5 \pm \sqrt{61})$

68 $4, \frac{10}{11}$ 69 $7, 3\frac{14}{15}$ 70. $4, \frac{12}{17}$ 71 $0, 4$. 72 $0, \frac{5}{3}$

73 $1\frac{7}{25}, -1$. 74 $a+b, 2b$. 75 $1, \frac{2b}{a-b}$. 76 $\frac{b^2}{ac}$

77. $3a, \frac{3a}{2}$. 76. a, b. 79. $a+b, \frac{1}{2}(a+b)$ 80 $a, \frac{1}{a}$.

81. $1, -\frac{a+b+c}{b+c}$ 82 $1, \frac{a-b}{b-c}$ 83 $1, \frac{c(a-b)}{a(b-c)}$ 84 $\frac{b}{a+b}, -\frac{a}{a+b}$.

85 $-1 \pm \frac{\sqrt{a^2-1}}{a}$ 86. $\frac{1}{2}(-3 \pm \sqrt{3})ab$ 87. $\frac{a}{b}, -\frac{b}{a}$ 88 $-a, -b$.

89 $4, -1$. 90 $5, -\frac{5}{4}$ 91 $4, \frac{3}{2}$ 92 $3, -\frac{1}{8}$ 93 $1, \frac{b+c-2a}{c+a-2b}$.

94 $0, \frac{1}{2}\{-(a+b+c) \pm \sqrt{a^2+b^2+c^2-2bc-2ca-2ab}\}$.

95 $\{bc+ca+ab \pm \sqrt{bc^2+c^2a^2+a^2b^2-2abc(a+b+c)}\} - 2(a+b+c)$.

96 $\frac{1}{3}\{a+b+c \pm \sqrt{a^2+b^2+c^2-bc-ca-ab}\}$

97. $\{bc+ca+ab \pm \sqrt{b^2c^2+c^2a^2+a^2b^2-abc(a+b+c)}\} - (a+b+c)$.

98 $\pm a, \frac{a}{2}, -2a$ 99 $a+b, \frac{ab(a+b)}{a^2+b^2}$.

100 $0, \pm\sqrt{\frac{1}{2}(b^2+bc+ab+2ac)}$ 101 $a, -\frac{b^2+c^2}{b+c}$.

102 $1, -\frac{2ab}{a^2+2ab-b^2}$ 103 $1, 3$. 104 $15, 7$ 105. $13, 193$

106 $5, -\frac{4}{3}$ 107 $9, -3\frac{3}{5}$ 108 $4, -3\frac{1}{5}$ 109 a, b 110 $\frac{7}{5}a, -a$.

CLXIII [p 452]

1 $-\frac{bc}{a^2}$ 2 $\frac{3bc}{a^2} - \frac{b^3}{a^3}$ 3 $\frac{2b^2}{a^2} + \frac{c}{a}$ 4 $\sqrt{\frac{b^2}{a^2} - \frac{4c}{a}}$

5 $x^2-5x+6=0$. 6. $x^2+x-20=0$ 7 $3x^2-8x-3=0$

8 $8x^2+10x+3=0$ 9 $x^2-(p+q)x+pq=0$.

10. $acx^2+(2ac-b^2)x+ac=0$, $a^2x^2-abx+9ac-2b^2=0$.

CLXIV. [p 454]

1 $\pm 1, \pm\sqrt{-5}$ 2 $\pm\sqrt{2}, \pm\frac{1}{3}\sqrt{3}$ 3. $2, \sqrt[3]{-12}$

4. $\pm\sqrt{-1}, \pm\frac{1}{3}\sqrt{5}$ 5 $2, 3$ 6 $\pm 2, \pm\frac{1}{2}$ 7 $25, 1$

8 $\frac{1}{4}, 4$ 9 $9, \frac{1}{4}$ 10 $\frac{81}{16}, 1$ 11 $3^6, (-\frac{1}{7})^6$ 12 $64, 729$.

13 $8, \frac{8}{9}$ 14 $27, \frac{3}{16}$ 15 $7, -1, 3 \pm 2\sqrt{2}$

16 $2, -\frac{1}{3}, \frac{1}{4}(3 \pm \sqrt{505})$ 17. $4 \pm \sqrt{34 \pm 18\sqrt{2}}$

18 $0, a, \frac{1}{2}(a \pm \sqrt{a^2-16a+16})$ 19 $1, 7$ 20. $4, -5, \frac{1}{2}(-1 \pm \sqrt{21})$.

21. $2, \frac{1}{3}, \frac{1}{6}(7 \pm \sqrt{-23})$ 22. $4, 2, \frac{1}{2}(6 \pm \sqrt{17})$

23 $3, -\frac{1}{2}, \frac{1}{4}(5 \pm \sqrt{1329})$ 24. $1, -4, \frac{1}{2}(-3 \pm \sqrt{109})$.

25 $12, -8, 2 \pm \sqrt{85}$ 26. $1, 2, \frac{1}{2}(3 \pm \sqrt{-13})$.

27. $3, -2, \frac{1}{2}(1 \pm \sqrt{-19})$ 28. $5, -2, \frac{1}{2}(3 \pm \sqrt{-39})$.

29 $0, -5a, \frac{a}{2}(-5 \pm \sqrt{-15})$ 30 $1, -2$. 31. $\frac{1}{2}(-5 \pm \sqrt{19})$.

32. $3, \frac{1}{3}, \frac{1}{3}(-2 \pm \sqrt{-5})$ 33 $2, -\frac{1}{2}$.

CLXV. [p 456]

1. $-1, \pm\sqrt{-1}$ 2 $-1, \frac{-1 \pm \sqrt{-15}}{4}$. 3 $1, \frac{-8 \pm \sqrt{55}}{3}$

4. $1, \frac{b-a \pm \sqrt{b^2-2ab-3a^2}}{2a}$. 5 $x+\frac{1}{x}=2$ or -3

6. $x+\frac{1}{x}=-1 \pm \sqrt{2}$ 7 $x+\frac{1}{x}=0$ or $-\frac{3}{2}$.

8 $x+\frac{1}{x}=\frac{-b \pm \sqrt{b^2-4ac+8a^2}}{2a}$.

CLXVI. [p 457]

1 $6, -3 \pm \sqrt{-2}$ 2 $5, \frac{1}{2}(1 \pm \sqrt{-23})$. 3 $a, \frac{1}{2}\{-(a+3)$ $\pm\sqrt{1-6a-3a^2}\}$. 4 $-1, \frac{1}{4}(3 \pm \sqrt{41})$ 5 $4\frac{1}{2}, 10, -1$.

6 $4a, \frac{a}{14}(-9 \pm \sqrt{-3})$ 7. $a, b, \frac{1}{2}(a+b)$ 8 $1, \frac{1}{2}(-1 \pm \sqrt{-33})$.

9 $0, \frac{1}{4}, -2$, 10. By Art 292, $x+1$ is factor of $4x^3+6x^2+x-1$, thus $(x+1)(4x^2+2x-1)=0$, &c., $-1, \frac{1}{4}(-1 \pm \sqrt{5})$

11. $\pm a, \pm\frac{1}{a}$ 12 $3, \frac{1}{3}, \frac{1}{2}(1 \pm \sqrt{-3})$. 13 $a, \frac{b}{2a}(a+b \pm \sqrt{5a^2+2ab+b^2})$

14. $0, a+b, \frac{a^2+b^2}{a+b}$. 15 $x^2+x+2=\pm 4x$, $1, 2, \frac{1}{2}(-5 \pm \sqrt{17})$

16. Divide by x^2, thus $x^2+ax+b+\frac{c}{x}+\frac{c^2}{a^2x^2}=0$,

or $\left(x^2+\frac{c^2}{a^2x^2}\right)+a\left(x+\frac{c}{ax}\right)+b=0$

CLXVII [pp. 460—468]

1. 48, 36 2 15, 21 3 8 or $1\frac{1}{2}$. 4 20, 30 5 8, 12
6. 121. 7 17, 32 8 Rs. 20 or Rs 80 9 £60 10 3 or 12
11. 4 12. 12 13 38 or 83 14 76 or 43 15 62 16 42. or 51 17 1, 6 18 $\frac{3}{4}, \frac{1}{4}$ 19 $\frac{5}{8}$ 20 $\frac{5}{3}$ 21 65 and 35 yrs
22 48 and 24 yrs 23 15 24 18, Rs. 4. 25 12 mi per hr.
26. 20 27. 32 bighas, R 1. 8a 28. 25, £1 10s

29. 11 yd., 14 yd 30 18 31. Sheep Rs. 4, lamb R. 1. 8*a*
32. 72. 33 14. 34 $5\frac{1}{2}$ hr 35 24 yd., 18 yd.
36 15 yd., 12 yd., 20 yd., 9 yd. 37. 4 hr., 12 hr.
38 12 hr, 15 hr. 39 784 40 6 and 9 mi. per hr.
41. 4 mi. per hr. 42. 6 ft., 8 ft. 43 $1\frac{3}{4}$ mi. per hr.
44. 5 mi. per hr. 45 5 mi per hr 46 150 and 120 mi.
47. 8*d.* 48 Rs. 2. 8*a* 49 6*a* 50 £2. 5*s* 51. 16
52. 20 ft., 15 ft. 53 15 ft, 8 ft. 54. $17\frac{1}{2}$ ft., $13\frac{1}{2}$ ft.
55 Sherry £2, Claret £3 56 4 and $3\frac{3}{4}$ mi. per hr.
57. *A*, 6 hr, *B*, 3 hr. 58 180 mi. 59. 64

CLXVIII [*p.* 469]

1. 8 58 2 9 7 3 Solution impossible. 4. Sol. imp
5. 3 45 or -1 45 6 Sol. imp 7. Sol. imp 8 Sol. imp
9 Sol. imp 10 1·33 in.

CLXIX. [*p* 471]

N. B.—The corresponding values of x and y are put together.

1. $4, 3; 3, 4.$ 2 $5, 3; -2\frac{3}{5}, -4\frac{3}{5}$ 3 $8, 3; -6, -4$
4. $2, 1; -1\frac{2}{13}, 2\frac{9}{13}$ 5 $-1, 2, -5, -2$ 6 $8, -3; -3\frac{3}{4}, -6\frac{11}{12}.$
7. $2, 1, -\frac{2}{3}, -3$ 8. $2, 3, -\frac{5}{2}, \frac{27}{8}$ 9 $-3, 2; -\frac{1}{2}, -\frac{1}{2}$
10 $1, 4; 1\frac{3}{16}, 4\frac{5}{16}$ 11. $5, -2; 4\frac{1}{4}, -2\frac{1}{2}$ 12 $6, 2, 8, 3$

CLXX. [*pp* 472—473]

N. B.—The corresponding values of x and y are put together

1. $\pm 1, \pm 3; \pm 4, \pm 2$ 2. $\pm 3, \pm 1, \pm 2\sqrt{2}, \pm\sqrt{2}.$
3 $\pm 1, \pm 2, \pm\frac{5}{\sqrt{2}}, \pm\frac{8\sqrt{2}}{3}$ 4. $\pm 1, \pm 2; \pm\frac{3}{\sqrt{2}}, \mp\frac{1}{\sqrt{2}}.$
5 $\pm 6, \pm 5, \pm\sqrt{2}, \mp 2\sqrt{2}$ 6 $\pm 3, \pm 2; \pm\sqrt{2}, \mp 4\sqrt{2}$
7. $0, \pm 2, \pm 3, \pm 1.$ 8 $\pm 2, \pm 3; \pm\frac{16}{9}\sqrt{3}, \mp\frac{13}{9}\sqrt{3}$
9. $\pm 2, \mp 3; \pm\frac{1}{\sqrt{3}}, \pm\frac{4}{\sqrt{3}}$ 10 $\pm 3, \pm 2; \pm 1, \pm 2$

CLXXI [*p* 475]

N. B.—The corresponding values of x and y are put together.

1. $10, 7; 7, 10$ 2 $5, 5; 3, 5$ 3 $18, 5; -5, -18$
4. $16, 12, -12, -16$ 5 $6, 2; -\frac{4}{5}, -15.$ 6 $4, 1; -1\frac{2}{7}, -2\frac{4}{5}.$

7 $\frac{3}{8}, \frac{2}{9}, \frac{1}{4}, \frac{1}{3}$ 8 $\frac{3}{2}, 5; -\frac{15}{16}, -8.$ 9 $4, \frac{2}{3}, -\frac{4}{5}, -3\frac{1}{3}.$
10. 6, −3, −3, 6 11 2, −5, 5, −2 12 7, −4, −4, 7.
13 3, 2, −2, −3 14 6, −8, 8, −6 15 9, −6, −6, 9
16 5, −2, −2, 5, 2, −5, −5, 2 17 5, 3, 3, 5, −3, −5, −5, −3.

CLXXII [*pp* 476—477.]

1. $\pm 8, \pm 3, \pm 5.$ 2 $\pm\frac{bc}{a}, \pm\frac{ca}{b}, \pm\frac{ab}{c}$ 3 2, 3, 4, 6

4 $\pm 2, \pm 3, \pm 5, \pm 2\sqrt{-1}, \pm 3\sqrt{-1}, \pm 4\sqrt{-1}$
5 $x^3=64, y^3=8, z^3=27.$ 6 $\pm 5, \pm 4, \pm 6$ 7. $x=y=z=\sqrt{\frac{1}{2}}.$
8. $\pm 2\frac{1}{2}, \pm 1, \pm 2$ 9 $\pm 4, \pm 9, \pm 11$ 10 $\pm 2, \pm 3, \pm 5$
11 $\pm 2, \pm 5, \pm 3$ 12 $\pm 6\sqrt{\frac{2}{5}}, \pm 2\sqrt{\frac{2}{5}}, \pm 2\sqrt{10}$ 13. $\pm 3, \pm 2, \pm 9.$
14 $\pm 4, \pm 2, \pm 3$ 15 $\pm 6\sqrt{\frac{3}{35}}, \pm 2\sqrt{\frac{7}{15}}, \pm 4\sqrt{\frac{5}{21}}$
16 $\pm a\sqrt{\frac{2(b+c)}{(c+a)(a+b)}}, \pm b\sqrt{\frac{2(c+a)}{(a+b)(b+c)}}, \pm c\sqrt{\frac{2(a+b)}{(b+c)(c+a)}}$

CLXXIV [*p* 480]

1. 20 units 2 (i) i) 13, (iii) 17. 3 13 units.
4 10 units 5 49 19 units

CLXXV [*p* 483]

[*Centre and radius are given in order*]

1 (0, 0), 1 2 (0, 0), 3 3 (0, 0), 8. 4. (0, 0), 0
5. (0, 0), $\sqrt{2}$ 6 (0, 0), $\sqrt{24}$. 7 (0, 0), 1 5 8 (0, 0), $\frac{2}{3}\sqrt{5}$.
9 (1, 2), 5 10 (4, −4), 2 11 (−2, −3), 9 12. (3, 4), 5.
13 (2, 3), $\sqrt{13}$ 14. (3, 5), 7. 15 (−4, −6), 6 16 (7, 0) 4.
17 (0, −5), 6 18. (3, 0), 3 19 (4, 0), 4 20 (−1, 0), 5.
21 (4, 0), 3 22 (2, 0), 6 23 3 and 4
25 5 and 11, 13 and 5 26 5 7 and 1 7, 1 7 and 5 7.
27 0 and 4, 0, −4 28 5 and 3, −1·6 and 4 7.

CLXXVII [*p* 492]

1 2, −3 2 −2, 3 3 2, −2 5. 4 1, 4
5 2 56, −1 56 6 1 5, 4 7 Roots imaginary.
8 2 77, −1 27 9 1, −1 25 10 Roots imaginary
11. 1 5, 0 12 4, 0. 13 1, 66 14. Roots imaginary.

CLXXVIII [p 493]

1. 44 ; 64 , 220 2 33 , 0 , −445 3 0 , 552 ; 780 4. 359.
5 −1041 6 −51

CLXXIX. [p. 494]

1. 106 , $7n+1$. 2 68 3 0 4 3 , 5
5 11, 10, 9, , 0 6. $-10, \frac{1}{2}, -\frac{2}{3}, -1\frac{5}{6}, \ldots$
7 77·91 ; 20·09 8 0 , 1 9 $\dfrac{2q}{m-n}$

CLXXX [p 495]

1. 3 2 10 3 49 4. 55. 5. 44 6 9
7. 17. 8 49 9 8 10 2 11 1 12 −9
13 $2, 3, \frac{4}{5}, 2 ; 1, 2$ 15 20th 16 50th 17 6th

CLXXXI. [p. 497]

1 405 2 60 3 620 4 $209\frac{1}{4}$. 5 −333 6 $-1\frac{1}{2}$.
7 $-234\frac{1}{2}$ 8. 0 9 0 10 $24(a+5b)$ 11. −11 12 $3n^2$
13. $-2n(n+9)$ 14 $\frac{5}{12}n(9-n)$ 15. $\frac{1}{2}(n-1)$ 16 345.
17. 530 18 2300 19. $n(n+2)$ 20. $\frac{n}{2}(9n-37)$ 21 $\frac{n}{2}(3n-23)$.
22 $\frac{1}{3}, \frac{5}{6}, {}^{4}, \ldots$

CLXXXII [p. 500]

1 20 2 45 3 $1\frac{3}{5}$, −2 4. 2475. 5 $2\frac{6}{17}$ 6. 7
7 3 or −1 , 10 or 12 8. $3\frac{3}{14}$. 9. 3 , 8 10. −4, −2, 0, 2, . .

CLXXXIII [pp 501—502]

1. (i) $12\frac{1}{2}$, (ii) −5 , (iii) $\frac{5}{12}$, (iv) a , (v) $\frac{3}{2}(a-b)$, (vi) $2xy$
2 6, 11, 16 3 $7\frac{2}{3}, 8\frac{1}{3}, 9, \ldots$ 4 $3\frac{1}{2}, 2, \frac{1}{2}, \ldots$
5 $-\frac{3}{4}, -\frac{1}{2}, -\frac{1}{4}, \ldots, 5\frac{1}{2}$ 6 $4x-5y, 3x-4y, 2x-3y, \ldots$

CLXXXIV [pp. 505—507]

1. 583 2 12 3 , −6, −4, −2, 6 2120 8 79
9. 4, 2, 0, −2, . 10 $p+q+(m-1)2q$, $p+q, p+3q, p+5q,$

11 1, 5, 9, 14 $\frac{ma-nb}{a-b}(2n+1)$ 16 $m+n$ or $m+n-1$, 0 or 1

20 $-2n, 2n+1$ 21. $-n, n+1$ 22 $20(2a-b)$

23 26, 19, 12 or $-12, -19, -26$ 24 5, 7, 9, 11

25. 9, 15, 21, 27, 33 27 £51 28 $6\frac{15}{44}$ miles

CLXXXV [pp 508—509]

1 2×3^9 2. $\frac{81}{256}$. 3 -128 4. $\frac{2^8}{5^9}$ 5 $-\frac{3^{n-6}}{2^{2n-5}}, -\frac{3^{n-1}}{2^{2n-1}}$

CLXXXVI. [p 509]

1. 486 2 2^{10} 3 $\frac{3}{4}, \frac{1}{2}, \frac{1}{3}, \frac{2}{9}$, . 4. $\frac{1}{3}$, 3. 5. $-\frac{1}{128}$.

CLXXXVII. [p 510]

1 $5^4=$&c. 2. $2^5-3=$&c 3 $4^{n-2}-1$ 4. 1, 1, 3, 9, 27, ...

5. 1, 7, -5, 19, ...

CLXXXVIII [p 511]

1 32767. 2 4921 3 $49\frac{493}{1215}$ 4. $-5\frac{261}{2048}$ 5. $-\frac{31}{2}(2-\sqrt{2})$.

6. $\frac{a^{21}-a}{a-1}$ 7 $\frac{1}{3}\{1-2^{2n}\}$ 8 $2\{1-(-\frac{1}{2})^n\}$.

CLXXXIX. [p 513]

1 $\frac{1}{2}$ 2 $5\frac{2}{5}$ 3 $\frac{81}{40}$. 4 -32. 5 $-2\frac{2}{3}$ 6 $2+\sqrt{2}$.

7. $\frac{1}{1-2x}$ 8 $\frac{ax+b}{x^2-1}$. 9. (i) $\frac{14}{37}$ (ii) $\frac{213}{148}$. (iii) $1\frac{59}{1100}$. (iv) $3\frac{5}{13}$

CXC. [p 515]

1 1, $\frac{2}{3}$ 2 6, 9 $13\frac{1}{2}$ 3 $\frac{1}{3}$, 1, 3 4 $\frac{1}{2}, \frac{1}{4}, \ldots, \frac{1}{32}$

5 $-1\frac{1}{2}$, 3, -6, 12, 6 $-7, \frac{7}{2}, \ldots \frac{7}{32}$

CXCI [pp 517—519]

1 $t_n=2^n-1$; $2(2^n-1)-n$ 2 2^n+1, $2(2^n-1)+n$

3 3^n-1, $\frac{3}{2}(3^n-1)-n$ 4 3^n+2, $\frac{3}{2}(3^n-1)+2n$

5 2^n+n, $2(2^n-1)+\frac{1}{2}n(n+1)$ 6 3^n-n, $\frac{3}{2}(3^n-1)-\frac{1}{2}n(n+1)$

7 (i) $n(n+1)+2(2^n-1)$ (ii) $\frac{3}{2}(3^n-1)-n(n+1)$.
(iii) $\frac{1}{2}an(n+1)+\frac{r(r^n-1)}{r-1}$.

8. (i) $\frac{10}{9}(10^n-1)-n$ (ii) $\frac{50}{81}(10^n-1)-\frac{5n}{9}$.

9 $4\left(1-\frac{1}{\sqrt{2}}\right)=2(2-\sqrt{2})$ 14. 3 and 27, or 27 and 3.

15 9 and 12, or 1 and -4 17. 3, 6, 12. 18. 2, 5, 8, or 26, 5, -16.

25. $\frac{1}{3}$ 28 $\frac{ar(r^n-1)}{(r-1)^2}-\frac{an}{r-1}$.

CXCII. [*p*. 525]

1. $\frac{1}{3}n(n+1)(n+2)$ 2 $\frac{1}{4}(n-1)n(n+1)(n+2)$ 3 $\frac{1}{3}n(n^2+6n+11)$
4 $n^2(n+1)$ 5 $\frac{1}{3}n(4n^2+6n-1)$ 6 $\frac{1}{12}n(n+1)(n+2)(3n+13)$
7. $\frac{1}{3}n(4n^2-1)$ 8 $\frac{1}{2}n(6n^2-3n-1)$ 9. $n(n+1)(2n^2+6n+1)-3n$.
10 $12n^3+12n^2+n$ 11. $t_n=\frac{3}{2}n^2+\frac{1}{2}n$, $s=\frac{1}{2}n(n+1)^2$
12 $t_n=3n^2-2n$, $\frac{1}{2}n(n+1)(2n-1)$
13 $t_n=n^2+4n+3$, $s=\frac{1}{6}n(2n^2+15n+31)$
14 $t_n=2^n-1$, $s=2(2^n-1)-n$
15. $t_n=\frac{3^n}{2}-\frac{1}{2}$, $s=\frac{3}{4}(3^n-1)-\frac{n}{2}$ 16. $\frac{1}{12}n(n+1)^2(n+2)$
17. $t_n=\frac{3^n+1}{2}+\frac{9}{2}$, $s=\frac{3^n-1}{4}+\frac{9n}{2}$ 18. $1-\frac{1}{n+1}=\frac{n}{n+1}$
19 $\frac{1}{2}\left(1-\frac{1}{2n+1}\right)=\frac{n}{2n+1}$. 20. $t_n=\frac{1}{(3n-1)(3n+2)}$, $s=\frac{n}{2(3n+2)}$.
21 $\frac{1}{12}(n+1)(2n^2+n+3)$ 22 $t_n=\frac{x^{n-1}}{1-a}-\frac{a^nx^{n-1}}{1-a}$, $s=\frac{1}{(1-x)(1-ax)}$.
23 $\frac{2}{3}n+1$ 25 $\frac{1}{5}(n-1)n(2n-1)$ yds

Appendix. [*pp* 536—551]

1. 8. 2 $(x^2-4)^2(x+3)^2$. 3 $5(x^2-13x-1)$ 4 $(a-1)x+a$
10 $x+4-8x^{-1}$ 11 $\sqrt{2x-3}+\sqrt{x+2}$ 12. $2\frac{1}{2}$. 13 $x=8\frac{4}{5}$,
$y=-11$ 15. 1220 16 91. 17 $2x-\frac{1}{2x}$ 18 $x(x+1)-(x^2+4x+1)$

23. 18. 24. $3\sqrt{5}-2\sqrt{3}$. 25. 4. 26. $x=\frac{ac}{a+b}, y=\frac{bc}{a+b}$.

28. 35. 29 1. 31. $2a^6, -6a^5; 0$. 32 $2\left(\frac{bc}{a}+\frac{ca}{b}+\frac{ab}{c}+\frac{1}{abq}\right)$.

33 $bx-ay$ 37 $\frac{2a}{\sqrt{(x+a)}}$. 39. $\frac{5}{9}$ 40. $x=b-c, y=c-a,$ $z=a-b$. 42. 90, 670, 55. 43 $2\frac{1}{2}$. 44. $4a^2-9b^2+24bc-16c^2$.

45. $(1+x)(1-x)\{1+y+(1-y)x\}\{1+y-(1-y)x\}$. 51. $\frac{1}{5}$

52. $x=\frac{(d-b)(d-c)}{(a-b)(a-c)}$, similar values for y and z. 54. At 4. 12 P. M.

55. a. 56 $(a-b)(x-c)^2(2x-a-b)$ 59. $\frac{2}{3}(3^{\frac{4}{3}}+3^{\frac{2}{3}}+1)$.

60. 1. 61 1 62 $\frac{1}{abcdf}$ 64. 6. 65. $x=a, y=b, z=c$.

67. 72; 60, 30 68 2 213... 69 $(ax+by-1)(bx+ay+1)$.

70 $(2a)^4$ 72 $\frac{x^2-3x+1}{x^2-4x+1}$. 73. $\frac{a^3(a+x)}{x(a^2+ax+x^2)}$. 74. 1

77 $a^x b^y$ 78 $1\frac{3}{7}$ 79 $x=2\frac{1}{2}, y=1\frac{1}{2}$ 81. 5 miles per hr, 15 miles 82 162. 83 $4(ax+by+cz)$ 84 $x(x+y)(x+2y)$.

85 $(x^6-1)^2=$&c 86. $(x-1)(x-2)(x-3)$ 88. $\frac{1}{(\sqrt{x}+1)\sqrt{(x+1)}}$

89. $a^3-3ab^2, a^4-4a^2b^2+2b^4$. 92. 1 93 $x=3, y=4$

95 $10\frac{5}{13}$ hours 96 3 97. $(a-1)x^3-3ax^2-a^2(a-1)x$ $-(a^3+2a^2+2a+1)$ 99. $(a+b)(a-b)(a^2+b^2)(x+y)(x-y)(x^2+y^2)$

102 $\frac{a}{bc}+\frac{b}{ca}+\frac{c}{ab}$ 103 $\frac{x^2}{(x-a)^n}$. 104 1 106 $\frac{\sqrt{(a-x)}}{\sqrt{a}+\sqrt{x}}$.

108 $\frac{3a}{a-1}$. 109. $x=\frac{1}{3}(2a+b), y=\frac{1}{3}(2a-b)$ 112 5.

113. $\frac{11}{5}a-2b$ 114 $4z(x+y-xy)(xyz^2+xyz+1)$

115. $x^2+x(y+z)+yz$. 116. $\frac{7}{2}\sqrt{3}$ 117 $\frac{1-x^2}{2x^2-1}$. 121. $\frac{1}{3}(a+b+c)$

122 $x=y=1, z=0$ 124 7 5 125 68 126 $a^2-3ab+b^2$.

127. $3(x-a)(x-b)(x-c)(c-a)(a-b)(b-c)$ 128. a 129. $\frac{1}{2}(3x-y)$

130. $x^2-\frac{3a^2x}{4b}$ 131. 0 133 b^3 134. $\frac{ac^2}{b^2}$. 135. $x=\frac{a}{m}\cdot\frac{a^2+b^2}{a^2-b^2}$,

$y=\frac{b}{m}\cdot\frac{a^2+b^2}{a^2-b^2}$ 137 240 apples; 60 oranges, worth $1\frac{1}{2}d.$ each

138. $(m^2-4n^2)(2x-y)$, 9 139. $\dfrac{3x^2-ax+a^2}{2x^2+a^2}$ 140. $\frac{1}{2}x^5-\frac{4}{3}x^4+\frac{27}{24}x^3$ $-\frac{43}{12}x^2-\frac{11}{4}x+9$ 141. $(2x-11y+1)(x+2y-3)$ 142 1. 144 $\dfrac{a^{-m}}{b^{-4p}}$ $-\dfrac{c^{-2n}}{a^{-6q}}$ 146 $-\frac{5}{6}$ 147. $x=bc(b-c)$, $y=ca(c-a)$, $z=ab(a-b)$

149 50, 120 150 550 151. $x^8-2x^7a+2x^5a^3-2x^4a^4+2x^3a^5$ $-2xa^7+a^8$ 152 $9a^2+6ab+4b^2$ 153 $x^3-14x^2y+49xy^2-36y^3$.

154. $3\sqrt{\dfrac{x}{y}}-4+3\sqrt{\dfrac{y}{x}}$ 156 $\dfrac{2a}{b}$, $2\sqrt{\left(\dfrac{a^2}{b^2}+1\right)}$

158. $\frac{1}{2}$ 159 $x=a-b$, $y=b-c$, $z=c-a$.

161 A, 4s 8d, B, 6s. 3d. 163 $x^4+\dfrac{x^3}{y}+\dfrac{x^2}{y^2}+\dfrac{x}{y^3}+\dfrac{1}{y^4}$ 164. 10

166 $9x^2+33x+19$ 167 $a+b+c$ 168 $\frac{5}{3}\sqrt{3}-2$ 171. $c-b$, $-a$

172. $\dfrac{2x}{a+2b+c}=\dfrac{2y}{a+b+2c}=\dfrac{2z}{2a+b+c}=a+b+c$ 174 $2\frac{1}{2}$ miles per hour

175 $(4a^2-9c^2)y^4-2acy^3-(a^2-16ac-c^2)y^2+2(2a^2+c^2)y-(4a^2-c^2)$

176 $\frac{5}{16}$ 177 $-9(b-c)(c-a)(a-b)$ 178 $a^2+b^2+c^2+bc+ca+ab$.

179 $\dfrac{xc^y+ye^x}{xe^x+yc^y}$ 180 y^5-5y^3+5y 183. $5\frac{1}{2}$ 184. $x=\frac{4}{3}a$, $y=\frac{5}{4}a$.

185 30 minutes 186 $1+(a+b+c)x+(a^2+b^2+c^2+bc+ca+ab)x^2+\ldots$

187. $\dfrac{x-q}{qx+p}$ 191 $\left\{\dfrac{mn(a+b)}{ab(m+n)}\right\}^{12}$. 193 $a+b$. 194. $x=8$, $y=13$

195 36 196 $ad-bc$ 197 $(a^2-b^2)x^4+2ab^2x^3y+(2a^2+2b^2-a^2b^2)x^2y^2$ $-2ab^2xy^3+(a^2-b^2)y^4$ 198 1 199 $(2x-y-z)(2y-z-x)(2z-x-y)$.

200 $x^2+y^2+z^2+yz+zx+xy$. 201. 0.

202 a^2 or $\dfrac{1}{a^2}$. 204. (1) $\dfrac{a+1}{2b-1}$; (2) 2.

205 $x=a^{\frac{n-2m}{n-m}}b^{\frac{n}{n-m}}$, $y=a^{\frac{n}{n-m}}b^{\frac{n-2m}{n-m}}$ 207. 16 and 20 tons.

209 $(\sqrt{x}+\sqrt{y}+\sqrt{z})(\sqrt{x}-\sqrt{y}-\sqrt{z})(\sqrt{x}+\sqrt{y}-\sqrt{z})(\sqrt{x}-\sqrt{y}+\sqrt{z})$

210 $\dfrac{1-x^2}{1+x^2}$ 211. $\dfrac{(b+c)(c+a)(a+b)}{bc+ca+ab}$ 212 $\frac{3}{2}x^{-\frac{2}{m}}-\frac{1}{3}a^{-\frac{1}{n}}x^{-\frac{1}{a}}$

214. $\left\{\dfrac{\sqrt{(x+y)}-\sqrt{(x-y)}}{\sqrt{(z+y)}-\sqrt{(z-y)}}\right\}^{\frac{3}{2}}$ 215 $\dfrac{a\sqrt{(b-4\sqrt{b})}}{\sqrt{b}-2}$.

216 $x=a^{\frac{m}{m+n}}b^{\frac{n}{m+n}}c^{\frac{m}{m^2-n^2}}d^{\frac{n}{n^2-m^2}}$, $y=a^{\frac{m}{m+n}}b^{\frac{n}{m+n}}c^{\frac{n}{n^2-m^2}}d^{\frac{m}{m^2-n^2}}$.

218 1 mile 220 $(a^2-bc)(b^2-ca)(c^2-ab)$ 221 $1+xyz$.

222 $\frac{1}{abc}$ 223 $\frac{x}{1-x^2}$ 226 $-a$. 227. $-1\frac{1}{3}$. 229. $\frac{7}{3}(x^2+xy+y^2)$.

231. $\frac{a^2+2ac-c^2}{2\sqrt{(2ab-c^2)}}$ 232 Since left side$=3(x-a)(x-b)(x-c)(b-c)$ $\times(c-a)(a-b)$[see *App*, Ex. 127], we get $x=a$ or b or c.

234 1, 2, or 3, 3,